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Factor Model Estimation By Using the Alpha-EM Algorithm

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ABSTRACT. In this paper, we apply the alpha-EM algorithm to factor model estimation. The alpha-EM includes the traditional log-EM as a special case. It has been shown that the convergence speed of the alpha-EM algorithm is much faster than log-EM algorithm. The alpha-EM also allows us to choose different alphas to achieve the fastest convergence speed and more accurate factor model estimation for different problems. In practice the update equations from the alpha-EM are not tractable. We can apply causal approximation and series expansion to those update equations to get practital update equations. With these update equations we can show that the alpha-EM can save us in total computation time. Empirical results from real financial data are given.

Key words : log-EM, alpha-EM, factor model, convergence speed, alpha-logarithm

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1 Introduction

The expecatation-maximization (EM) algorithm is a well known iterative method for finding maximum likelihood estimates of parameters in statistical models, where the model depends on unobserved latent variables. Typically these models involve latent variables in addition to unknown parameters and known data observations. That is, either there are missing values among the data, or the model can be formulated more simply by assuming the existence of additional unobserved data points.

Factor analysis is a statistical method used to describe variability among observed, correlated variables in terms of a potentially lower number of unobserved variables called factors. Factor analysis searches for such joint variations in response to unobserved latent variables. The observed variables are modelled as linear combinations of the potential factors, plus "error" terms. This factor model has been widely used to construct portfolios with certain characteristics, such as risk, because it has many useful properties that sample covariance doesn't have. One advantage of factor model is reduction of number of variables, by combining two or more variables into a single factor. Another advantage is the identification of groups of inter-related variables, to see how they are related to each other.

A. P. Dempster, N. M. Laird and D. B. Rubin presented both the general theory of EM algorithms and a general approach to iterative computation of maximum-likelihood estimates in 1977 [1]. D. B. Rubin and D. T. Thayer applied log-EM algorithm for maximum likelihood factor model analysis in 1982 [2]. After that, a number of methods have been proposed to accelerate the sometimes slow convergence of the EM algorithm, such as those utilising conjugate gradient and modified Newton–Raphson techniques. Xiaoli Meng and D. B. Rubin introduced a class of generalized EM algorithms which they call the ECM algorithm in 1993 [3]. Expectation conditional maximization (ECM) replaces each M step with a sequence of conditional maximization (CM) steps in which each parameter is maximized individually, conditionally on the other parameters remaining fixed. Chuanhai Liu and Donald B. Rubin introduced a simple estension of EM and ECM with faster covergence in 1994 [4], which they call the ECME algorithm. They applied ECEM algorithm for maximum likelihood estimation of factor analysis in 1998 [5]. This idea is further extended in generalized expectation maximization (GEM) algorithm, in which one only seeks an increase in the objective function for both the E step and M step under the alternative description. The Q-function used in the EM algorithm is based on the log-likelihood. Thus in this paper we call it the log-EM algorithm.

The alpha-EM algorithm was introduced by Yasuo Matsuyama [6], [7], [8], who also proved the covergence speed of the alpha-EM is faster than the log-EM if the incomplete data comes from an exponential family. Logarithms have important roles besides simplifying the likelihood maximazation. In information measures, logarithmic is correspond to Kullback-Leibler divergence which is a key for realizing the maximization transfer in the EM algorithm [1]. The alpha-EM derived by the maximization transfer which uses more general surrogate

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functions than the logarithmic one. The use of the log-likelihood ratio can be generalized to that of the alpha-log-likelihood ratio. The log-EM corresponds to the special case of $\alpha = -1$. Yasuo Matsuyama also applied alpha-EM to clustering and the results showed that it is better than the log-EM algorithm in terms of both the number of iterations and the total computation time. In 2010 and 2011 Yasuo Matsuyama applied alpha-EM algorithm to hidden markov model estimation [9], [10]. It had been expected that the alpha-EM for factor model estimation would exist. On one hand, the complete data of factor model comes from exponential family, so theoretically alpha-EM can be applied to factor analysis. On the other hand, the convergence speed of the log-EM for factor model can be slow when the problem is not well conditioned. However for applications such as high frequency trading, problems may be ill condition and require fast computation. Since the log-EM is a subclass of the alpha-EM, the alpha-EM can only do better than the log-EM. However, there are several hurdles to implement the alpha-EM for factor model in practice. In this paper, we presented a way to use the alpha-EM for factor model estimation.

The orgnization of the main text is as follows. In Section II, factor model, the log-EM algorithm and the alpha-EM algorithm are presented. In Section III, the update equations for factor model parameters are presented. In Section IV, we applied alpha-EM algorithm for factor model estimation on real financail data which is the S&P 500 from 2007 to 2012. Section V gives concluding remarks.

2 Model Description

2.1 Factor Model

Suppose we have a set of p observable random variables, $x_1, x_{2,\dots} x_p$ with means, $\mu_1, \mu_2, \dots, \mu_p$ and each variable has n observations. Suppose for some unknown factor loading matrix $\Lambda_{p\times d}$ and d unobserved factor-score $z_1, z_2, \dots, z_d$ and each factor-score has n unobserved factors. We have:

$$x_i - \mu_i = \Lambda_{i,1} z_1 + \Lambda_{i,2} z_2 + \cdots + \Lambda_{i,d} z_d + u_i \ , \ 1 < i < p \tag{1}$$

where u_i are independently distributed error terms with zero mean and finite variance, which may not be the same for all i. Let X be the $p \times n$ observed data matrix with zero mean, if the mean is not zero we can always subtract it from X, and Z be the $d \times n$ unoberserved factor-score matrix where $d < p$. In matrix terms the generative model is given by:

$$X = \Lambda Z + U \tag{2}$$

We also will impose the following assumptions on Z and U.

1. Z and U are independent.
2. $E[Z] = 0$ and $Cov[Z] = I$ (to make sure that the factors are uncorrelated)

3. $E[U] = 0$ and $Cov[U] = Diag(\phi_1, \phi_2, \ldots, \phi_n) \stackrel{def}{=} \Phi$

Suppose $Cov[X] = \Sigma$ and we have $Cov[X] = Cov[\Lambda Z + U]$, so we can get $\Sigma = \Phi + \Lambda\Lambda'$. So factor model estimation means that given observed data X we need to estimate Φ and Λ.

Note that for any orthogonal matrix Q if we set $\Lambda^* = \Lambda Q$ and $Z^* = Q'Z$, the criteria for being factors and factor loadings still hold. Hence a set of factors and factor loadings is identical only up to orthogonal transformation.

2.2 Log-EM Algorithm

Given a statistical model consisting of observed data X, unobserved latent data Z and unknown parameters Φ, Λ, along with log-likelihood function, the maximum likelihood estimate of the unknown parameters is determined by the marginal likelihood of the observed data. Then the incomplete data log-likelihood is $L(X, \Phi, \Lambda) = \log \prod_i^n p(x_i|\Phi, \Lambda)$. On the other hand, the complete data log-likelihood is $L_C(X, Z, \Phi, \Lambda) = \log \prod_i^n p_C(x_i, z_i|\Phi, \Lambda)$. The log-EM algorithm seeks to maximize the marginal likelihood by iteratively applying the following two steps: (The subscripts of Φ_0, Λ_0 mean the current estimates and the subscripts of Φ_1, Λ_1 mean the next estimates)

- Expectation step (E step): Calculate the expected value of the log likelihood function, with respect to the conditional distribution of z given x under the current estimate of the parameters Φ_0, Λ_0

$$Q(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) = E_{Z|X,\Phi_0,\Lambda_0}[L_C(X, Z, \Phi, \Lambda)]$$

- Maximization step (M step): Find the parameter that maximizes this quantity:

$$\Phi_1, \Lambda_1 = \arg \max_{\Phi_1,\Lambda_1} Q(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$$

Given Φ_0, Λ_0 and x_i, i means the ith observation, the expected value of the factors z_i can be computed and this computation is in fact necessary for log-EM algorithm. For the distribution of the observed variable $p(x_i)$ we have $E[x_i] = 0$ and $Cov[X] = \Sigma = \Phi_0 + \Lambda_0\Lambda_0'$. For the distribution of the complete data $p(x_i, z_i)$, let $y_i = \begin{bmatrix} x_i \\ z_i \end{bmatrix}$ and $Y = \begin{bmatrix} X \\ Z \end{bmatrix}$, we have $E[y_i] = 0$ and $Cov[Y] = \begin{bmatrix} \Phi_0 + \Lambda_0\Lambda_0' & \Lambda_0 \\ \Lambda_0 & I \end{bmatrix}$. For the distribution $p(z_i|x_i)$ we have $p(z_i|x_i) = \frac{p(x_i,z_i)}{p(x_i)}$. Since we know the distribution of $p(x_i)$ and $p(x_i, z_i)$, we will have $E[z_i|x_i] = \beta x_i$ and $Var[z_i|x_i] = C$ where:

$$\begin{aligned} \beta &= \Lambda_0'(\Phi_0 + \Lambda_0\Lambda_0')^{-1} \\ C &= I - \Lambda_0'(\Phi_0 + \Lambda_0\Lambda_0')^{-1}\Lambda_0 \end{aligned}$$

2.3 Alpha-EM Algorithm

The alpha-logarithm function is defined as follows [8]:

$$L^{(\alpha)}(r) \stackrel{def}{=} \frac{2}{1+\alpha}\left(r^{\frac{1+\alpha}{2}} - 1\right) \tag{3}$$

where $r \in (0, \infty)$. $L^{(\alpha)}(r)$ is strictly concave for $\alpha < 1$, a straight line $r - 1$ for $\alpha = 1$ and strictly convex for $\alpha > 1$. Especially when $\alpha = -1$ we have $L^{(-1)} = \log(r)$.

Let $P_I(X|\Phi, \Lambda)$ be a probability density for the observed(incomplete) data X parameterized by Φ and Λ. Let $P_C(X, Z|\Phi, \Lambda)$ be a probability density for the complete data. Then the incomplete data alpha-log-likelihood ratio is:

$$L_X^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) = \frac{2}{1+\alpha}\left[\left(\frac{P_I(X|\Phi_1, \Lambda_1)}{P_I(X|\Phi_0, \Lambda_0)}\right)^{\frac{1+\alpha}{2}} - 1\right] \tag{4}$$

On the other hand, the complete data alpha-log-likelihood ratio is :

$$L_{X,Z}^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) = \frac{2}{1+\alpha}\left[\left(\frac{P_C(X, Z|\Phi_1, \Lambda_1)}{P_C(X, Z|\Phi_0, \Lambda_0)}\right)^{\frac{1+\alpha}{2}} - 1\right] \tag{5}$$

by taking the conditional expectation in terms of $P_{Z|X,\Phi_0,\Lambda_0}$ we can get

$$Q_{X,Z|X}^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) = E\left[L_{X,Z}^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)\right]$$

by computing the alpha-divergence between $P_{Z|X,\Phi_0,\Lambda_0}(Z|X, \Phi_0, \Lambda_0)$ and $P_{Z|X,\Phi_1,\Lambda_1}(Z|X, \Phi_1, \Lambda_1)$ we have the following basic equality for the alpha-EM algorithm [6], [7], [8].

$$L_X^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) = Q_{X,Z|X}^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) + \frac{1-\alpha}{2}\left\{\frac{P(X|\Phi_1, \Lambda_1)}{P(X|\Phi_0, \Lambda_0)}\right\}^{\frac{1+\alpha}{2}} D^{(\alpha)}(\Phi_1, \Lambda_1||\Phi_0, \Lambda_0) \tag{6}$$

Therefore, the alpha-log likelihood ratio of the observed data can be exactly expressed in terms of the Q-function of the alpha-log likelihood ratio and the alpha-divergence. The alpha-divergence is an information measure. When $\alpha = \pm 1$, it is the Kullback-Leibler divergence. When $\alpha = 0$, it is the well known Hellinger distance. Equation (6) is the core of the alpha-EM algorithm. The second term on right-hand side is nonnegative for $\alpha < 1$, this also ensures positivity of the alpha-information matrix. So the algorithm to increase $L_X^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$ is obtained by increasing the $Q_{X,Z|X}^{(\alpha)}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$ function with respect to the argument Φ_1 and Λ_1.

Obtaining this $Q^{(\alpha)}_{X,Z|X}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$ function is a generalized E step. Its maximization is a generalized M step. This pair is called the alpha-EM algorithm which contains the log-EM algorithm as its subclass. Thus, the alpha-EM algorithm of Yasuo Matsuyama is an exact generalization of the log-EM algorithm. The alpha-EM shows faster convergence than the log-EM algorithm by choosing an appropriate alpha. For possible choices of alpha, we already have $\alpha < 1$, and on the other hand the alpha-EM requires $\alpha > -1$ for the exponential family.

3 Estimation Using Alpha-EM

3.1 Non-Causal Update Equations

Here, by non-causal we mean that given the current estimations we can not use the update equations to get the next estimations directly. But the non-causal update equations are the most accurate equations you can get them after applying the alpha-EM for factor model estiamtion. In order to be able to use the alpha-EM we use a causal approximation of non-causal update equations.

For factor model, we have

$$P_C(X, Z|\Phi_0, \Lambda_0) = \prod_{i=1}^{N} P_c(x_i, z_{i,}|\Phi_0, \Lambda_0) = \prod_{i=1}^{N} P(x_i|z_i, \Phi_0, \Lambda_0) * P(z_i)$$

so the $Q^{(\alpha)}_{X,Z|X}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$ function is:

$$\begin{aligned} Q^{(\alpha)}_{X,Z|X}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0) &= E_{P(Z|X,\Phi_0,\Lambda_0)}[L^{(\alpha)}_{X,Z}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)] \qquad (7) \\ &= E_{P(Z|X,\Phi_0,\Lambda_0)}\left[\frac{2}{1+\alpha}\left(\frac{P_C(X, Z|\Phi_1, \Lambda_1)}{P_C(X, Z|\Phi_0, \Lambda_0)}\right)^{\frac{1+\alpha}{2}} - 1\right] \\ &= \frac{2}{1+\alpha}\left(\prod_{i=1}^{N} E_{P(z_i|x_i,\Lambda_0,\Phi_0)}\left[\left(\frac{Pc(x_i, z_i|\Lambda_1, \Phi_1)}{Pc(x_i, z_i|\Lambda_0, \Phi_0)}\right)^{\frac{1+\alpha}{2}}\right] - 1\right) \\ &= \frac{2}{1+\alpha}\left(S^{(\alpha)}_{Z|X,\Phi_0,\Lambda_0} - 1\right) \end{aligned}$$

where

$$\begin{aligned} S^{(\alpha)}_{Z|X,\Phi_0,\Lambda_0} &= \prod_{i=1}^{N} W^{(\alpha)}_i \\ W^{(\alpha)}_i &= E\left[P_i^{\frac{1+\alpha}{2}}\right] \\ P_i &= \frac{Pc(x_i, z_i|\Lambda_1, \Phi_1)}{Pc(x_i, z_i|\Lambda_0, \Phi_0)} \\ E[\cdot] &= E_{P(z|x,\Lambda_0,\Phi_0)}[\cdot] \end{aligned}$$

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After the E-step we need to do the M-step. The update equations can be obtained by differentiating $Q^{(\alpha)}_{X,Z|X}(\Phi_1, \Lambda_1|\Phi_0, \Lambda_0)$ with respect to the update parameters Φ_1 and Λ_1 and setting differentiation to zero solve for maximization. For Λ_1 we have

$$\begin{aligned}
\frac{\partial Q^{(\alpha)}}{\partial \Lambda_1} &= 0 \Rightarrow \frac{\partial S^{(\alpha)}}{\partial \Lambda_1} = 0 \qquad (8)\\
\frac{\partial S^{(\alpha)}}{\partial \Lambda_1} &= \sum_{j=1}^{N} \frac{\partial W_j^{(\alpha)}}{\partial \Lambda_1} \prod_{i=1, i\neq j}^{N} W_i^{(\alpha)} = \sum_{j=1}^{N} \frac{\partial W_j^{(\alpha)}}{\partial \Lambda_1} \frac{S^{(\alpha)}}{W_j^{(\alpha)}} = \sum_{j=1}^{N} \frac{\frac{\partial W_j^{(\alpha)}}{\partial \Lambda_1}}{W_j^{(\alpha)}} S^{(\alpha)}\\
S^{(\alpha)} &\neq 0 \Rightarrow \sum_{i=1}^{N} \frac{\frac{\partial W_i^{(\alpha)}}{\partial \Lambda_1}}{W_i^{(\alpha)}} = 0 \Rightarrow \sum_{j=1}^{N} \frac{\frac{\partial E\left[P^{\frac{1+\alpha}{2}}\right]}{\partial \Lambda_1}}{E\left[P^{\frac{1+\alpha}{2}}\right]} = \sum_{j=1}^{N} \frac{E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Lambda_1}\right]}{E\left[P^{\frac{1+\alpha}{2}}\right]} = 0
\end{aligned}$$

likewise for Φ_1 we have

$$\frac{\partial Q^{(\alpha)}}{\partial \Phi_1^{-1}} = 0 \Rightarrow \frac{\partial S^{(\alpha)}}{\partial \Phi_1^{-1}} = 0 \Rightarrow \sum_{j=1}^{N} \frac{\frac{\partial E\left[P^{\frac{1+\alpha}{2}}\right]}{\partial \Phi_1^{-1}}}{E\left[P^{\frac{1+\alpha}{2}}\right]} = \sum_{j=1}^{N} \frac{E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Phi_1^{-1}}\right]}{E\left[P^{\frac{1+\alpha}{2}}\right]} = 0 \qquad (9)$$

In order to solve equations (8) and (9) we need to calculate $E\left[P^{\frac{1+\alpha}{2}}\right]$, $E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Lambda_1}\right]$ and $E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Phi_1^{-1}}\right]$. By the definition of expectation:

$$\begin{aligned}
E\left[P^{\frac{1+\alpha}{2}}\right] &= \int P(z_i|x_i, \Lambda_0, \Phi_0) \left(\frac{Pc(x_i, z_i|\Lambda_1, \Phi_1)}{Pc(x_i, z_i|\Lambda_0, \Phi_0)}\right)^{\frac{1+\alpha}{2}} dz_i \qquad (10)\\
E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Lambda_1}\right] &= \frac{1+\alpha}{2} E\left[P^{\frac{1+\alpha}{2}} * \left(\Phi_1^{-1} x_i z_i' - \Phi_1^{-1} \Lambda_1 z_i z_i'\right)\right] \qquad (11)
\end{aligned}$$

$$E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Phi_1^{-1}}\right] = \frac{1+\alpha}{2} E\left[P^{\frac{1+\alpha}{2}} * \frac{1}{2}\left(\Phi_1 - x_i x_i' + x_i z_i' \Lambda_1' + \Lambda_1 z_i x_i' - \Lambda_1 z_i z_i' \Lambda_1'\right)\right] \qquad (12)$$

After calculate the expectations, we have the update equations:

$$\begin{aligned}
\Lambda_1 &= \sum_{i=1}^{N} x_i E[z_i'] \left(\sum_{i=1}^{N} E[z_i z_i']\right)^{-1} \qquad (13)\\
\Phi_1 &= diag\left(\frac{1}{n}\left(\sum_{i=1}^{N} x_i x_i' - \sum_{i=1}^{N} x_i E[z_i'] \Lambda_1'\right)\right) \qquad (14)
\end{aligned}$$

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However, the expectation here is w.r.t. a new distribution:

$$\begin{aligned} E[z_i] &= \Sigma W' x_i \Rightarrow E[z_i'] = x_i' W \Sigma \\ E[z_i z_i'] &= Var[z_i] + E[z_i]E[z_i]' = \Sigma + \Sigma W' x_i x_i' W \Sigma \\ \Sigma^{-1} &= \frac{1+\alpha}{2}\Lambda_1'\Phi_1^{-1}\Lambda_1 - \frac{1+\alpha}{2}\Lambda_0'\Phi_0^{-1}\Lambda_0 + C^{-1} \\ W &= \frac{1+\alpha}{2}\Phi_1^{-1}\Lambda_1 - \frac{1+\alpha}{2}\Phi_0^{-1}\Lambda_0 + \beta' C^{-1} \end{aligned}$$

if we Assume the sample covariance is $C_{xx} = \sum_{j=1}^{N} \frac{x_i x_i'}{N}$, we get:

$$\Lambda_1 = C_{xx} W \Sigma \left(\Sigma + \Sigma W' C_{xx} W \Sigma\right)^{-1} \tag{15}$$

$$\Phi_1 = diag(C_{xx-} C_{xx} W \Sigma \Lambda_1') \tag{16}$$

and we notice that we have Φ_1 and Λ_1 on both right hand side and left hand side of these update equations. It is hard to put either Φ_1 or Λ_1 on one side of the equations and this is what we mean by non-causal. So we can not use these update quations directly.

For general case $-1 < \alpha < 1$, the update equations:

$$\Lambda_1 = F(\Lambda_1, \Phi_1, \Phi_0, \Lambda_0) \tag{17}$$

$$\Phi_1 = G(\Lambda_1, \Phi_1, \Phi_0, \Lambda_0) \tag{18}$$

are non-causal but they illustrate two important things. First, we can iteratively update Λ_1 and Φ_1 through (17) and (18) until Λ_1 and Φ_1 converge and we call it one major iteration which is the iterations we count in the log-EM. Then replace Φ_0, Λ_0 with Λ_1, Φ_1, do the same thing for the next major iteration Λ_2, Φ_2. In practice, the convergence speed is much faster than the log-EM for the same major iteration. Second, each major iteration here contains many minor iterations which will take a large amount of time. Using this updating method the alpha-EM can not save us in total computation time in practice. Therefore, on one hand we know that the alpha-EM is better than the log-EM in convergence speed, on the other hand we need effective update equations otherwise we can't use the alpha-EM in practice.

Let's consider two special cases first:

Case 1. $\alpha = -1$:

$$\begin{aligned} \Sigma^{-1} &= C^{-1} \\ W &= \beta' C^{-1} \end{aligned}$$

then we have $W\Sigma = \beta' = (\Phi_0 + \Lambda_0\Lambda_0')^{-1}\Lambda_0$. Assume that

$$\begin{aligned}\delta_0 &= (\Phi_0 + \Lambda_0\Lambda_0')^{-1}\Lambda_0 \\ \Delta_0 &= I_d - \Lambda_0'(\Phi_0 + \Lambda_0\Lambda_0')^{-1}\Lambda_0\end{aligned}$$

we get

$$\Lambda_1 = C_{xx}\delta_0(\Delta_0 + \delta_0' C_{xx}\delta_0)^{-1} = f(\Lambda_0, \Phi_0) \tag{19}$$
$$\Phi_1 = diag(C_{xx-}C_{xx}\delta_0\Lambda_1') = g(\Lambda_1, \Lambda_0, \Phi_0) \tag{20}$$

these update equations are the same as Rubin and Thayer [2]. This shows that the log-EM algorithm is a subset of the alpha-EM algorithm.

Case 2. $\alpha = 1$:

$$\Lambda_1 = C_{xx}\delta_1(\Delta_1 + \delta_1' C_{xx}\delta_1)^{-1} = f(\Lambda_1, \Phi_1) \tag{21}$$
$$\Phi_1 = diag(C_{xx-}C_{xx}\delta_1\Lambda_1') = g(\Lambda_1, \Phi_1) \tag{22}$$

where

$$\begin{aligned}\delta_1 &= (\Phi_1 + \Lambda_1\Lambda_1')^{-1}\Lambda_1 \\ \Delta_1 &= I_d - \Lambda_1'(\Phi_1 + \Lambda_1\Lambda_1')^{-1}\Lambda_1\end{aligned}$$

here we have two equations and two unknown parameters Λ_1 and Φ_1 but it is impossible to solve for Λ_1 and Φ_1 directly. Because if we assume Λ_1 and Φ_1 are the optimal solutions, we have

$$C_{xx} = \Phi_1 + \Lambda_1\Lambda_1' \tag{23}$$

If we substitute (23) back to (21) and (22), we get:

$$\begin{aligned}\Lambda_1 &= \Lambda_1 \\ \Phi_1 &= diag(\Phi_1) = \Phi_1\end{aligned}$$

Since we can not solve (21) and (22), we can iteratively update Λ_1 and Φ_1 through (21) and (22) until Λ_1 and Φ_1 don't change. In practice this method takes exactly the same computation time as when $\alpha = -1$ because they have identical f and g. In order to have a practical solution we need to solve the non-causality.

3.2 Causal update equations

In order to solve the non-causality, we need to know why we have non-causality in the first place. The reason is the expectations, equation (10), (11) and

(12). We need to calculate the three expectaions $E\left[P^{\frac{1+\alpha}{2}}\right]$, $E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Lambda_1}\right]$ and $E\left[\frac{\partial P^{\frac{1+\alpha}{2}}}{\partial \Phi_1^{-1}}\right]$ in a causal way. These three expectations have integral of the form:

$$\int P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_1,\Phi_1)}{Pc(x_i,z_i|\Lambda_0,\Phi_0)}\right)^{\frac{1+\alpha}{2}} dz_i \tag{24}$$

in common and we need to calculate (24) without using Λ_1 and Φ_1.

3.2.1 Causal approximation

We have

$$\begin{aligned} P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_1,\Phi_1)}{Pc(x_i,z_i|\Lambda_0,\Phi_0)}\right)^{\frac{1+\alpha}{2}} &\approx P(z_i|x_i,\Lambda_1,\Phi_1)\left(\frac{Pc(x_i,z_i|\Lambda_1,\Phi_1)}{Pc(x_i,z_i|\Lambda_0,\Phi_0)}\right)^{-\frac{1-\alpha}{2}} \quad (25)\\ &\approx P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}\right)^{-\frac{1-\beta}{2}} \end{aligned}$$

around the region of $P(x_i|\Lambda_1,\Phi_1) = P(x_i|\Lambda_0,\Phi_0)+o(1)$ and the last term is the causal approximation w.r.t. the iteration index shift or shift of time [9], [10]. We can use the relationship:

$$\frac{1+\alpha}{2} = -\frac{1-\beta}{2} \Rightarrow \beta = \alpha+2 \tag{26}$$

and because we have $\alpha \in (-1,1)$, then we also have $\beta \in (1,3)$.

Within the first few interations $\beta = \alpha+2$ is not always a good approximation. For example, when $\alpha = 1$, $P_1/P_0 \approx P_0/P_{-1}$ is not a good approximation in the first a few iterations. This bad approximation will cause us numerical problems in practice. Another example is when $\alpha = 0$, $(P_1/P_0)^{1/2} \approx (P_0/P_{-1})^{1/2}$ is a better approximation during the first a few iterations since P_1/P_0 and P_0/P_{-1} are greater than 1. So for choice of α close to 1, we can choose β starting at 2 and approaching $\alpha+2$ as iteration increases. We will see this is sometimes necessary in practice.

Now, we can approximately calculate (24) without knowing Λ_1,Φ_1. However, this requires a power computation of a likelihood ratio. This is computationally expensive and becomes intractability as time increases. So another approximation is necessary in view of computational complexity.

3.2.2 Series Expansion

A taylor expansion can simplify this without discarding merit of the alpha-log likelihood ratio.

$$P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}\right)^{-\frac{1-\beta}{2}} = P(z_i|x_i,\Lambda_{-1},\Phi_{-1})\left(\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}\right)^{\frac{1+\beta}{2}} \tag{27}$$

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Let's assume that $f(x) = x^{\frac{1+\beta}{2}}$, according to Taylor expansion we have $f(x) = f(r) + \frac{f'(r)}{1!}(x-r) + o(1)$ For our case $x = \frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}$ and assume $r = 1$, so we get [9], [10]:

$$\left(\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}\right)^{\frac{1+\beta}{2}} \approx \frac{1-\beta}{2} + \frac{1+\beta}{2}\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})} \tag{28}$$

now we substitute the right hand side of equation (25) with equation (27) and (28) then we get:

$$P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_1,\Phi_1)}{Pc(x_i,z_i|\Lambda_0,\Phi_0)}\right)^{\frac{1+\alpha}{2}} \approx \frac{1-\beta}{2}P(z_i|x_i,\Lambda_{-1},\Phi_{-1}) + \frac{1+\beta}{2}P(z_i|x_i,\Lambda_0,\Phi_0)$$

So, now we can calculate the expectaions in a causal way without using Λ_1 and Φ_1. We get the update equations:

$$\Lambda_1 = \frac{\left(\frac{1-\beta}{2}\sum_{j=1}^{N} x_i E_{-1}[z_i'] + \frac{1+\beta}{2}\sum_{j=1}^{N} x_i E_0[z_i']\right)}{\left(\frac{1-\beta}{2}\sum_{j=1}^{N} E_{-1}[z_i z_i'] + \frac{1+\beta}{2}\sum_{j=1}^{N} E_0[z_i z_i']\right)} \tag{29}$$

$$\Phi_1 = diag\left(C_{xx-}C_{xx}\left(\frac{1-\beta}{2}\sum_{j=1}^{N} x_i E_{-1}[z_i'] + \frac{1+\beta}{2}\sum_{j=1}^{N} x_i E_0[z_i']\right)\Lambda_1'\right) \tag{30}$$

where

$$\begin{aligned} E_{-1}[z_i] &= \beta_{-1}x_i \text{ and } E_{-1}[z_i z_i'] = C_{-1}^{-1} + \beta_{-1}x_i x_i' \beta_{-1}' \\ E_0[z_i] &= \beta_0 x_i \text{ and } E_0[z_i z_i'] = C_0^{-1} + \beta_0 x_i x_i' \beta_0' \end{aligned}$$

with

$$\begin{aligned} \beta_{-1} &= \Lambda_{-1}'(\Phi_{-1} + \Lambda_{-1}\Lambda_{-1}')^{-1} \\ C_{-1} &= I - \Lambda_{-1}'(\Phi_{-1} + \Lambda_{-1}\Lambda_{-1}')^{-1}\Lambda_{-1} \\ \beta_0 &= \Lambda_0'(\Phi_0 + \Lambda_0\Lambda_0')^{-1} \\ C_0 &= I - \Lambda_0'(\Phi_0 + \Lambda_0\Lambda_0')^{-1}\Lambda_0 \end{aligned}$$

For the first a few iterations that $a = 1$ is not a very accurate estimation, we should choose a close to 1, so that:

$$\left(\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}\right)^{\frac{1+\beta}{2}} \approx \frac{1-\beta}{2}a^{\frac{1+\beta}{2}} + \frac{1+\beta}{2}a^{\frac{1+\beta}{2}-1}\frac{Pc(x_i,z_i|\Lambda_0,\Phi_0)}{Pc(x_i,z_i|\Lambda_{-1},\Phi_{-1})}$$

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and

$$P(z_i|x_i,\Lambda_0,\Phi_0)\left(\frac{Pc(x_i,z_i|\Lambda_1,\Phi_1)}{Pc(x_i,z_i|\Lambda_0,\Phi_0)}\right)^{\frac{1+\alpha}{2}} \approx \frac{1-\beta}{2}a^{\frac{1+\beta}{2}}P(z_i|x_i,\Lambda_{-1},\Phi_{-1})+\frac{1+\beta}{2}a^{\frac{1+\beta}{2}-1}P(z_i|x_i,\Lambda_0,\Phi_0)$$

So the update equations are:

$$\Lambda_1 = \frac{\left(\frac{1-\beta}{2}a^{\frac{1+\beta}{2}}\sum_{j=1}^{N} x_i E_{-1}[z_i'] + \frac{1+\beta}{2}a^{\frac{1+\beta}{2}-1}\sum_{j=1}^{N} x_i E_0[z_i']\right)}{\left(\frac{1-\beta}{2}a^{\frac{1+\beta}{2}}\sum_{j=1}^{N} E_{-1}[z_i z_i'] + \frac{1+\beta}{2}a^{\frac{1+\beta}{2}-1}\sum_{j=1}^{N} E_0[z_i z_i']\right)} \tag{31}$$

$$\Phi_1 = diag\left(C_{xx} - C_{xx}\left(\frac{1-\beta}{2}a^{\frac{1+\beta}{2}}\sum_{j=1}^{N} x_i E_{-1}[z_i'] + \frac{1+\beta}{2}a^{\frac{1+\beta}{2}-1}\sum_{j=1}^{N} x_i E_0[z_i']\right)\Lambda_1'\right) \tag{32}$$

Now we get two causal update equations and we are able to use to do the factor model estimation. At the kth iteration to obtain Λ_k and Φ_k, we use $\Lambda_{k-1}, \Phi_{k-1}$ and $\Lambda_{k-2}, \Phi_{k-2}$ on the right-hand sides of (29) and (30). We need to calculate the $E[z]$ and $E[zz']$ of the previuous state and the one before previous state. It seems that the alpha-EM needs to do more calcuation in each update but in practice acutully we can save $E[z]$ and $E[zz']$ of the previous state for the next iteration, so we don't need to recalculate it. For example, for Λ_1 and Φ_1 we need to calculate $E_{-1}[z_i']$ and $E_0[z_i']$, when we calculate Λ_2 and Φ_2 we can reuse $E_0[z_i']$ and we only need to calcualte $E_1[z_i']$. We only compute $E[z]$ and $E[zz']$ once for each state and this is exactly the same with the log-EM. Thus, the total computation time per iteration of the alpha-EM or the log-EM are almost the same. We will see the numerical results in the following section.

4 Empirical Results

4.1 Factor analysis from complete observations

Here, we applied both the log-EM and the alpha-EM to the same data used in Rubin and Thayer (1982) with $p = 9$ and $d = 4$.

$$C_{xx} = \begin{bmatrix} 1.0 & 0.554 & 0.227 & 0.189 & 0.461 & 0.506 & 0.408 & 0.280 & 0.241 \\ 0.554 & 1.0 & 0.296 & 0.219 & 0.479 & 0.530 & 0.425 & 0.311 & 0.311 \\ 0.227 & 0.296 & 1.0 & 0.769 & 0.237 & 0.243 & 0.304 & 0.718 & 0.730 \\ 0.189 & 0.219 & 0.769 & 1.0 & 0.212 & 0.226 & 0.291 & 0.681 & 0.661 \\ 0.461 & 0.479 & 0.237 & 0.212 & 1.0 & 0.520 & 0.514 & 0.313 & 0.245 \\ 0.506 & 0.530 & 0.243 & 0.226 & 0.520 & 1.0 & 0.473 & 0.348 & 0.290 \\ 0.408 & 0.425 & 0.304 & 0.291 & 0.514 & 0.473 & 1.0 & 0.374 & 0.306 \\ 0.280 & 0.311 & 0.718 & 0.681 & 0.313 & 0.348 & 0.374 & 1.0 & 0.672 \\ 0.241 & 0.311 & 0.730 & 0.661 & 0.245 & 0.290 & 0.306 & 0.672 & 1.0 \end{bmatrix}$$

To do the 1st iteration to obtain Λ_2 and Φ_2, we require previous two estimates which are Λ_0, Φ_0 and Λ_1, Φ_1. For Λ_{-1} and Φ_{-1}, we can use a random guess as:

$$\begin{aligned} \Phi_{-1} &= diag(C_{xx}) \\ Vr &= \text{rand}(p, d) \\ V_{-1} &= Vr * \sqrt{\|C_{xx}\|_F \,/\, \|V\|_F} \end{aligned}$$

where $p = 9$, $d = 4$ and then for Λ_0, Φ_0 and Λ_1, Φ_1 we can do the log-EM by using (19) and (20) with Λ_{-1} and Φ_{-1}. With Λ_0, Φ_0 and Λ_1, Φ_1 we can apply the alpha-EM now. Fig. 1 illustrates the different convergence curves of the alpha-EM in previous section with different alpha values. Remember that when $\alpha = -1$, it is the log-EM.

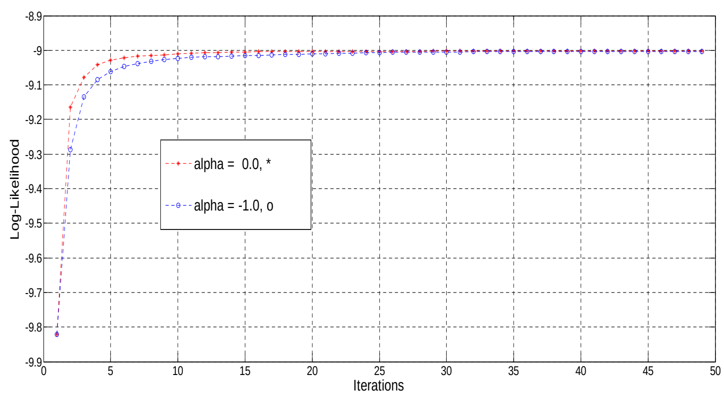

Fig.1 Convergence speed for various α

The log-likelihood on the y-axis is calculated by

$$LL = \log\det((\Phi + \Lambda\Lambda')^{-1}C_{xx}) - trace((\Phi + \Lambda\Lambda')^{-1}C_{xx}) \tag{33}$$

so for the optimal results we have $C_{xx} = \Phi + \Lambda\Lambda'$ then $LL = -p$ where p is the dimentionality of the problem. Whether you can reach the optimal results depends on the condition of the problem. Factor analysis can be only as good as the data allows.

Table I shows a speedup comparision. The second column shows that the alpha-EM (the case of $\alpha = 0$) is 30/15=2.00 times faster than the log-EM (the case of $\alpha = -1$) for the same convergence. The third and fourth columns show a more practical comparision based upon CPU time. We use t denotes the total time per iteration. The alpha-EM didn't require more CPU time per iteration. So we can see that the alpha-EM is much faster than the log-EM by a total CPU-time speedup ratio of 30t/15t=2.00.

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TABLE I Speedup Ratio For Factor Model Estimation(p=9, d=4)

α	Iterations	Time per Iteration	Total CPU-Time	Speedup Ratio
-1.00	30	1t	30t	1.00
0	15	1t	15t	2.00

Besides the log-likelihood, we can compare the Hellinger distance between different alpha. The definition of Hellinger distance is $H^2 = 1/2\int \left(\sqrt{P_0} - \sqrt{P_1}\right)^2 dx$ where P_0 and P_1 are probability density functions. In our case $P_0 \sim N(0, C_{xx})$ and $P_1 \sim N(0, \Phi + \Lambda\Lambda')$ then we have:

$$\begin{aligned} H^2 &= 1/2\int \left(\sqrt{P_0} - \sqrt{P_1}\right)^2 dx = 1 - \int \sqrt{P_0}\sqrt{P_1} dx \\ &= 1 - \det\left(\left(C_{xx}^{-1} + (\Phi + \Lambda\Lambda')^{-1}\right)/2\right)^{-\frac{1}{2}} \det(C_{xx})^{-\frac{1}{4}} \det(\Phi + \Lambda\Lambda')^{-\frac{1}{4}} \end{aligned}$$

Fig.2 illustrates the decreasing speed in Hellinger distance as the number of iteration increases for various α.

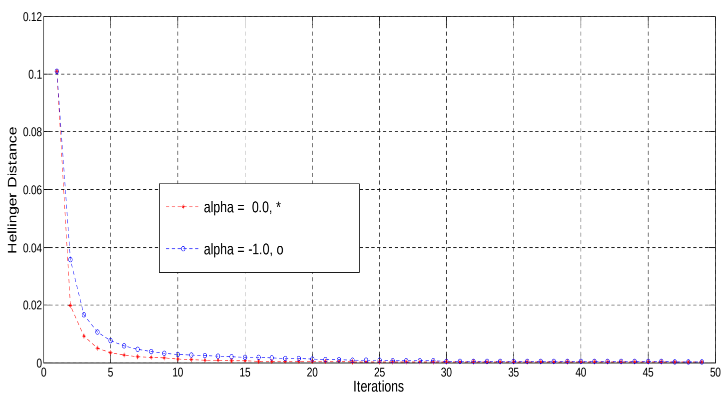

Fig.2 Hellinger distance for various α

From [11] we know that Hellinger distance is actually bounded by the Information distance.

$$H(\Theta_0, \Theta_1) \leq \frac{1}{\sqrt{8}} I(\Theta_0, \Theta_1)$$

Especially for sufficiently samll $I(\Theta_0, \Theta_1)$, we have:

$$\frac{K}{\sqrt{8}} I(\Theta_0, \Theta_1) \leq H(\Theta_0, \Theta_1)$$

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where $0 < K < 1$. This shows that the upper bound of the Hellinger distance is the best possible. So if we can get the the sample covariance and factor model close in Hellinger distance, it also implies that they are close in Information distance.

4.2 Factor analysis on financial data

Here, we download Yahoo daily close prices of each member of S&P500 from Jan-03-2007 to May-31-2013. Then we select members which have prices since Jan-03-2007 and we got 471 members left. We calculate the sample covariance C first, then with the sample covariance we can calcualte the factor model by both log-EM algorithm and alpha-EM algorithm. In order to do the 1st iteration to obtain Λ_1 and Φ_1, it requires previous two states which are Λ_0, Φ_0 and Λ_{-1}, Φ_{-1}. For Λ_{-1} and Φ_{-1}, we use the a random guess method but $d = 20$ and then we use the log-EM for Λ_0 and Φ_0. after we got Λ_0, Φ_0 and Λ_{-1}, Φ_{-1} we can apply the alpha-EM. Fig.3 also illustrates the different convergence curves of the alpha-EM in previous section with different alpha values. Remember that when $\alpha = -1$, it is the log-EM.

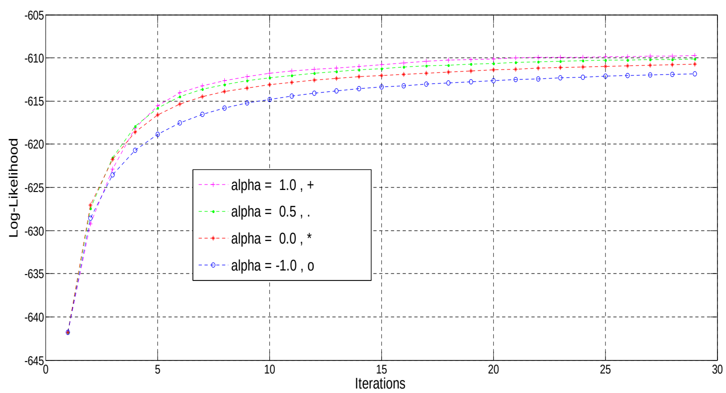

Fig.3 Convergence speed for various α

Table II also shows a speedup comparision. The second column shows that the alpha-EM (the case of $\alpha = 1$) is 30/10=3.00 times faster than the log-EM (the case of $\alpha = -1$). The third and fourth columns show a more practical comparision based upon CPU time. Again, the alpha-EM takes the same CPU time as the log-EM per iteration. We can see that the alpha-EM is still faster than the log-EM by a CPU-time speedup ratio of 30t/10t=3.00.

TABLE II Speedup Ratio For Factor Model Estimation(p=471, d=20)

α	Iterations	Time per Iteration	Total CPU-Time	Speedup Ratio
-1.00	30	1t	30t	1.00
0	16	1t	16t	1.875
+0.5	12	1t	12t	2.50
+1.00	10	1t	10t	3.00

We can see that during the first few iterations $\alpha = 1$ is not the fastest, this is as mentioned in Section 3.2.1 that $\beta = \alpha+2$ is not always a good approximation. We should choose α increases to 1 as iteration increases. For exmaple we can choose $\alpha = 0, 0.25, 0.5$ for the first three interations and $\alpha = 1$ from the fourth interation. Fig.4 also illustrates the different convergence curves of the alpha-EM in previous section with different alpha values.

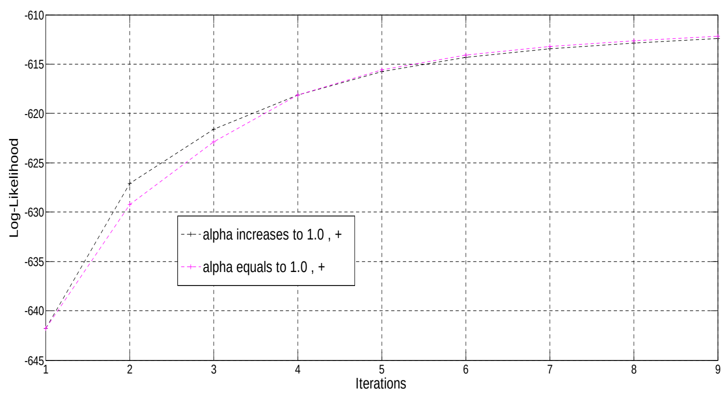

Fig.4 Convergence speed for various α

In addition to the improved convergence speed, as we mentioned in Section 3.2.2, for $r \neq 1$ in series expansion, for kth iteration we can choose $r = 1-0.1\hat{}k$ which is more approperate than $r = 1$. So as iteration increases the value of r gets closer to 1 eventually. Fig.5 illustrates the different convergence curves of the alpha-EM with the same α but different r in series expansion.

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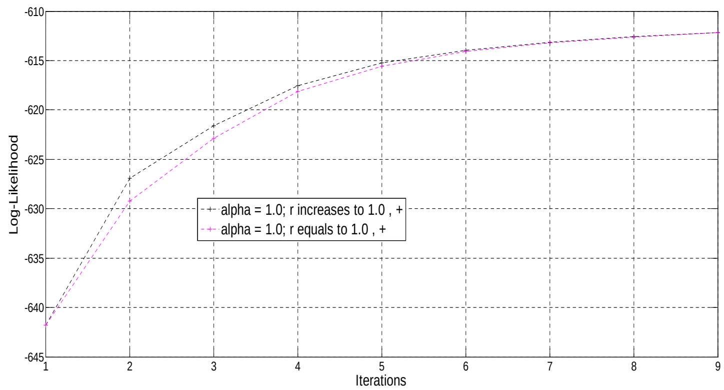

Fig.5 Convergence speed for various r

We can see that they are almost the same, but if you amplify the first a few iterations we will notice that $r = 1 - 0.1\hat{}k$ is an improvment compared to $r = 1$. But the improvement is not large.

Here we consider a smaller p. We just pick the first 100 members out of 471 members of S&P500. For the number of factors we choose $d = 10$, all other parts remain the same. Fig.6 also illustrates the different convergence curves of the alpha-EM in previous section with different alpha values. Remember that when $\alpha = -1$, it is the log-EM.

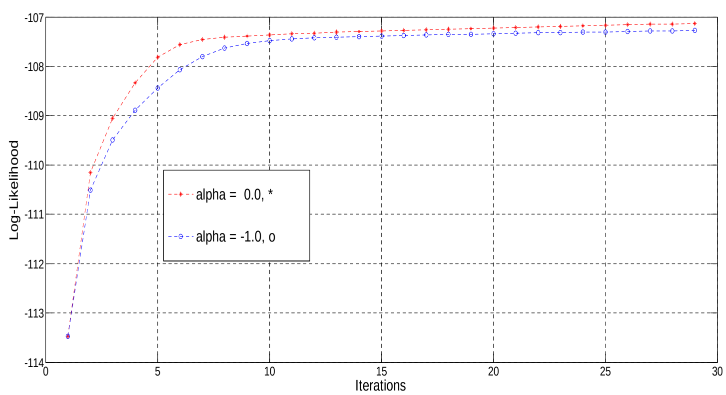

Fig.6 Convergence speed for various α

Table III also shows a speedup comparision. The second column shows that

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the alpha-EM (the case of $\alpha = 0$) is 30/16=1.875 times faster than the log-EM (the case of $\alpha = -1$) for the same convergence. The third and fourth columns show a more practical comparision based upon CPU time. CPU time per iteration is the same for the alpha-EM and the log-EM. So we can see that the alpha-EM is much faster than the log-EM by a total CPU-time speedup ratio of 30t/16t=1.875.

TABLE III Speedup Ratio For Factor Model Estimation(p=100, d=10)

α	Iterations	Time per Iteration	Total CPU-Time	Speedup Ratio
-1.00	30	1t	30t	1.00
0	16	1t	16t	1.875

Therefore, for small dimension problems or large dimension problems the alpha-EM will not require more CPU time per iteration than the log-EM. So as long as the number iterations of the alpha-EM is smaller than the log-EM for the same accuracy, the alpha-EM will save us the total computation time. That's we should choose the alpha-EM over the log-EM.

5 Concluding Remarks

In this paper, we applied the alpha-EM to factor model estimation. Through calculation we found it is hard to get causal update equations directly. Instead we get two non-causal update equations. From those non-causal update equations we learned that the alpha-EM has faster convergence speed than the log-EM but each major iteration for the alpha-EM takes large amount of time. We can't use the alpha-EM in practice without solving the non-causality. Then, we did causal approximation and series expansion in order to get the approximate causal update equations. By choosing proper alpha, we showed that the alpha-EM algorithm converges much faster than the log-EM algorithm in factor model estimation and also gives more accurate estimations. In CPU-Time, as long as we save some results from the previous updates for the next updates, the alpha-EM doesn't require more CPU time than the log-EM. However more importantly, the speedup in convegence is significant, so the alpha-EM can save us the total computation time for the same accuracy.

In order to make the alpha-EM work in practice, causal apporximation and series expansion played very important roles. In causal apporximation, there are actually many choices about how α increases to 1. Which choice is better usually related to what problem you have. In series expansion, there are actually many other choices of a, such as $a = 1 - 0.1\hat{}(0.9 + k/10)$ for kth iteration, which works better than $a = 1 - 0.1\hat{}k$ in the first few iterations. But the improvement is not significant in factor model estimation. Also, there must be other methods to solve the non-causality, such as moving all the future states on one side of the original update equations (15) and (16). That would be the most accurate method but it is also harder than the apporximation method. Thus, further exploration of practical issues pertaining to the alpha-EM family is needed.

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For the alpha-EM we used, we focus our attention on the convex divergenvce (3) because of its general capacity on convex optimization. We would like to consider other possibility of different types of surrogate functions.

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GENERALIZED POLYNOMIAL CHAOS EXPANSIONS WITH WEIGHTS

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Abstract: Polynomial chaos is used as an alternative to Monte Carlo methods for the propagation of uncertainty through dynamical systems. By truncating the infinite series of the polynomial chaos expansion to a finite order, the positivity of the approximate solution may be lost. We show in general how the positivity can be preserved by introducing weights into the finite polynomial approximation, where the polynomial systems are assumed to have compact support. In the case of Legendre polynomials examples of such weights are given explicitly.

Key words: polynomial chaos, orthogonal polynomials, Legendre polynomials, positive operators, dynamical systems

Running head: Polynomial chaos with weights

1. Introduction

Norbert Wiener was the first one who used the term homogenous chaos as a generalization of the measure induced by the increments of the Wiener process [15]. Later, Cameron and Martin proved that every square integrable functional with respect to the Wiener process can be expanded in a series of Hermite polynomials in Gaussian random variables [4]. The theory gained a lot of attention due to the work of Ghanem and Spanos who used expansions in Hermite polynomials of Gaussian random variables in combination with Galerkin methods for the solution of stochastic

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partial differential equations [9]. Xiu and Karniadakis generalized this idea and proposed the use of polynomials other than the Hermite in order to speed up the convergence rate [16]. The conditions under which the generalized polynomial chaos expansions actually converge was examined in [7].

Polynomial chaos is being used for the propagation of uncertainty through dynamical systems. Uncertain parameters and functions in the system dynamics are modeled as random variables and stochastic processes respectively. They admit a polynomial chaos expansion as does the solution of the system equations. The expansion of the solution is truncated to the desired approximation order and its coefficients are determined by stochastic Galerkin or non-intrusive methods. Even when the original system preserves the positivity of solution, this property cannot be assured to be inherited to the finite series approximation.

The goal of this work is to provide theoretical methods, i.e. proper summability methods, which are positivity preserving. This is equivalent to introducing weights in the truncated polynomial approximation. In section 2 we give a short introduction in polynomial chaos theory and in section 3 we present suitable summability methods. Examples of such methods in the case of Legendre polynomials are presented in section 4. In section 5 and 6 we summarize how generalized polynomial chaos and weighted generalized polynomial chaos can be applied to dynamical systems. We end our investigations with some conclusions.

2. Generalized Polynomial Chaos

Let $(\Omega, \mathcal{A}, P)$ be a probability space and denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$. Moreover, assume that there exists independent random variables

$$\Xi_i : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),\ i = 1, \dots, d, \tag{1}$$

such that

$$\Xi_i \in L^p(\Omega, \mathcal{A}, P) \text{ for all } i = 1, \dots, d,\ 1 \le p < \infty, \tag{2}$$

and the support S_i of the push-forward measure μ_i of Ξ_i is of infinite cardinality for any i. We note here that the random variables need not be identically distributed. Then for any i there exists a

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sequence of orthogonal polynomials $\{P_{i;n}\}_{n=0}^{\infty}$ such that $P_{i;n}$ is a polynomial of degree n and

$$\int P_{i;n}(x)P_{i;m}(x)d\mu_i(x) = \frac{1}{h_{i;n}}\delta_{n,m}, \tag{3}$$

with $h_{i;n} > 0$. The quantities $h_{i;n}$ ensure that our concept does not depend on a special normalization of the polynomials. Nevertheless, for simplicity we assume $P_{i;0}(x) = 1$ for any i, that is $h_{i;0} = 1$. For details with respect to orthogonal polynomials we refer to [6, 8].

Let us now draw our attention to the random vector

$$\boldsymbol{\Xi} = (\Xi_1, \ldots, \Xi_d) : (\Omega, \mathcal{A}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)). \tag{4}$$

The push-forward measure of $\boldsymbol{\Xi}$ is determined by $\boldsymbol{\mu} = \mu_1 \times \cdots \times \mu_d$ with support $\boldsymbol{S} = S_1 \times \cdots \times S_d$ due to independence. In the following we repeatedly take advantage of the fact that for a measurable function $f : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we have f is integrable with respect to $\boldsymbol{\mu}$ if and only if $f(\boldsymbol{\Xi})$ is integrable with respect to P, and in such cases $\int f d\boldsymbol{\mu} = \int f(\boldsymbol{\Xi})dP$.

The set of multivariate polynomials $\{P_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}_0^d\}$ with

$$P_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^{d} P_{i;n_i}(x_i) \text{ for all } \boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d,\ \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{5}$$

is an orthogonal set with respect to $\boldsymbol{\mu}$, that is

$$\int P_{\boldsymbol{n}}(\boldsymbol{x})P_{\boldsymbol{m}}(\boldsymbol{x})d\boldsymbol{\mu}(\boldsymbol{x}) = \frac{1}{h_{\boldsymbol{n}}}\delta_{\boldsymbol{n},\boldsymbol{m}} \text{ for all } \boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}_0^d, \tag{6}$$

where $h_{\boldsymbol{n}} = h_{1;n_1} \cdots h_{d;n_d}$. Denote by $\mathcal{P} = \operatorname{lin}\{P_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}_0^d\}$ the space of multivariate polynomials.

Let $\sigma(\boldsymbol{\Xi}) \subset \mathcal{A}$ be the smallest σ-algebra such that $\boldsymbol{\Xi}$ is measurable, and let

$$X : (\Omega, \sigma(\boldsymbol{\Xi})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \tag{7}$$

be a random variable. According to the Doob-Dynkin lemma [14] there exists a measurable function

$$f_X : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

such that

$$X = f_X(\boldsymbol{\Xi}). \tag{8}$$

There is an expansion

$$X = f_X(\boldsymbol{\Xi}) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \widehat{f_X}_{\boldsymbol{n}} P_{\boldsymbol{n}}(\boldsymbol{\Xi}) h_{\boldsymbol{n}} \text{ for all } X \in L^2(\Omega, \sigma(\boldsymbol{\Xi}), P), \tag{9}$$

where

$$\widehat{f_X}_{\boldsymbol{n}} = \int X(\omega) P_{\boldsymbol{n}}(\boldsymbol{\Xi})(\omega) dP(\omega) = \int f_X(\boldsymbol{x}) P_{\boldsymbol{n}}(\boldsymbol{x}) d\boldsymbol{\mu}(\boldsymbol{x}), \tag{10}$$

if and only if $\{P_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}_0^d\}$ is a complete orthogonal set in $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. Sufficient conditions for the completeness are given in [7]. In practice, one is working with an Nth order approximation

$$\sum_{|\boldsymbol{n}|=0}^{N} \widehat{f_X}_{\boldsymbol{n}} P_{\boldsymbol{n}}(\boldsymbol{\Xi}) h_{\boldsymbol{n}},\ N \in \mathbb{N}_0, \tag{11}$$

of X, where $|\boldsymbol{n}| = n_1 + \ldots + n_d$.

One disadvantage of this approach is that $X \geq 0$ almost everywhere does not imply positivity of the Nth order approximation. Therefore, our goal is to develop more general summability methods which may be positivity preserving. Moreover, we are also interested in the case $X \in L^p(\Omega, \sigma(\boldsymbol{\Xi}), P)$, $p \neq 2$.

3. Summability methods based on kernels

From now on we assume $\boldsymbol{S}$ to be compact. Then $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu}) \subset L^q(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ for all $1 \leq q \leq p \leq \infty$, and

$$\hat{f}_{\boldsymbol{n}} = \int f(\boldsymbol{x}) P_{\boldsymbol{n}}(\boldsymbol{x}) d\boldsymbol{\mu}(\boldsymbol{x}) \tag{12}$$

exists for all $f \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$, $\boldsymbol{n} \in \mathbb{N}_0^d$. Let $\tau_i : \mathbb{N}_0 \to \mathbb{N}_0$, $i = 1, \ldots, d$, be sequences with $\lim_{N\to\infty} \tau_i(N) = \infty$, and

$$T(N) = \{\boldsymbol{n} \in \mathbb{N}_0^d : n_i \leq \tau_i(N), i = 1, \ldots, d\} \quad \text{for all } N \in \mathbb{N}_0. \tag{13}$$

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Furthermore, choose $\omega_{N,\boldsymbol{n}} \in \mathbb{R}$ for all $N \in \mathbb{N}_0$, $\boldsymbol{n} \in T(N)$. Then

$$F_N(f) = \sum_{\boldsymbol{n} \in T(N)} \omega_{N,\boldsymbol{n}} \hat{f}_{\boldsymbol{n}} P_{\boldsymbol{n}} h_{\boldsymbol{n}} \tag{14}$$

is a continuous linear operator from $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ into $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ for all $N \in \mathbb{N}_0$, $1 \leq p \leq \infty$. If we define multivariate kernels by

$$K_N(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{n} \in T(N)} \omega_{N,\boldsymbol{n}} P_{\boldsymbol{n}}(\boldsymbol{x}) P_{\boldsymbol{n}}(\boldsymbol{y}) h_{\boldsymbol{n}} \tag{15}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $N \in \mathbb{N}_0$, then

$$F_N(f)(\boldsymbol{x}) = \int f(\boldsymbol{y}) K_N(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{y}). \tag{16}$$

We recall that an operator $F : L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu}) \to L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$, $1 \leq p \leq \infty$, is called positive if $f(\boldsymbol{x}) \geq 0$ $\boldsymbol{\mu}$-almost everywhere implies that $F(f)(\boldsymbol{x}) \geq 0$ $\boldsymbol{\mu}$-almost everywhere. Furthermore, F is called boundedness preserving if $a \leq f(\boldsymbol{x}) \leq b$ $\boldsymbol{\mu}$-almost everywhere implies that $a \leq F(f)(\boldsymbol{x}) \leq b$ $\boldsymbol{\mu}$-almost everywhere, where $a, b \in \mathbb{R}$. It is elementary to show that the operator F_N is positive if and only if $K_N(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$. There is a simple characterization of positive operators F_N which are additionally boundedness preserving.

Theorem 1. *Let F_N be a positive operator defined by (16). Then F_N is boundedness preserving if and only if $\omega_{N,\mathbf{0}} = 1$.*

Proof. Let $\omega_{N,\mathbf{0}} = 1$ and $a \leq f(\boldsymbol{x}) \leq b$ $\boldsymbol{\mu}$-almost everywhere. Then $F_N(f)(\boldsymbol{x}) - a = F_N(f - a)(\boldsymbol{x}) \geq 0$ and $b - F_N(f)(\boldsymbol{x}) = F_N(b - f)(\boldsymbol{x}) \geq 0$ $\boldsymbol{\mu}$-almost everywhere. Conversely, assume F_N is boundedness preserving and let $f(\boldsymbol{x}) = a \in \mathbb{R} \setminus \{0\}$ for all $\boldsymbol{x} \in \mathbb{R}^d$. Then $F_n(f)(\boldsymbol{x}) = \omega_{N,\mathbf{0}} a$ for all $\boldsymbol{x} \in \mathbb{R}^d$, which implies $a \leq \omega_{N,\mathbf{0}} a \leq a$. □

Denote by $\|F\|^{L^p}$ the operator norm of a continuous operator F from $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ into $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. The following lemma provides us with upper bounds for the corresponding operator norms of F_N.

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Lemma 1. *Let $1 \leq p \leq \infty$ and F_N be an operator defined by (16). Then*

$$\|F_N\|^{L^p} \leq \|F_N\|^{L^1} = \|F_N\|^{L^\infty} = \sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}).$$

If F_N is positive, then $\|F_N\|^{L^\infty} = \omega_{N,\mathbf{0}}$.

Proof. If $p = \infty$, then by trivial means

$$\|F_N(f)\|_\infty \leq \sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\mu(\boldsymbol{y}) \ \|f\|_\infty \text{ for all } f \in L^\infty(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu}).$$

Let $1 \leq p < \infty$, $1/p + 1/q = 1$, and $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. Hölder's inequality with respect to the measure $|K_N(\boldsymbol{x},\boldsymbol{y})|d\mu(\boldsymbol{y})$ yields

$$(\int |f(\boldsymbol{y})||K_N(\boldsymbol{x},\boldsymbol{y})| \mid K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}))^{p/q} \int |f(\boldsymbol{y})|^p |K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}).$$

Since $K_N(\boldsymbol{x},\boldsymbol{y}) = K_N(\boldsymbol{y},\boldsymbol{x})$, Fubini's theorem implies that

$$\begin{aligned} \|F_N(f)\|_p^p &\leq (\sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}))^{p/q} \int\int |f(\boldsymbol{y})|^p |K_N(\boldsymbol{y},\boldsymbol{x})| d\boldsymbol{\mu}(\boldsymbol{x}) d\boldsymbol{\mu}(\boldsymbol{y}) \\ &\leq (\sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\mu(\boldsymbol{y}))^p \ \|f\|_p^p. \end{aligned}$$

Therefore,

$$\|F_N\|^{L^p} \leq \sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\mu(\boldsymbol{y}) \text{ for all } 1 \leq p \leq \infty.$$

Denote by $S_{L^p} = \{f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu}) : \ \|f\|_p = 1\}$ the unit sphere in $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. Riesz's representation theorem implies that

$$\begin{aligned} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}) &= \sup_{f\in S_{L^\infty}} |\int f(\boldsymbol{y}) K_N(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{y})| = \sup_{f\in S_{L^\infty}} |F_N(f)(\boldsymbol{x})| \\ &\leq \sup_{f\in S_{L^\infty}} \| F_N(f) \|_\infty = \|F_N\|^{L^\infty} \text{ for all } \boldsymbol{x} \in \boldsymbol{S}. \end{aligned}$$

Thus,

$$\|F_N\|^{L^\infty} = \sup_{\boldsymbol{x}\in\boldsymbol{S}} \int |K_N(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{\mu}(\boldsymbol{y}).$$

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Finally,

$$\begin{aligned}
\|F_N\|^{L^1} &= \sup_{g\in S_{L^1}} \|F_N(g)\|_1 \\
&= \sup_{g\in S_{L^1}} \sup_{f\in S_{L^\infty}} |\int f(\boldsymbol{x}) \int g(\boldsymbol{y}) K_N(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{x})| \\
&= \sup_{g\in S_{L^1}} \sup_{f\in S_{L^\infty}} |\int g(\boldsymbol{y}) \int f(\boldsymbol{x}) K_N(\boldsymbol{y},\boldsymbol{x}) d\boldsymbol{\mu}(\boldsymbol{x}) d\boldsymbol{\mu}(\boldsymbol{y})| \\
&= \sup_{f\in S_{L^\infty}} \sup_{g\in S_{L^1}} |\int g(\boldsymbol{y}) F_N(f)(\boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{y})| \\
&= \sup_{f\in S_{L^\infty}} \|F_N(f)\|_\infty = \| F_N\|^{L^\infty}.
\end{aligned}$$

If F_N is positive, then $K_N(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for all $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{S}$. Therefore,

$$\|F_N\|^{L^\infty} = \sup_{\boldsymbol{x}\in\boldsymbol{S}} \int K_N(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{\mu}(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in\boldsymbol{S}} \omega_{N,\mathbf{0}} P_{\mathbf{0}}(\boldsymbol{x}) = \omega_{N,\mathbf{0}}.$$

□

Our aim is to construct a sequence of operators $\{F_N\}_{N=0}^\infty$ in such a way that $F_N(f)$ is converging towards f for all $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. The following theorem is basic for that purpose.

Theorem 2. *Let $1 \leq p < \infty$ and $\{F_N\}_{N=0}^\infty$ be a sequence of operators defined by (16). Then $\lim_{N\to\infty} F_N(f) = f$ for all $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ if and only if*

(i) *$\lim_{N\to\infty} \omega_{N,\boldsymbol{n}} = 1$ for all $\boldsymbol{n} \in \mathbb{N}_0^d$, and*

(ii) *there exists $C > 0$ such that $\|F_N\|^{L^p} < C$ for all $N \in \mathbb{N}_0$.*

If $\{F_N\}_{N=0}^\infty$ is a sequence of positive operators, then $\lim_{N\to\infty} F_N(f) = f$ for all $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ if and only if $\lim_{N\to\infty} \omega_{N,\boldsymbol{n}} = 1$ for all $\boldsymbol{n} \in \mathbb{N}_0^d$.

Proof. Note that $\lim_{N\to\infty} F_N(P_{\boldsymbol{n}}) = P_{\boldsymbol{n}}$ if and only if $\lim_{N\to\infty} \omega_{N,\boldsymbol{n}} = 1$.
Suppose that $\lim_{N\to\infty} F_N(f) = f$ for all $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. Then (i) follows immediately and (ii) is a consequence of the Banach-Steinhaus theorem.
Conversely, assume that (i) and (ii) hold. Then (i) implies that $\lim_{N\to\infty} F_N(Q) = Q$ for all polynomials $Q \in \mathcal{P}$. Let $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ and $\epsilon > 0$. Since $\mathcal{P}$ is dense in $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$, we

are able to choose $Q \in \mathcal{P}$ with $\| Q - f \|_p < \epsilon$. Finally, (ii) implies

$$\begin{aligned} \| f - F_N(f) \|_p &\leq \| f - Q \|_p + \| Q - F_N(Q) \|_p + C \| Q - f \|_p \\ &\leq (2+C)\epsilon \end{aligned}$$

for sufficiently large N.

If $\{F_N\}_{N=0}^{\infty}$ is a sequence of positive operators, then Lemma 1 yields $\|F_N\|^{L^p} \leq \omega_{N,\mathbf{0}}$. Hence, (i) implies (ii). □

Corollary 1. *Let $1 \leq p < \infty$ and $\{F_N\}_{N=0}^{\infty}$ be a sequence of operators defined by (16) with $\lim_{N\to\infty} F_N(f) = f$ for all $f \in L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. Then $\sup_{N\in\mathbb{N}_0, \boldsymbol{n}\in T(N)} |\omega_{N,\boldsymbol{n}}| < \infty$.*

Proof. Let $N \in \mathbb{N}_0$. Since $F_N(P_{\boldsymbol{n}}) = \omega_{N,\boldsymbol{n}} P_{\boldsymbol{n}}$ for all $\boldsymbol{n} \in T(N)$ we get $\|F_N\|^{L^p} \geq \sup_{\boldsymbol{n}\in T(N)} |\omega_{N,\boldsymbol{n}}|$. Hence, (ii) of Theorem 2 implies $\sup_{N\in\mathbb{N}_0, \boldsymbol{n}\in T(N)} |\omega_{N,\boldsymbol{n}}| < \infty$. □

If $p = 2$, then condition (ii) of Theorem 2 can be replaced by a much simpler condition.

Corollary 2. *Let $p = 2$ and $\{F_N\}_{N=0}^{\infty}$ be a sequence of operators defined by (16). Then $\lim_{N\to\infty} F_N(f) = f$ for all $f \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$ if and only if*

(i) $\lim_{N\to\infty} \omega_{N,\boldsymbol{n}} = 1$ *for all* $\boldsymbol{n} \in \mathbb{N}_0^d$, *and*

(ii) $\sup_{N\in\mathbb{N}_0, \boldsymbol{n}\in T(n)} |\omega_{N,\boldsymbol{n}}| < \infty$.

Proof. It remains to prove that $\sup_{N\in\mathbb{N}_0, \boldsymbol{n}\in T(N)} |\omega_{N,\boldsymbol{n}}| = C < \infty$ implies (ii) of Theorem 2. Concerning this matter Parseval's identity yields

$$\begin{aligned} \|F_N(f)\|_2^2 &= \int \sum_{\boldsymbol{n}\in T(N)} \omega_{N,\boldsymbol{n}} \hat{f}_{\boldsymbol{n}} P_{\boldsymbol{n}} h_{\boldsymbol{n}} \sum_{\boldsymbol{n}\in T(N)} \overline{\omega_{N,\boldsymbol{n}} \hat{f}_{\boldsymbol{n}}} P_{\boldsymbol{n}} h_{\boldsymbol{n}} \, d\boldsymbol{\mu} \\ &= \sum_{\boldsymbol{n}\in T(N)} |\omega_{N,\boldsymbol{n}}|^2 |\hat{f}_{\boldsymbol{n}}|^2 h_{\boldsymbol{n}} \leq C^2 \|f\|_2^2 \end{aligned}$$

for all $N \in \mathbb{N}_0$, $f \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \boldsymbol{\mu})$. □

The following approach for the construction of multivariate kernels is obvious. Assume that for $i = 1, \ldots, d$ there are univariate kernels

$$K_{i;N}(x_i, y_i) = \sum_{n_i=0}^{\tau_i(N)} \omega_{i;N,n_i} P_{i;n_i}(x_i) P_{i;n_i}(y_i) h_{i;n_i} \tag{17}$$

with $x_i, y_i \in S_i$. For $N \in \mathbb{N}_0$ and $\boldsymbol{n} = (n_1, \ldots, n_d) \in T(N)$ define

$$\omega_{N,\boldsymbol{n}} = \omega_{1;N,n_1} \cdots \omega_{d;N,n_d}, \tag{18}$$

and a multivariate kernel by

$$\begin{aligned} K_N(\boldsymbol{x}, \boldsymbol{y}) &= K_{1;N}(x_1, y_1) \cdots K_{d;N}(x_d, y_d) \\ &= \sum_{\boldsymbol{n} \in T(N)} \omega_{N,\boldsymbol{n}} P_{\boldsymbol{n}}(\boldsymbol{x}) P_{\boldsymbol{n}}(\boldsymbol{y}) h_{\boldsymbol{n}}. \end{aligned} \tag{19}$$

It is easily seen that the multivariate kernel $K_N(\boldsymbol{x}, \boldsymbol{y})$ is positive if and only if the univariate kernels $K_{i;N}(x_i, y_i)$ are positive for all $i = 1, \ldots, d$. Therefore, for our purpose it is sufficient to construct positive univariate kernels $K_{i;N}(x_i, y_i)$ with $\lim_{N\to\infty} \omega_{i;N,n_i} = 1$ for all $n_i \in \mathbb{N}_0$, $i = 1, \ldots, d$. Next, we give some examples of positive kernels in the case of Legendre polynomials.

4. Positive kernels in the case of Legendre polynomials

In the classical polynomial chaos theory the representation of random variables is with respect to normal distributed random variables and Hermite polynomials. Here we propagate a representation with respect to uniformly distributed random variables. Therefore let us assume that the random variables Ξ_i are uniformly distributed on $[-1, 1]$, that is with push-forward measures

$$d\mu_i(x) = dx/2. \tag{20}$$

For the sake ok brevity we will drop the subscript i in the following. The sequence of Legendre polynomials $\{L_n\}_{n=0}^{\infty}$ normalized by $L_n(1) = 1$ are orthogonal with respect to the measure above,

that is

$$\int_{-1}^{1} L_n(x)L_m(x)dx/2 = \frac{1}{2n+1}\delta_{n,m}. \tag{21}$$

Hence, we have $h_n = 2n+1$. We also make use of the so-called linearization coefficients $c_{n,m,k}$ which are uniquely defined by

$$P_n(x)P_m(x) = \sum_{k=|n-m|}^{n+m} c_{n,m,k}P_k(x). \tag{22}$$

If $k \in \{|m-n|, |m-n|+2, \ldots, m+n\}$, then

$$c_{n,m,k} = \frac{(1+2k)}{2\pi}\frac{\Gamma(\frac{1+k+m-n}{2})\Gamma(\frac{1+k-m+n}{2})\Gamma(\frac{1-k+m+n}{2})\Gamma(\frac{2+k+m+n}{2})}{\Gamma(\frac{2+k+m-n}{2})\Gamma(\frac{2+k-m+n}{2})\Gamma(\frac{2-k+m+n}{2})\Gamma(\frac{3+k+m+n}{2})}, \tag{23}$$

and $c_{n,m,k} = 0$ else. Based on former research we list some examples of positive univariate kernels.

Example 1 (De la Vallée-Poussin kernel for Legendre polynomials)**.**

Set

$$\omega_{N,n}^{(1)} = \frac{\Gamma(N+1)\Gamma(N+2)}{\Gamma(N-n+1)\Gamma(N+n+2)} \tag{24}$$

for all $N \in \mathbb{N}_0$ *and* $n \in \{0, \ldots, N\}$, *see* [2, 10, 12].

Example 2 (Fejér Kernel I for Legendre polynomials)**.**

Set

$$\omega_{N,n}^{(2)} = \frac{\sum_{m=0}^{N}\sum_{k=0}^{N} c_{n,m,k}(2m+1)}{(N+1)^2} \tag{25}$$

for all $N \in \mathbb{N}_0$ *and* $n \in \{0, \ldots, 2N\}$, *see* [11].

Example 3 (Fejér kernel II for Legendre polynomials)**.**

Set

$$\omega_{N,n}^{(3)} = \frac{\sum_{k=n}^{N}\frac{1}{1+k}}{\sum_{k=0}^{N}\frac{1}{1+k}} \tag{26}$$

for all $N \in \mathbb{N}_0$ *and* $n \in \{0, \ldots, N\}$, *see* [13].

Example 4 (Jackson kernel for Legendre polynomials)**.**

Set

$$\omega_{N,n}^{(4)} = \frac{\sum_{m=0}^{N} \sum_{k=0}^{N} c_{n,m,k} \omega_{N,k}^{(3)} \omega_{N,m}^{(3)} (2m+1)}{\sum_{m=0}^{N} (\omega_{N,m}^{(3)})^2 (2m+1)} \tag{27}$$

for all $N \in \mathbb{N}_0$ *and* $n \in \{0, \ldots, 2N\}$*, see* [13].

The corresponding univariate kernels are given by

$$K_N^{(j)}(x,y) = \sum_{n=0}^{\tau^{(j)}(N)} \omega_{N,n}^{(j)} L_n(x) L_n(y)(2n+1), \quad j = 1, \ldots, 4. \tag{28}$$

Note that $\tau^{(1)}(N) = \tau^{(3)}(N) = N$ and $\tau^{(2)}(N) = \tau^{(4)}(N) = 2N$. For the proof of positivity and $\lim_{N\to\infty} \omega_{N,n}^{(j)} = 1$ for all $n \in \mathbb{N}_0$, $j = 1, \ldots, 4$, we refer to the cited literature. There are also stated reasons for the naming of the kernels above. Concerning Theorem 1 we have $\omega_{N,0}^{(j)} = 1$ for all $N \in \mathbb{N}_0$, $j = 1, \ldots, 4$. Note that the examples can be generalized for a wider range of Jacobi polynomials, see [10, 11, 13].

5. Application of gPC in dynamical systems

We present here briefly how the polynomial chaos theory is being applied for the propagation of uncertainty through dynamical systems. For simplicity we present the general procedure on the example of an ordinary differential equation of a scalar quantity. Systems of differential equations or partial differential equations are treated similarly. We refer to [17] for a nice introduction to the field and to the references therein.

Let $u(t, \mathbf{\Theta})$ be a quantity of interest satisfying

$$\dot{u}(t, \mathbf{\Theta}) = g(t, u, \mathbf{\Theta}),\ u(t_0, \mathbf{\Theta}) = u_0, \tag{29}$$

where $\mathbf{\Theta} = (\Theta_1, \ldots, \Theta_d)$ is a d-dimensional vector representing unknown parameters. Let $J \subset \mathbb{R}$, $U \subset \mathbb{R}$ and $D \subset \mathbb{R}^d$ be open sets with $(t_0, u_0) \in J \times U$ such that the function g is smooth on $J \times U \times D$. Then, there is a unique smooth solution of (29), possibly defined on subsets of J, U and D, see [5].

Choose a d-dimensional vector of basis random variables $\boldsymbol{\Xi}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ with push-forward measure $\boldsymbol{\mu}$ such that the corresponding orthogonal set of multivariate polynomials $\{P_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}_0^d\}$ with respect to $\boldsymbol{\mu}$ (as in (5)) is complete [7]. For instance, this is the case when the support of $\boldsymbol{\mu}$ is compact. Assume that $\Theta_i \in L^2(\Omega, \sigma(\boldsymbol{\Xi}), P)$ for all $i = 1, \ldots, d$. Due to (9) for each component of $\boldsymbol{\Theta}$ we have that

$$\Theta_i = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} a_{i;\boldsymbol{n}} P_{\boldsymbol{n}}(\boldsymbol{\Xi}) h_{\boldsymbol{n}},\ i = 1, \ldots, d. \tag{30}$$

For our demonstration here we assume known expansions of the Θ_i.

The solution $u(t, \boldsymbol{\Theta})$ of (29) is a stochastic process measurable with respect to $\sigma(\boldsymbol{\Xi})$ for each fixed time t. If we assume $u(t, \boldsymbol{\Theta}) \in L^2(\Omega, \sigma(\boldsymbol{\Xi}), P)$ for all times t, then

$$u(t, \boldsymbol{\Theta}) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} u_{\boldsymbol{n}}(t) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) h_{\boldsymbol{n}}, \tag{31}$$

where the coefficients, according to (10), are theoretically determined by

$$u_{\boldsymbol{n}}(t) = \int u(t, \boldsymbol{\Theta}) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) dP. \tag{32}$$

Equation (32) is of no practical use as it includes the unknown solution. In practice, one looks for the best approximation

$$u_N(t, \boldsymbol{\Xi}) = \sum_{|\boldsymbol{n}|=0}^{N} u_{\boldsymbol{n}}(t) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) h_{\boldsymbol{n}} \tag{33}$$

of the solution in a finite dimensional subspace $\mathcal{P}_N = lin\{P_{\boldsymbol{n}} : |\boldsymbol{n}| = 0, \ldots, N\}$, $N \in \mathbb{N}_0$, of $\mathcal{P}$. These approximately summarize all the stochastic information of the solution. For example one can compute its moments and estimate its density by sampling [1].

The two main numerical approaches to compute these coefficients are intrusive stochastic Galerkin methods [1, 9] and non-intrusive methods like Monte Carlo integration [3].

The stochastic Galerkin methods depend on the fact that

$$\dot{u}_{\boldsymbol{n}}(t) = \int \dot{u}(t, \boldsymbol{\Theta}) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) dP = \int g(t, u, \boldsymbol{\Theta}) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) dP \text{ for all } \boldsymbol{n} \in \mathbb{N}_0^d, \tag{34}$$

and approximately

$$\dot{u}_{\boldsymbol{n}}(t) \approx \int g(t, \sum_{|\boldsymbol{k}|=0}^{N} u_{\boldsymbol{k}}(t) P_{\boldsymbol{k}}(\boldsymbol{\Xi}) h_{\boldsymbol{k}}, (\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} a_{1;\boldsymbol{n}} P_{\boldsymbol{k}}(\boldsymbol{\Xi}) h_{\boldsymbol{k}}, \ldots, \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} a_{d;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\Xi}) h_{\boldsymbol{k}})) P_{\boldsymbol{n}}(\boldsymbol{\Xi}) dP \tag{35}$$

for all $|\boldsymbol{n}| = 0, \ldots, N$. Since the stochastic is integrated out, (35) results in a deterministic differential equation system for $u_{\boldsymbol{n}}(t)$. The corresponding initial conditions are given by the coefficients in the polynomial chaos expansion (31) of $u(t_0, \boldsymbol{\Theta})$. In the case $u(t_0, \boldsymbol{\Theta})$ does not depend on the stochastic we get $u_{\boldsymbol{0}}(t_0) = u_0$ and $u_{\boldsymbol{n}}(t_0) = 0$ for $\boldsymbol{n} \neq \boldsymbol{0}$. If there is a dependence $u(t_0, \boldsymbol{\Theta}) = \Theta_i$, then we get $u_{\boldsymbol{n}}(t_0) = a_{i;\boldsymbol{n}}$. A solution of the deterministic differential equation system (35) is taken for the unknown coefficients $u_{\boldsymbol{n}}(t)$ in (33). A difficulty associated with this approach is that the derivation of the Galerkin system can be nontrivial in situations in which the original system is nonlinear. Furthermore, the method results in a system of higher dimension than the initial one and to determine the coefficients one has to solve a coupled system of equations.

In order to calculate the coefficients $u_{\boldsymbol{n}}(t)$ one can also apply non-intrusive methods which are based on a random sample $\{\boldsymbol{\xi}_m\}_{m=0}^M$ of the random vector $\boldsymbol{\Xi}$. The basic idea behind using so-called non-intrusive spectral projections (NISP) is to numerically approximate the multidimensional integrals in (32). For instance, by Monte Carlo integration one gets the following approximation

$$u_{\boldsymbol{n}}(t) \approx \frac{1}{M} \sum_{m=1}^{M} u(t, (\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} a_{1;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}}, \ldots, \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} a_{d;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}})) P_{\boldsymbol{n}}(\boldsymbol{\xi}_m), \tag{36}$$

where, due to the law of large numbers, $M \in \mathbb{N}$ has to be taken sufficiently large. In practice, of course, a further step of approximation is necessary, that is

$$u_{\boldsymbol{n}}(t) \approx \frac{1}{M} \sum_{m=1}^{M} u(t, (\sum_{|\boldsymbol{k}|=0}^{N} a_{1;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}}, \ldots, \sum_{|\boldsymbol{k}|=0}^{N} a_{d;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}})) P_{\boldsymbol{n}}(\boldsymbol{\xi}_m). \tag{37}$$

Note that $u(t, (\sum_{|\boldsymbol{k}|=0}^{N} a_{1;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}}, \ldots, \sum_{|\boldsymbol{k}|=0}^{N} a_{d;\boldsymbol{k}} P_{\boldsymbol{k}}(\boldsymbol{\xi}_m) h_{\boldsymbol{k}}))$ is a solutions of the corresponding deterministic system (29).

6. Application of positive summability methods in dynamical systems

Recognize that in many applications, the solution of (29) describes a quantity such as a chemical concentration or a population density. Thus positivity is a natural property to require for the solution. Even when the initial system is positivity preserving, i.e. solutions starting from non-negative initial data remain non-negative in their existence interval, it cannot be assured that any finite approximation $u_N(t, \Xi)$ remains positive for all realizations of Ξ and all times t. In the case the push-forward measure of Ξ does have compact support, the problem can be solved if one applies a weighted PC expansion related to a positive kernel defined as in (19). Instead of (33) one can work with

$$u_N(t, \Xi) = \sum_{\boldsymbol{n} \in T(N)} \omega_{N,\boldsymbol{n}} u_{\boldsymbol{n}}(t) P_{\boldsymbol{n}}(\Xi) h_{\boldsymbol{n}}, \; N \in \mathbb{N}_0, \tag{38}$$

see (14). As demonstrated in section 3, the weights are pre-computed real numbers and they do not depend on the solution $u(t, \Xi)$ and the coefficients $u_{\boldsymbol{n}}(t)$. As before, the coefficients $u_{\boldsymbol{n}}(t)$ are computed by numerical methods mentioned in the previous section. Note that in the case of using Galerkin methods (35) changes into

$$\dot{u}_{\boldsymbol{n}}(t) \approx \int g(t, \sum_{\boldsymbol{k} \in T(N)} u_{\boldsymbol{k}}(t) P_{\boldsymbol{k}}(\Xi) h_{\boldsymbol{k}}, (\sum_{\boldsymbol{k} \in \mathbb{N}_0^d} a_{1;\boldsymbol{n}} P_{\boldsymbol{k}}(\Xi) h_{\boldsymbol{k}}, \ldots, \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} a_{d;\boldsymbol{k}} P_{\boldsymbol{k}}(\Xi) h_{\boldsymbol{k}})) P_{\boldsymbol{n}}(\Xi) dP \tag{39}$$

for all $\boldsymbol{n} \in T(N)$.

Applying the methods of the previous section, there may arise instabilities if for some positive random variable Θ_i the truncated expansion $\sum_{|\boldsymbol{k}|=0}^{N} a_{i;\boldsymbol{k}} P_{\boldsymbol{k}}(\Xi) h_{\boldsymbol{k}}$ also produces negative values. The problem can be solved by using instead of

$$\sum_{\boldsymbol{k} \in T(N)} \omega_{N,\boldsymbol{k}} a_{i;\boldsymbol{k}} P_{\boldsymbol{k}}(\Xi) h_{\boldsymbol{k}}. \tag{40}$$

7. Conclusions

While there are benefits, there are also drawbacks in applying weighted generalized polynomial chaos. One disadvantage is concerning complexity. A sum indexed by $|\boldsymbol{n}| = 0, \ldots, N$ does have $\binom{N+d}{N}$ terms, whereas a sum in the weighted case indexed by $\boldsymbol{n} \in T(N)$ does have $\prod_{i=1}^{d} \tau_i(N)$

terms. For instance in the case $\tau_i(N) = N$, $i = 1, \ldots, d$, asymptotically there are d times more terms in the weighted case as compared to the procedure without weights. Do also note that the Galerkin system increases in dimension. Another drawback is concerning the approximation error. For $f \in L_2$ the approximation error in the usual case is $\sum_{|\boldsymbol{n}|=N+1}^{\infty} \hat{f}_{\boldsymbol{n}}^2 h_{\boldsymbol{n}}$, which is optimal, whereas the approximation error in the weighted case is $\sum_{n \notin T(N)} \hat{f}_{\boldsymbol{n}}^2 h_{\boldsymbol{n}} + \sum_{n \in T(N)} (\omega_{N,\boldsymbol{n}} - 1)^2 \hat{f}_{\boldsymbol{n}}^2 h_{\boldsymbol{n}}$. Therefore, the error in the weighted case depends on the convergence rate of $\lim_{N\to\infty} \omega_{N,\boldsymbol{n}} = 1$. Our goal here is only to provide new theoretical tools for uncertainty quantification which extend general polynomial chaos. We are aware that further work is necessary to develope applicable algorithms for uncertainty quantification based on generalized polynomial chaos with weights. With this in mind, we hope our investigations can act as a kick-off.

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A NOTE ON THE HIGH ORDER GENOCCHI POLYNOMIALS BY MEANS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract In this paper, we construct the Nth order nonlinear ordinary differential equation with respect to the generating function of Genocchi numbers. From these relationships, we derive some identities on Genocchi polynomials of higher order.

1. Introduction

There are many methodologies to obtain identities on Genocchi numbers and polynomials. Our methodology depends on a differential equation which has the generating function of Genocchi numbers as a solution. The crucial point of our methodology is to solve an another differential equation to determine coefficients of the first differential equation. There are some researches about another numbers by almost same methodology (see [2-5]).

The generating function of Eulerian polynomial $H_n(x|u)$ is defined by

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(x|u)\frac{t^n}{n!},$$

where $u\in\mathbb{C}$ with $u\neq 1$. In the special case, $x=0$, $H_n(0|u)=H_n(u)$ is called the nth Eulerian number (see [1,2,5]). Sometimes that is called the nth Frobenius-Euler number.

In [5], Kim constructed a nonlinear ordinary differential equation with respect to t which was related to the generating function of Eulerian polynomial. Some identities on Eulerian polynomials of higher order were derived from the differential equation. In [2], Choi considered nonlinear ordinary differential equations with respect to u not t to obtain different identities on Eulerian polynomial.

As is well known, the Bernoulli polynomials $B_n(x)$ are defined by the generating function as follows:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

In the special case, $x=0$, $B_n(0)=B_n$ is the nth Bernoulli number (see [4,8]).

In [4], Choi et al. constructed a nonlinear differential equation related to the generating function of Bernoulli numbers. They found that the construction had been more difficult than the case of Euler number.

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Considering the similarity of Bernouli numbers and Genocchi numbers, the target ordinary differential equations are similar with each other. By our methodology, we derive some interesting identities on the high order Genocchi numbers and polynomials.

The Genocchi numbers are defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi).$$

The Genocchi polynomials $G_n(x)$ are defined by the generating function as follows:

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{1}$$

In the special case, $x = 0$, $G_n(0) = G_n$ is the nth Genocchi number (see [7,8]).

From (1), we have the following recurrence relations

$$(G+1)^n + G_n = 2\delta_{1,n} \quad \text{and} \quad G_0 = 0,$$

with the usual convention about replacing G^n by G_n (see [7]). Here $\delta_{1,n}$ is Kronecker symbol and $n \in \mathbb{N} \cup \{0\}$.

In this paper, we consider the Nth order nonlinear differential equation associated with the generating function of Genocchi numbers. The purpose of this paper is to investigate some identities on the high order Genocchi polynomials by using nonlinear differential equations.

2. Differential equations related to Genocchi numbers

In this section, we derive the Nth order nonlinear differential equation whose solution is the generating function of Genocchi numbers.

Definition 2.1. *For $N \in \mathbb{N}$, the Genocchi polynomial of order N is defined by the generating function as follows:*

$$\begin{aligned} G^N(t,x) &= \underbrace{\left(\frac{2t}{e^t+1}\right) \times \cdots \times \left(\frac{2t}{e^t+1}\right)}_{N-times} e^{xt} \\ &= \sum_{n=0}^{\infty} G_n^{(N)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2}$$

Taking $x = 0$ into (2), $G_n^{(N)}(0) = G_n^{(N)}$ is called the nth Genocchi number of order N.

Henceforth we define

$$G = G(t) = \frac{2t}{e^t+1}. \tag{3}$$

By differentiating (3) with respect to t, we have

$$\frac{1}{2} G^2 = tG^{(1)} + (t-1)G. \tag{4}$$

By differentiating (4) with respect to t repeatedly, we have that

$$\begin{aligned}\frac{1}{2}G^3 &= t^2G^{(2)} + (3t^2-2t)G^{(1)} + (2t^2-3t+2)G,\\ \frac{3}{2^2}G^4 &= t^3G^{(3)} + (6t^3-3t^2)G^{(2)} + (11t^3-12t^2+6t)G^{(1)}\\ &\quad + (6t^3-11t^2+12t-6)G.\end{aligned} \tag{5}$$

Continuing this process, we see by mathematical induction that

$$\frac{{}_NP_{N-2}}{2^{N-1}}G^{N+1} = \sum_{m=1}^{N+1} a_m(N,t)G^{(N-m+1)}, \tag{6}$$

where $G^{(n)} = \dfrac{d^nG(t)}{dt^n}$ and ${}_nP_r = \dfrac{n!}{(n-r)!}$.

Now, let us investigate the derivative of (6) with respect to t to find the recurrence relation of the coefficient $a_m(N,t)$ in (6).

Theorem 2.2. *For $N \in \mathbb{N}$, one has*

$$\begin{aligned}\frac{{}_{N+1}P_{N-1}}{2^N}G^{N+2} &= (N+1)(t-1)\sum_{m=1}^{N+1} a_m(N,t)G^{(N-m+1)}\\ &\quad + \sum_{m=0}^{N} ta_{m+1}(N,t)G^{(N-m+1)} + \sum_{m=1}^{N+1} t\frac{d}{dt}a_m(N,t)G^{(N-m+1)}.\end{aligned} \tag{7}$$

Proof. By differentiating (6) with respect to t and multiplying by t, we get

$$t\frac{{}_NP_{N-2}}{2^{N-1}}(N+1)G^NG^{(1)} = t\sum_{m=1}^{N+1}\left(a_m(N,t)G^{(N-m+2)} + \frac{d}{dt}a_m(N,t)G^{(N-m+1)}\right). \tag{8}$$

By (4), we see that

$$\begin{aligned}&\frac{{}_NP_{N-2}}{2^{N-1}}(N+1)G^NtG^{(1)}\\ &\quad = \frac{{}_NP_{N-2}}{2^N}(N+1)G^N\big(G^2 + 2(1-t)G\big)\\ &\quad = \frac{{}_NP_{N-2}}{2^N}(N+1)G^{N+2} - (N+1)(t-1)\frac{{}_NP_{N-2}}{2^{N-1}}G^{N+1}.\end{aligned} \tag{9}$$

On the other hand, we have that the right hand side of (8) equals

$$\sum_{m=0}^{N} ta_{m+1}(N,t)G^{(N-m+1)} + \sum_{m=1}^{N+1} t\frac{d}{dt}a_m(N,t)G^{(N-m+1)}. \tag{10}$$

By (6), (9) and (10), we get

$$\begin{aligned}
&\frac{{}_NP_{N-2}}{2^N}(N+1)G^{N+2}\\
&= (N+1)(t-1)\frac{{}_NP_{N-2}}{2^{N-1}}G^{N+1}\\
&\quad+\sum_{m=0}^{N} ta_{m+1}(N,t)G^{(N-m+1)} + \sum_{m=1}^{N+1} t\frac{d}{dt}a_m(N,t)G^{(N-m+1)}\\
&= (N+1)(t-1)\sum_{m=1}^{N+1} a_m(N,t)G^{(N-m+1)}\\
&\quad+\sum_{m=0}^{N} ta_{m+1}(N,t)G^{(N-m+1)} + \sum_{m=1}^{N+1} t\frac{d}{dt}a_m(N,t)G^{(N-m+1)}.
\end{aligned}$$

Since $\frac{{}_NP_{N-2}}{2^N}(N+1)G^{N+2} = \frac{{}_{N+1}P_{N-1}}{2^N}G^{N+2}$, we have the result. □

Therefore, by comparing coefficients on both sides (7) in Theorem 2.2, we obtain the following corollary (see [4]).

Corollary 2.3. *For $N \in \mathbb{N}$, one has*

(i) $a_1(N+1,t) = ta_1(N,t)$,

(ii) $a_{N+2}(N+1,t) = (N+1)(t-1)a_{N+1}(N,t) + t\frac{d}{dt}a_{N+1}(N,t)$,

(iii) $a_{m+1}(N+1,t) = (N+1)(t-1)a_m(N,t)+ta_{m+1}(N,t)+t\frac{d}{dt}a_m(N,t),\ 1 \le m \le N$,

and

(iv) $a_m(N,t) = 0, \quad m > N+1 \quad \text{or} \quad m < 1.$

By (4), (5) and mathematical induction, we note that $a_m(N,t)$ is the Nth order polynomial of t containing m terms and its the lowest order is $N-m+1$.

Let us consider another recurrence relations as follow:

$$a_m(N,t) = b_m(N)t^N - (N-m+2)a_{m-1}(N,t)t^{-1}, \text{ for } 1 \le m \le N+1. \tag{11}$$

$$b_{m+1}(N+1) = (N+1)b_m(N) + b_{m+1}(N), \quad \text{for } 1 \le m \le N. \tag{12}$$

$$a_m(N,t) = b_m(N) = 0, \qquad \text{for } m > N+1 \text{ or } m < 1. \tag{13}$$

By (11), we have the following proposition (see [4]).

Proposition 2.4. *For $N \in \mathbb{N}$, one has*

$$a_m(N,t) = \sum_{l=0}^{m-1} (-1)^l\ {}_{N-m+l+1}P_l\, b_{m-l}(N)t^{N-l}.$$

Proposition 2.4 means that the recurrence relations (11)-(13) imply Corollary 2.3. To determine coefficients $a_m(N,t)$, it is sufficient what $b_m(N)$ are.

Now we set a function f to find $b_m(N)$ in Proposition 2.4 as follows:

$$f(x,y) = \sum_{N=1}^{\infty}\sum_{m=1}^{N+1} b_m(N)\frac{x^N}{N!}y^m, \quad \text{where} \quad |xy| < 1. \tag{14}$$

Therefore, by (14) and Proposition 2.4, we have the following proposition (see [4] for details).

From this, we set

$$\sum_{l_1+\cdots+l_j=N} \frac{1}{l_1\cdots l_j} = C_{jN}.$$

Proposition 2.5. *For $N \in \mathbb{N}$, one has*

$$a_m(N,t) = \sum_{l=0}^{m-1}\sum_{j=1}^{N} C_{jN}\frac{(-1)^l N!}{(j-l)!}\binom{j-l}{N-m+1}t^{N-l},$$

where $(-n)! = 0$, and $\binom{n}{k} = 0$ for $n > 0$ and $n < k$.

Proof. Basically, the contents of this proof are similar with those of the proof in [4]. Hence we give the sketch of the proof.

From (12) and (14), f satisfies the first order linear ordinary differential equation:

$$f' + \frac{y+1}{xy-1}f = \frac{y(y+1)}{1-xy}, \quad |xy| < 1,$$
$$f(0,y) = 0, \quad y \in \mathbb{R}.$$

By the integrating factor method, we obtain that the solution is

$$f(x,y) = y\left((1-xy)^{-\frac{y+1}{y}} - 1\right).$$

From the series expansion of f, we have

$$b_m(N) = \sum_{j=1}^{N} C_{jN}\frac{N!}{j\,!}\binom{j}{m-(N+1)+j}.$$

Combining this fact and Proposition 2.4, we have the result. □

Therefore, by (6) and Proposition 2.5, we have the following theorem.

Theorem 2.6. *For $N \in \mathbb{N}$, we consider the following nonlinear Nth order differential equation with respect to t:*

$$G^{N+1} = \sum_{m=1}^{N+1}\sum_{l=0}^{m-1}\sum_{j=1}^{N} C_{jN}\frac{(-1)^l 2^N}{(j-l)!}\binom{j-l}{N-m+1}t^{N-l}G^{(N-m+1)}, \tag{15}$$

where $G^{(n)} = \frac{d^nG(t)}{dt^n}$ and $G^N = \underbrace{G(t)\times\cdots\times G(t)}_{N-times}$. Then $\frac{2t}{e^t+1}$ is a solution of (15).

Let us define $G^{(n)}(t,x) = G^{(n)}e^{xt}$. Then we see the following corollary by (2).

Corollary 2.7. *For $N \in \mathbb{N}$, we consider*

$$G^{N+1}(t,x) = \sum_{m=1}^{N+1}\sum_{l=0}^{m-1}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N+1+l}2^N}{(j-l)!}\binom{j-l}{N-m+1}t^{N-l}G^{(N-m+1)}(t,x). \tag{16}$$

Then $\frac{2t}{e^t+1}e^{xt}$ is a solution of (16).

3. MAIN RESULTS

Let us find some identities on the high order Genocchi numbers and polynomials. From (2), we have that

$$G^{N+1} = \underbrace{\left(\frac{2t}{e^t+1}\right) \times \cdots \times \left(\frac{2t}{e^t+1}\right)}_{(N+1)-\text{times}} = \sum_{n=0}^{\infty} G_n^{(N+1)} \frac{t^n}{n!}, \tag{17}$$

and

$$G(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{18}$$

From (18), we see

$$G^{(k)} = \frac{d^k G(t)}{dt^k} = \sum_{n=0}^{\infty} G_{n+k} \frac{t^n}{n!}. \tag{19}$$

Theorem 3.1. *For $N \in \mathbb{N}$, one has the following idendities:*
If $n = 0, 1, \ldots, N-1$, then

$$G_n^{(N+1)} = \sum_{m=0}^{n} \sum_{l=m}^{n} \sum_{j=1}^{N} C_{jN} \frac{(-1)^{N-l} 2^N n!}{(j-N+l)!} \binom{j-N+l}{m} \frac{G_{n-l+m}}{(n-l)!}.$$

If $n = N, N+1, \ldots,$ then

$$G_n^{(N+1)} = \sum_{m=0}^{N} \sum_{l=m}^{N} \sum_{j=1}^{N} C_{jN} \frac{(-1)^{N-l} 2^N n!}{(j-N+l)!} \binom{j-N+l}{m} \frac{G_{n-l+m}}{(n-l)!}.$$

Proof. By the case of $m = 1$ in (15), we get

$$\begin{aligned}
&\left(\sum_{j=1}^{N} C_{jN} \frac{2^N}{j\,!} \binom{j}{N} t^N\right) \left(\sum_{p=0}^{\infty} G_{p+N} \frac{t^p}{p!}\right) \\
&= \sum_{n=N}^{\infty} \sum_{j=1}^{N} C_{jN} \frac{2^N}{j\,!} \binom{j}{N} \frac{G_n}{(n-N)!} t^n.
\end{aligned} \tag{20}$$

If $m = 2$ in (15), we have

$$\begin{aligned}
&\left(\sum_{l=0}^{1} \sum_{j=1}^{N} C_{jN} \frac{(-1)^l 2^N}{(j-l)!} \binom{j-l}{N-1} t^{N-l}\right) \left(\sum_{s=0}^{\infty} G_{s+N-1} \frac{t^s}{s!}\right) \\
&= \left(\sum_{l=N-1}^{N} \sum_{j=1}^{N} C_{jN} \frac{(-1)^{N-l} 2^N}{(j-N+l)!} \binom{j-N+l}{N-1} t^l\right) \left(\sum_{s=0}^{\infty} G_{s+N-1} \frac{t^s}{s!}\right) \\
&= \sum_{j=1}^{N} C_{jN} \frac{2^N}{(j-1)!} \binom{j-1}{N-1} G_{N-1} t^{N-1} \\
&\quad + \sum_{n=N}^{\infty} \sum_{l=N-1}^{N} \sum_{j=1}^{N} C_{jN} \frac{(-1)^{N-l} 2^N}{(j-N+l)!} \binom{j-N+l}{N-1} \frac{G_{n-l+N-1}}{(n-l)!} t^n.
\end{aligned} \tag{21}$$

If $m = N + 1$ in (15), we get

$$\begin{aligned}
&\left(\sum_{l=0}^{N}\sum_{j=1}^{N} C_{jN}\frac{(-1)^l 2^N}{(j-l)!}\binom{j-l}{0}t^{N-l}\right)\left(\sum_{s=0}^{\infty} G_s\frac{t^s}{s!}\right)\\
&=\left(\sum_{l=0}^{N}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l} 2^N}{(j-N+l)!}\binom{j-N+l}{0}t^{l}\right)\left(\sum_{s=0}^{\infty} G_s\frac{t^s}{s!}\right)\\
&=\sum_{n=0}^{N-1}\sum_{l=0}^{n}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l} 2^N}{(j-N+l)!}\binom{j-N+l}{0}\frac{G_{n-l}}{(n-l)!}t^n\\
&\quad+\sum_{n=N}^{\infty}\sum_{l=0}^{N}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l} 2^N}{(j-N+l)!}\binom{j-N+l}{0}\frac{G_{n-l}}{(n-l)!}t^n.
\end{aligned} \tag{22}$$

Adding the case of $m = 1, 2, \ldots, N+1$ included (20), (21) and (22), we obtain that the right hand of (15) equals

$$\begin{aligned}
&\sum_{n=0}^{N-1}\sum_{m=0}^{n}\sum_{l=m}^{n}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l} 2^N}{(j-N+l)!}\binom{j-N+l}{m}\frac{G_{n-l+m}}{(n-l)!}t^n\\
&\quad+\sum_{n=N}^{\infty}\sum_{m=0}^{N}\sum_{l=m}^{N}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l} 2^N}{(j-N+l)!}\binom{j-N+l}{m}\frac{G_{n-l+m}}{(n-l)!}t^n.
\end{aligned}$$

Therefore, we have the result. □

From (17), we have

$$\begin{aligned}
G^{(N+1)}(t,x) = G^{(N+1)}e^{xt} &= \left(\sum_{n=0}^{\infty} G_n^{(N+1)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{x^n t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}x^{n-s}G_s^{(N+1)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{23}$$

Therefore, by (2), Theorem 2.6, Theorem 3.1 and (23), we have the following corollary.

Corollary 3.2. *For $N \in \mathbb{N}$, one has the following identities: If $n = 0, 1, \ldots, N-1$, then*

$$G_n^{(N+1)}(x) = \sum_{s=0}^{n}\sum_{m=0}^{s}\sum_{l=m}^{s}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l}2^N s!}{(j-N+l)!(s-l)!}\binom{j-N+l}{m}\binom{n}{s}x^{n-s}G_{s-l+m}.$$

If $n = N, N+1, \ldots,$ then

$$\begin{aligned}
G_n^{(N+1)}(x) = G_{N-1}^{(N+1)}(x) + \sum_{s=N}^{n}\sum_{m=0}^{N}\sum_{l=m}^{N}\sum_{j=1}^{N} C_{jN}\frac{(-1)^{N-l}2^N s!}{(j-N+l)!(s-l)!}\binom{j-N+l}{m}\binom{n}{s}\\
\times x^{n-s}G_{s-l+m}.
\end{aligned}$$

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New bound on eigenvalue of the Hadamard product of matrices *

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Abstract

In this paper, we study the new lower bound for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse. The new lower bound improves the results of Li et al.[Y. T. Li, F. B. Chen, D. F. Wang, New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse, Lin. Alg. Appl. 430(2009)1423-1431.] and generalizes some corresponding results.

Keywords: M-matrix; Hadamard product; Minimum eigenvalue.

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1 Introduction

We first give some basic concepts for latter use. $|\cdot|$ stands for the modulus. For a positive integer n, N denotes the set $\{1, 2, \cdots, n\}$. Matrix $A \geq 0$ $(A > 0)$ means its entries $a_{ij} \geq 0$ $(a_{ij} > 0)$. The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$ and $R^{n\times n}$ denotes the set of all $n \times n$ real matrices throughout this paper.

A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an M-matrix, if $A = \alpha I - P$ with $P \geq 0$ and $\alpha \geq \rho(P)$, where $\rho(P)$ is the spectral radius of P.

An $n \times n$ matrix A is reducible if there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where B is an $r \times r$ submatrix and D is an $(n-r) \times (n-r)$ submatrix, for $1 \leq r \leq n$. If no such permutation matrix exists, then A is irreducible.

If A is an M-matrix, then there exists a minimum eigenvalue $\tau(A)$ of A, where $\tau(A) = min\{|\lambda| : \lambda \in \sigma(A)\}$, $\sigma(A)$ denotes the spectrum of A(See [1]).

For two matrices $A = (a_{ij}) \in C^{n\times n}$ and $B = (b_{ij}) \in C^{n\times n}$, the Hadamard product of A and B is denoted by $A \circ B = (c_{ij}) = (a_{ij} \times b_{ij})$. If A and B are M- matrices, then it was proved that $A \circ B^{-1}$ is an M-matrix in [1].

Let $A = (a_{ij})$ be an M-matrix, Fiedler and Markham [1] proved that $\tau(A \circ A^{-1}) \geq \frac{1}{n}$ and conjectured that $\tau(A \circ A^{-1}) \geq \frac{2}{n}$. Yong [4] has proved the conjecture.

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Let $A = (a_{ij})$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout this paper, For any $i, j, k \in N$, We denote

$$\begin{aligned}
R_i &= \sum_{k \neq i} |a_{ik}|; \\
r_{ji} &= \frac{|a_{ji}|}{|a_{jj}| - \sum\limits_{k \neq j,i} |a_{jk}|}, j \neq i, r_i = \max_{j \neq i}\{r_{ji}\}; \\
m_{ji} &= \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| r_i}{|a_{jj}|}, j \neq i, m_i = \max_{j \neq i}\{m_{ji}\}; \\
h_{ji} &= \frac{|a_{ji}|}{|a_{jj}| r_{ji} - \sum\limits_{k \neq j,i} |a_{jk}| r_{ki}}, j \neq i, h_i = \max_{j \neq i}\{h_{ji}\}; \\
s_{ji} &= \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| r_{ki} h_i}{|a_{jj}|}, j \neq i, s_i = \max_{j \neq i}\{s_{ji}\}.
\end{aligned}$$

If $A = (a_{ij})$ and $B = (b_{ij})$ are M-matrices, Huang [2] proved the following inequality:

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + (\rho(J_B))^2} \min_i \frac{a_{ii}}{b_{ii}},$$

where $\rho(J_A)$ and $\rho(J_B)$ is the spectral radius of J_A and J_B, respectively. When $A = B$, the above inequality gave another lower bound of $\tau(A \circ A^{-1})$ that is

$$\tau(A \circ A^{-1}) \geq \frac{1 - (\rho(J_A))^2}{1 + (\rho(J_A))^2}.$$

Li [3] proved the following result:

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum\limits_{j \neq i} m_{ji}} \right\},$$

where A is an M-matrix and A^{-1} is doubly stochastic matrix.

The remainder of this paper is organized as follows. In Sections 2, some Lemmas and notations are given. In Section 3, our main results are presented. In Section 4, an example is shown.

2 Some Lemmas and notations

In this section, we give some Lemmas that involve inequalities for the entries of A^{-1}.

Lemma 2.1 [4] *If $A = (a_{ij}) \in R^{n \times n}$ is a strictly diagonally dominant matrix, then $A^{-1} = (b_{ij})$ exists, and for all $j \neq i$,*

$$|b_{ji}| \leq \frac{\sum\limits_{k \neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}|.$$

Lemma 2.2 [3] *If $A = (a_{ij}) \in R^{n\times n}$ is a strictly diagonally dominant M-matrix, then $A^{-1} = (b_{ij})$ exists, and for all $j \neq i$,*

$$b_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|} b_{ii}.$$

Lemma 2.3 *If $A = (a_{ij}) \in R^{n\times n}$ is a strictly diagonally dominant M-matrix, then for $A^{-1} = (b_{ij})$ exists, and for all $j \neq i$,*

$$b_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_{ki} h_i}{|a_{jj}|} b_{ii}.$$

Proof. For $i \in N$, let $r_{ji}(\varepsilon) = \frac{|a_{ji}|+\varepsilon}{|a_{ji}|-\sum\limits_{k\neq j,i}|a_{jk}|}$, $h_i(\varepsilon) = \max\limits_{j\neq i}\{\frac{|a_{ji}|+\varepsilon}{|a_{jj}|r_{ji}(\varepsilon)-\sum\limits_{k\neq j,i}|a_{jk}|r_{ki}(\varepsilon)}\}$, since A is a strictly diagonally dominant, there exists $\varepsilon > 0$ such that $0 < r_{ji}(\varepsilon) \leq 1,\ 0 < h_i(\varepsilon) \leq 1$.

Let $X_i(\varepsilon) = diag(r_{1i}(\varepsilon)h_i(\varepsilon), r_{2i}(\varepsilon)h_i(\varepsilon), \cdots, r_{i-1,i}(\varepsilon)h_i(\varepsilon), r_{i+1,i}(\varepsilon)h_i(\varepsilon), \cdots, r_{ni}(\varepsilon)h_i(\varepsilon))$. For every $i, j \in N, j \neq i$, we have

$$h_i(\varepsilon) > \frac{|a_{ji}|}{|a_{jj}|r_{ji}(\varepsilon) - \sum\limits_{k\neq j,i} |a_{jk}| r_{ki}(\varepsilon)}.$$

It means that

$$|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_{ki}(\varepsilon) h_i(\varepsilon) < |a_{jj}| r_{ji}(\varepsilon) h_i(\varepsilon).$$

While, for $j = i$, we can get

$$\sum_{k\neq i} |a_{ik}| r_{ki}(\varepsilon) h_i(\varepsilon) \leq \sum_{k\neq i} |a_{ik}| < |a_{ii}|.$$

Therefore, $AX_i(\varepsilon)$ is strictly diagonally dominant, By Lemma 2.1, we obtain the following inequality:

$$\frac{b_{ji}}{r_{ji}(\varepsilon)h_i(\varepsilon)} < \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_{ki}(\varepsilon) h_i(\varepsilon)}{|a_{jj}| r_{ji}(\varepsilon) h_i(\varepsilon)} b_{ii},$$

By simple calculation, we have

$$b_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_{ki}(\varepsilon) h_i(\varepsilon)}{|a_{jj}|} b_{ii}.$$

Let $\varepsilon \to 0$, for all $i, j \in N,\ j \neq i$, we obtain

$$b_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_{ki} h_i}{|a_{jj}|} b_{ii}.$$

Hence, the proof of the Lemma 2.3 is completed. □

Remark 2.1 *Since $r_{ki}h_i \leq r_i$, we have*

$$b_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_{ki} h_i}{|a_{jj}|} b_{ii} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|} b_{ii} \leq \frac{\sum\limits_{k\neq j} |a_{jk}|}{|a_{jj}|} b_{ii}.$$

So Lemma 2.3 is improvement of Lemma 2.1 and Lemma 2.2.

Lemma 2.4 [5] *Let $A = (a_{ij}) \in C^{n\times n}$ and let $x_1, x_2, \cdots, x_n$ be positive real numbers, then all the eigenvalues of A lie in the region:*

$$\bigcup_{i\neq j}\left\{z : |z-a_{ii}||z-a_{jj}| \leq \left(x_i\sum_{k\neq i}^{n}\frac{1}{x_k}|a_{ki}|\right)\left(x_j\sum_{k\neq j}^{n}\frac{1}{x_k}|a_{kj}|\right)\right\}.$$

Lemma 2.5 [6] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^Te = e$, where $e = (1, 1, \cdots, 1)^T$*

3 Main results

In this section, we obtain new lower bound for $\tau(A \circ A^{-1})$, which improves the result of Li [3].

Theorem 3.1 *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix, and suppose $A^{-1} = (b_{ij})$ is doubly stochastic matrix, then $b_{ii} \geq \frac{1}{1+\sum\limits_{j\neq i} s_{ji}}$, $i \in N$.*

Proof. Since $A^{-1} = (b_{ij})$ is a doubly stochastic matrix, and A is an M-matrix, by Lemma 2.5, we have

$$a_{ii} = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq i}|a_{ki}| + 1 \; and \; b_{ii} + \sum_{j\neq i} b_{ji} = 1, i \in N.$$

Since the matrix A is strictly diagonally dominant, then, by Lemma 2.2, we obtain

$$1 = b_{ii} + \sum_{j\neq i} b_{ji} \leq b_{ii} + \sum_{j\neq i}\frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|r_{ki}h_i}{|a_{jj}|}b_{ii} = \left(1 + \sum_{j\neq i} s_{ji}\right) b_{ii}.$$

It means that

$$b_{ii} \geq \frac{1}{1+\sum\limits_{j\neq i} s_{ji}}, i \in N.$$

The proof of the Theorem 3.1 is completed. □

Remark 3.1 *Since $s_{ji} \leq m_{ji}$, we have*

$$b_{ii} \geq \frac{1}{1+\sum\limits_{j\neq i} s_{ji}} \geq \frac{1}{1+\sum\limits_{j\neq i} m_{ji}}.$$

Then, Theorem3.1 is improvement of Theorem 3.1 in [3].

Theorem 3.2 *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix, and suppose $A^{-1} = (b_{ij})$ is doubly stochastic matrix, then*

$$\tau(A \circ A^{-1}) \geq \min_{j\neq i}\frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_jb_{ii}b_{jj}R_iR_j]^{\frac{1}{2}}\}. \tag{1}$$

Proof. There are two cases which should be considered.

Case (1): First, we assume that A is irreducible, since A^{-1} is doubly stochastic, we have

$$a_{ii} = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq i}|a_{ki}| + 1 \; and \; a_{ii} \geq 1, i \in N.$$

so, we can get $\sum\limits_{k\neq i}|a_{ik}| = \sum\limits_{k\neq i}|a_{ki}|$. For convenience, we denote $R_j^r = \sum\limits_{k\neq j}|a_{jk}|r_{ki}h_i,\ j\neq i$. Then for $i,j\in N, j\neq i$, we have

$$R_j^r = \sum_{k\neq j}|a_{jk}|r_{ki}h_i \leq |a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_{ki}h_i \leq |a_{ji}| + \sum_{k\neq j}|a_{jk}| = R_j \leq a_{jj}.$$

Therefore, there exists a real number $\alpha_{ji}(0\leq\alpha_{ji}\leq 1)$ such that

$$|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_{ki}h_i = \alpha_{ji}R_j + (1-\alpha_{ji})R_j^r.$$

So, we obtain that

$$s_{ji} = \frac{\alpha_{ji}R_j + (1-\alpha_{ji})R_j^r}{a_{jj}}.$$

Let $\alpha_j = \max\limits_{i\neq j}\{\alpha_{ji}\}, (0<\alpha_j\leq 1)$(if $\alpha_j = 0$, then A is reducible, which is a contraction).

Let $s_j = \max\limits_{i\neq j}\{s_{ji}\} = \frac{\alpha_{ji}R_j+(1-\alpha_{ji})R_j^r}{a_{jj}}, j\in N$. Since A is irreducible, then $R_j>0, R_j^r>0$ and $0<s_j\leq 1$. Let $\tau(A\circ A^{-1)} = \lambda$, we can get $0\leq\lambda\leq a_{ii}b_{ii}, i\in N$, thus by Lemma 2.4, there is a pair (i,j) of positive integers with $i\neq j$ such that

$$|\lambda - a_{ii}b_{ii}||\lambda - a_{jj}b_{jj}| \leq \left(s_i\sum_{k\neq i}^{n}\frac{1}{s_k}|a_{ki}||b_{ki}|\right)\left(s_j\sum_{k\neq j}^{n}\frac{1}{s_k}|a_{kj}||b_{kj}|\right).$$

Observe that

$$\begin{aligned}
&\left(s_i\sum_{k\neq i}^{n}\frac{1}{s_k}|a_{ki}||b_{ki}|\right)\left(s_j\sum_{k\neq j}^{n}\frac{1}{s_k}|a_{kj}||b_{kj}|\right)\\
\leq&\left(s_i\sum_{k\neq i}^{n}\frac{a_{kk}}{\alpha_kR_k+(1-\alpha_kR_k^r)}|a_{ki}|\frac{|a_{ki}|+\sum\limits_{l\neq k,i}|a_{kl}|r_{li}h_i}{a_{kk}}b_{ii}\right)\\
&\times\left(s_j\sum_{k\neq j}^{n}\frac{a_{kk}}{\alpha_kR_k+(1-\alpha_kR_k^r)}|a_{kj}|\frac{|a_{kj}|+\sum\limits_{l\neq k,j}|a_{kl}|r_{lj}h_j}{a_{kk}}b_{ii}\right)\\
\leq&\left(s_i\sum_{k\neq i}^{n}|a_{ki}|b_{ii}\right)\left(s_j\sum_{k\neq j}^{n}|a_{kj}|b_{jj}\right)\\
=&\left(s_i\sum_{k\neq i}^{n}|a_{ik}|b_{ii}\right)\left(s_j\sum_{k\neq j}^{n}|a_{jk}|b_{jj}\right)\\
=&s_is_jR_iR_jb_{ii}b_{jj}.
\end{aligned}$$

By the above relation, we can get

$$|\lambda - a_{ii}b_{ii}||\lambda - a_{jj}b_{jj}| \leq s_is_jR_iR_jb_{ii}b_{jj}.$$

Then, we have

$$\lambda \geq \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_jb_{ii}b_{jj}R_iR_j]^{\frac{1}{2}}\}.$$

Hence, it is easy to see that inequality (1) is satisfied.

Case (2): when A is reducible, without loss of generality, we can assume that A has the block upper triangular form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ & A_{22} & \cdots & A_{2s} \\ & & \cdots & \cdots \\ & & & A_{ss} \end{pmatrix},$$

with irreducible diagonal block $A_{ii}, i = 1, 2, \cdots, s$. Then A^{-1} is block upper triangular with irreducible diagonal block A_{ii}^{-1}. Observing that $\tau(A \circ A^{-1}) = \min_{k} \tau(A_{kk} \circ A_{kk}^{-1})$, the inequality (1) is also satisfied. From what has been discussed above, the proof of the theorem is completed.□

Theorem 3.3 *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix, and $A^{-1} = (b_{ij})$ be doubly stochastic matrix, then*

$$\begin{aligned} &\min_{j\neq i} \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_jb_{ii}b_{jj}R_iR_j]^{\frac{1}{2}}\} \\ &\geq \min_{i}(a_{ii} - s_iR_i)b_{ii} \\ &\geq \min_{i} \frac{(a_{ii} - s_iR_i)}{1 + \sum\limits_{j\neq i} s_{ji}}. \end{aligned} \tag{2}$$

Proof. Without loss of generality, for $j \neq i$, we assume that $a_{ii}b_{ii} - s_iR_ib_{ii} \leq a_{jj}b_{jj} - s_jR_jb_{jj}$. Thus

$$s_jR_jb_{jj} \leq a_{jj}b_{jj} - a_{ii}b_{ii} + s_iR_ib_{ii}.$$

From the above inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_jb_{ii}b_{jj}R_iR_j]^{\frac{1}{2}}\right\} \\ &\geq \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_ib_{ii}R_i(a_{jj}b_{jj} - a_{ii}b_{ii}) + (2s_ib_{ii}R_i)^2]^{\frac{1}{2}}\right\} \\ &= \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} - (a_{jj}b_{jj} - a_{ii}b_{ii} + 2s_iR_ib_{ii})\} \\ &= a_{ii}b_{ii} - s_iR_ib_{ii}. \end{aligned}$$

Hence, we get

$$\min_{j\neq i} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_jb_{ii}b_{jj}R_iR_j]^{\frac{1}{2}}\right\} \geq \min_{i}(a_{ii} - s_iR_i)b_{ii}.$$

By Theorem 3.1, we obtain

$$\min_{i}(a_{ii} - s_iR_i)b_{ii} \geq \min_{i} \frac{(a_{ii} - s_iR_i)}{1 + \sum\limits_{j\neq i} s_{ji}}.$$

Based on the above discussion and analysis, we know that (2) is satisfied. We complete the proof of the theorem.□

Remark 3.2 *Since $s_{ji} \leq m_{ji}$, let $s_i = \max_{j\neq i} s_{ji}$, $m_i = \max_{j\neq i} m_{ji}$, then we have*

$$a_{ii} - s_i R_i \geq a_{ii} - m_i R_i \text{ and } \frac{1}{1+\sum\limits_{j\neq i} s_{ji}} \geq \frac{1}{1+\sum\limits_{j\neq i} m_{ji}}.$$

Then, for $i \in N$, we obtain

$$\min_i \frac{(a_{ii} - s_i R_i)}{1+\sum\limits_{j\neq i} s_{ji}} \geq \min_i \frac{(a_{ii} - m_i R_i)}{1+\sum\limits_{j\neq i} m_{ji}}.$$

Hence, Theorem3.3 shows that the result of Theorem 3.2 is better than the result $\tau(A\circ A^{-1}) \geq \min_i \frac{(a_{ii}-m_iR_i)}{1+\sum_{j\neq i} m_{ji}}$ of Theorem3.2 in [3].

4 Example

Consider the following M-matrix

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix},$$

If we apply the conjecture of Fiedler and Markham [1], we have $\tau(A\circ A^{-1}) \geq \frac{2}{n} = 0.5$;
If we apply Theorem 9 of [2] with $A = B$, we have $\tau(A\circ A^{-1}) \geq 0.2641$;
If we apply Theorem 2 of [3], we have $\tau(A\circ A^{-1}) \geq 0.7999$;
The bound in our Theorem 3.1 is better: $\tau(A\circ A^{-1}) \geq 0.8233$. In fact, $\tau(A\circ A^{-1}) = 0.9756$.

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OSCILLATION RESULTS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. By employing a generalized Riccati transformation and the averaging technique, some new oscillation criteria for second-order nonlinear differential equation of the form

$$\left(r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + f(t,x(t),x'(t)) = 0, \quad t \geq t_0,$$

are established. Several examples are also considered to illustrate the main results.

1. INTRODUCTION

In this paper, we are concerned with the oscillatory behavior of solutions of the second-order nonlinear differential equation

$$\left(r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + f(t,x(t),x'(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $t_0 \geq 0$, $\alpha > 0$ is a constant, $r \in C^1([t_0,\infty);(0,\infty))$, $\psi(x) \in C(R;R)$, and $f:[t_0,\infty)\times R\times R \to R$ is a continuous function. We shall consider the two cases

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} < \infty. \tag{1.3}$$

By a solution of (1.1), we mean a real-valued function $x(t) \in C^1[T_x,\infty)$, $T_x \geq t_0$ such that $r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1}x'(t) \in C^1[T_x,\infty)$ and satisfies equation (1.1) on $[T_x,\infty)$. Our attention is restricted to those solutions of (1.1) which exist on the half-line $[T_x,\infty)$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. A solution $x(t)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Since Sturm [1] introduced the concept of oscillation when he studied the problem of the heat transmission, oscillation theory has been an important area of research in the qualitative theory of ordinary differential and dynamic equations. Usually a qualitative approach is concerned with the behavior of solutions of a given differential equation and does not seek explicit solutions. Since then, oscillation behavior of solutions for various classes of second-order linear and nonlinear ordinary differential and dynamic equations has been discussed by many authors with several different methods (see, for

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example, ([4]-[36]) and the references quoted therein). Many papers deal with a special case of equation (1.1) such as the second order linear differential equations

$$x''(t) + q(t)x(t) = 0,$$

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0,$$

the second order nonlinear differential equations

$$x''(t) + q(t)\left|x\right|^{\alpha-1} x = 0,$$

$$\left(r(t)\left|x'(t)\right|^{\alpha-1} x'(t)\right)' + q(t)\left|x\right|^{\alpha-1} x = 0$$

and

$$\left(r(t)\psi(x(t))x'(t)\right)' + q(t)f(x) = 0.$$

Following this trend, in this paper, we will establish several new oscillation criteria for Eq. (1.1) under the cases when (1.2) and (1.3) hold. To obtain our results, we use a Riccati-type transformation and the integral averaging technique. Finally, some examples are provided to illustrate the main results.

2. Main results

Theorem 2.1. *Assume that (1.2) holds, and*

$$0 < L_1 \le \psi(x) \le L_2, \text{ where } L_1 \text{ and } L_2 \text{ are real numbers.} \tag{2.1}$$

Assume further that there exist positive functions $\rho \in C^1([t_0,\infty);R)$ and $q \in C([t_0,\infty);R)$ such that $\rho'(t) \ge 0$,

$$f(t,x,x')/\left|x\right|^{\alpha-1} x \ge q(t) \text{ for all } t \in [t_0,\infty),\ x \in R\setminus\{0\},\ x' \in R, \tag{2.2}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)q(s) - \frac{L_2^2}{L_1}\rho'(s)\varphi^{\alpha}(s)\right) ds = \infty, \tag{2.3}$$

where $\varphi(t) = \left(\int_{t_0}^{t} r^{-1/\alpha}(s)ds\right)^{-1}$.

Then Eq. (1.1) is oscillatory.

Proof. Assume (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Then there exists a $t_1 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge t_1$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \ge t_1$, for some $t_1 \ge t_0$. A similar argument holds for the case when $x(t)$ is eventually negative. From (1.1) and (2.2), we have, for all $t \ge t_1$,

$$\left(r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t)\right)' = -f(t,x(t),x'(t)) \le -q(t)x^{\alpha}(t) < 0. \tag{2.4}$$

Then $r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t)$ is strictly decreasing on $[t_1,\infty)$ and is eventually of one sign. We claim that

$$x'(t) > 0 \quad \text{for } t \ge t_1. \tag{2.5}$$

If this is not the case, then there exists $t_2 \in [t_1,\infty)$ such that $x'(t_2) \le 0$. Thus, we have $r(t_2)\psi(x(t_2))\left|x'(t_2)\right|^{\alpha-1} x'(t_2) \le 0$. Since $r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t)$ is strictly decreasing on $[t_1,\infty)$, there is a $t_3 \ge t_2$ such that

$$r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t) \le r(t_3)\psi(x(t_3))\left|x'(t_3)\right|^{\alpha-1} x'(t_3) := c < 0 \quad \text{for } t \in [t_3,\infty),$$

which gives

$$x'(t) \leq -(-c)^{1/\alpha} \frac{1}{r^{1/\alpha}(t)\psi^{1/\alpha}(x(t))} \leq -(-c)^{1/\alpha} \frac{1}{L_2^{1/\alpha} r^{1/\alpha}(t)}. \tag{2.6}$$

Integrating (2.6) from t_3 to t, and using (1.2), we obtain

$$x(t) \leq x(t_3) - (-\frac{c}{L_2})^{1/\alpha} \int_{t_3}^{t} \frac{ds}{r^{1/\alpha}(s)} \to -\infty \quad \text{as } t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Hence, (2.5) holds. Now, in view of (2.4) and (2.5), we conclude that for all $t \geq t_1$,

$$\left(r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha}\right)' \leq -q(t)x^{\alpha}(t) < 0. \tag{2.7}$$

We now define the function

$$w(t) = \rho(t) \frac{r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha}}{x^{\alpha}(t)} \qquad \text{for } t \geq t_1. \tag{2.8}$$

From (2.5), we see that $w(t) > 0$. Differentiating (2.8) and using (2.7), we obtain, for all $t \geq t_1$

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t) \left[\frac{\left(r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha}\right)' x^{\alpha}(t) - \alpha r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha} x^{\alpha-1}(t) x'(t)}{x^{2\alpha}(t)} \right] \\
&\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - \alpha\rho(t) \frac{r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha+1}}{x^{\alpha+1}(t)} \\
&= -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \alpha w(t) \frac{x'(t)}{x(t)} \\
&\leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) \qquad (2.9) \\
&= -\rho(t)q(t) + \rho'(t) r(t)\psi(x(t)) \left(\frac{x'(t)}{x(t)}\right)^{\alpha} \\
&\leq -\rho(t)q(t) + L_2 r(t)\rho'(t) \left(\frac{x'(t)}{x(t)}\right)^{\alpha}. \qquad (2.10)
\end{aligned}$$

From (2.7), we have

$$\begin{aligned}
x(t) &= x(t_1) + \int_{t_1}^{t} x'(s) ds \\
&= x(t_1) + \int_{t_1}^{t} r^{-1/\alpha}(s)\psi^{-1/\alpha}(x(s)) \left(r(s)\psi(x(s)) \left(x'(s)\right)^{\alpha}\right)^{1/\alpha} ds \\
&\geq \left(r(t)\psi(x(t)) \left(x'(t)\right)^{\alpha}\right)^{1/\alpha} \int_{t_1}^{t} r^{-1/\alpha}(s)\psi^{-1/\alpha}(x(s)) ds \\
&\geq \left(\frac{L_1}{L_2}\right)^{1/\alpha} r^{1/\alpha}(t) x'(t) \int_{t_1}^{t} r^{-1/\alpha}(s) ds,
\end{aligned}$$

which implies

$$\left(\frac{x'(t)}{x(t)}\right)^{\alpha} \leq \frac{L_2\varphi^{\alpha}(t)}{L_1 r(t)} \quad \text{for } t \geq t_1. \tag{2.11}$$

Using (2.11) in (2.10), we get

$$w'(t) \leq -\rho(t)q(t) + \frac{L_2^2}{L_1}\rho'(t)\varphi^{\alpha}(t) \quad \text{for } \; t \geq t_1. \tag{2.12}$$

Integrating (2.12) from t_1 to t, we obtain

$$\int_{t_1}^{t} \left(\rho(s)q(s) - \frac{L_2^2}{L_1}\rho'(s)\varphi^{\alpha}(s)\right) ds \leq -w(t) + w(t_1) \leq w(t_1).$$

Then, taking a limit superior on both sides, we obtain a contradiction to the condition (2.3). Therefore, equation (1.1) is oscillatory. □

Theorem 2.2. *Assume that (1.3) and (2.1) hold. Furthermore, the functions $\rho(t)$ and $q(t)$ be defined as in Theorem 2.1 such that (2.2) and (2.3) hold. If*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(z)} \int_{t_0}^{z} q(s)\chi^{\alpha}(s)ds\right)^{1/\alpha} dz = \infty, \tag{2.13}$$

where $\chi(t) = \int_t^{\infty} r^{-1/\alpha}(s)ds$,
then every solution of Eq. (1.1) is oscillatory.

Proof. Suppose to the contrary that equation (1.1) possesses a nonoscillatory solution x on an interval $[t_0, \infty)$. Then there exists a $t_1 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$ since the case when $x(t)$ is eventually negative can be treated analogously. Define again the function $w(t)$ by (2.8). There are two possible cases for the sign of $x'(t)$. The proof if $x'(t)$ is eventually positive is similar to that in the proof of Theorem 2.1, and hence is omitted.

Now, assume that $x'(t)$ is eventually negative. Then there exists $t_2 \geq t_1$ such that $x'(t) < 0$ for $t \in [t_2, \infty)$. From this, (1.1) and (2.2), we get for $t \geq t_2$

$$\left(r(t)\psi(x(t))\left(-x'(t)\right)^{\alpha}\right)' = f(t, x(t), x'(t)) \geq q(t)x^{\alpha}(t) > 0. \tag{2.14}$$

Thus, $r(t)\psi(x(t))\left(-x'(t)\right)^{\alpha}$ is increasing on $[t_2, \infty)$, and so

$$r(s)\psi(x(s))\left(-x'(s)\right)^{\alpha} \geq r(t)\psi(x(t))\left(-x'(t)\right)^{\alpha} \quad \text{for } s \geq t \geq t_2.$$

The last inequality yields, for $s \geq t \geq t_2$,

$$-x'(s) \geq \frac{1}{L_2^{1/\alpha}} r^{-1/\alpha}(s) r^{1/\alpha}(t)\psi^{1/\alpha}(x(t))\left(-x'(t)\right). \tag{2.15}$$

Integrating (2.15) from $t \geq t_2$ to $u \geq t$ and letting $u \to \infty$, we see that

$$\begin{aligned} x(t) &\geq \frac{1}{L_2^{1/\alpha}}\left(\int_t^\infty r^{-1/\alpha}(s)ds\right) r^{1/\alpha}(t)\psi^{1/\alpha}(x(t))\left(-x'(t)\right) \\ &= \frac{1}{L_2^{1/\alpha}}\chi(t)r^{1/\alpha}(t)\psi^{1/\alpha}(x(t))\left(-x'(t)\right) \\ &\geq \frac{1}{L_2^{1/\alpha}}\chi(t)r^{1/\alpha}(t_2)\psi^{1/\alpha}(x(t_2))\left(-x'(t_2)\right) := \beta\chi(t) \qquad (2.16) \end{aligned}$$

where $\beta := \frac{1}{L_2^{1/\alpha}}r^{1/\alpha}(t_2)\psi^{1/\alpha}(x(t_2))\left(-x'(t_2)\right) > 0$. Thus, by (2.14) and (2.16), we conclude that

$$\left(r(t)\psi(x(t))\left(-x'(t)\right)^\alpha\right)' \geq q(t)x^\alpha(t) > \beta^\alpha q(t)\chi^\alpha(t) \quad \text{for } t \geq t_2 \qquad (2.17)$$

Integrating (2.17) from t_2 to t, we obtain

$$r(t)\psi(x(t))\left(-x'(t)\right)^\alpha \geq r(t_2)\psi(x(t_2))\left(-x'(t_2)\right)^\alpha + \beta^\alpha \int_{t_2}^t q(s)\chi^\alpha(s)ds$$

or

$$-x'(t) \geq \frac{1}{L_2^{1/\alpha}}\left(\frac{1}{r(t)}\beta^\alpha \int_{t_2}^t q(s)\chi^\alpha(s)ds\right)^{1/\alpha} \quad \text{for } t \geq t_2.$$

A second integration yields

$$x(t) \leq x(t_2) - \frac{\beta}{L_2^{1/\alpha}}\int_{t_2}^t \left(\frac{1}{r(z)}\int_{t_2}^z q(s)\chi^\alpha(s)ds\right)^{1/\alpha} dz \quad \text{for } t \geq t_2.$$

Letting $t \to \infty$ and using (2.13), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. This completes the proof of the theorem. □

Theorem 2.3. *Assume (1.2), (2.1) and (2.2) hold, and let $H : D = \{(t,s) : t \geq s \geq t_0\} \to R$ be a continuous function such that*

$$H(t,t) \geq 0 \quad \textit{for } t \geq t_0, \quad H(t,s) > 0 \quad \textit{for } t > s \geq t_0$$

and H has a continuous partial derivative on D with respect to the second variable. Suppose also that there exists a positive function $\rho \in C^1([t_0,\infty); R)$ such that

$$\frac{\rho'(t)}{\rho(t)}H(t,s) + \frac{\partial H(t,s)}{\partial s} \leq 0 \quad \textit{for } t \geq s \geq t_0 \qquad (2.18)$$

and for any $t_1 \in [t_0,\infty)$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)}\int_{t_1}^t \rho(s)q(s)H(t,s)ds = \infty, \qquad (2.19)$$

then every solution of equation (1.1) is oscillatory.

Proof. As in Theorem 2.1, without loss of generality we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ on $[t_1, \infty)$, for some $t_1 \geq t_0$. Using the function $w(t)$ defined in (2.8) and proceeding similarly as in the proof of Theorem 2.1, we arrive at the inequality (2.9) . Multiplying (2.9) by $H(t,s)$ and integrating the resulting inequality from t_1 to t, we have,

$$\begin{aligned} \int_{t_1}^{t} \rho(s)q(s)H(t,s)ds &\leq -\int_{t_1}^{t} H(t,s)w'(s)ds + \int_{t_1}^{t} \frac{\rho'(s)}{\rho(s)} w(s)H(t,s)ds \\ &\leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) \right] w(s)ds \end{aligned} \tag{2.20}$$

Using (2.18) in (2.20), we obtain

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} \rho(s)q(s)H(t,s)ds \leq w(t_1) < \infty, \tag{2.21}$$

which contradicts condition (2.19). This completes the proof of the theorem. □

Theorem 2.4. *Suppose that (1.3), (2.1) and (2.2) hold. Let the functions ρ and H be defined as in Theorem 2.3 such that (2.18) and (2.19) hold. Suppose also that (2.13) holds. Then equation (1.1) is oscillatory.*

Proof. The proof of this theorem is similar to that of Theorem 2.2 and hence is omitted. □

In the remaining part of this section, we present another set of oscillation results which differ from Theorems 2.1 and 2.2. To obtain the results, beside the basic assumptions on r and ψ that appeared in equation (1.1), we shall also assume that $r'(t) \geq 0$, and ψ is a differentiable function such that $\psi'(x) \geq 0$ for all x.

Lemma 2.1. *Assume that (1.2), (2.1) and (2.2) hold. Furthermore, assume that $r'(t) \geq 0$, $\psi'(x) \geq 0$ for all x, and $x(t)$ is an eventually positive solution of (1.1). Then, there exists a $T_x \geq t_0$ such that*

$$x'(t) > 0,\ x''(t) < 0,\ \text{and}\ \left(r(t)\psi(x(t)) \left| x'(t) \right|^{\alpha-1} x'(t) \right)' < 0 \tag{2.22}$$

for $t \geq T_x$.

Proof. Since $x(t)$ is an eventually positive solution of (1.1), then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 2.1 , we see that (2.4) and (2.5) hold for $t \geq t_1$, and so $r(t)\psi(x(t)) (x'(t))^{\alpha}$ is decreasing on $[t_1, \infty)$. Let us show now that

$$x''(t) < 0 \quad \text{for } t \geq t_1. \tag{2.23}$$

From (2.7), we obtain

$$\begin{aligned} 0 > \left(r(t)\psi(x(t)) (x'(t))^{\alpha} \right)' = \quad & r'(t)\psi(x(t)) \left(x'(t) \right)^{\alpha} + r(t)\psi'(x(t)) \left(x'(t) \right)^{\alpha+1} \\ & + \alpha r(t)\psi(x(t)) \left(x'(t) \right)^{\alpha-1} x''(t), \end{aligned}$$

which implies that (2.23) holds. Hence, (2.22) holds and the proof is complete.

Theorem 2.5. *Assume that (1.2) and (2.1) hold, $r'(t) \geq 0$, and $\psi'(x) \geq 0$ for all x. Suppose also that there exist positive functions $\rho \in C^1([t_0, \infty); R)$ and $q \in C([t_0, \infty); R)$ such that $\rho'(t) \geq 0$, (2.2) holds and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)q(s) - L_2 r(s)\rho'(s)\left(\frac{2}{s}\right)^{\alpha}\right] ds = \infty. \tag{2.24}$$

Then every solution of equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1.1). Then there exists a $t_1 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$, for some $t_1 \geq t_0$. Define $w(t)$ as of Theorem 2.1. As in the proofs of Theorem 2.1, we arrive at the inequality (2.10). On the other hand, by Lemma 2.1, we know that $x'(t)$ is positive and decreasing on $[t_1, \infty)$. Using this, and fixing $t_2 \geq 2t_1$, we have for $t \in [t_2, \infty)$ that

$$x(t) = x(t_1) + \int_{t_1}^{t} x'(s)ds \geq \int_{t_1}^{t} x'(s)ds \geq (t - t_1)x'(t) \geq \frac{t}{2}x'(t),$$

which yields

$$\frac{x'(t)}{x(t)} \leq \frac{2}{t} \quad \text{for } t \geq t_2. \tag{2.25}$$

Substituting (2.25) into (2.10), we obtain

$$w'(t) \leq -\rho(t)q(t) + L_2 r(t)\rho'(t)\left(\frac{2}{t}\right)^{\alpha}. \tag{2.26}$$

Integrating (2.26) from t_2 to t, we obtain

$$\int_{t_2}^{t} \left[\rho(s)q(s) - L_2 r(s)\rho'(s)\left(\frac{2}{s}\right)^{\alpha}\right] ds \leq -w(t) + w(t_2) \leq w(t_2) < \infty,$$

which contradicts (2.24). This completes the proof of Theorem 2.5. □

Theorem 2.6. *Suppose that (1.3) and (2.1) hold, $r'(t) \geq 0$, and $\psi'(x) \geq 0$ for all x. Let ρ and q be defined as in Theorem 2.5 such that (2.2) and (2.24) are satisfied. Suppose further that (2.13) is satisfied. Then equation (1.1) is oscillatory.*

Proof. The proof is similar to that of Theorem 2.2, so we omit the details. □

3. Examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the nonlinear differential equation

$$\left(t^{\alpha}\left(2 + \frac{2 - e^{-x(t)}}{1 + e^{-x(t)}}\right)|x'(t)|^{\alpha-1}x'(t)\right)' + \left(t^2 + \frac{1}{t}\right)|x(t)|^{\alpha-1}x(t)\left(1 + (x'(t))^2\right) = 0, \tag{3.1}$$

for $t \in [1, \infty)$, where $r(t) = t^\alpha$, $\psi(x(t)) = 2 + \frac{2-e^{-x(t)}}{1+e^{-x(t)}}$, $q(t) = t^2 + \frac{1}{t}$ and $\alpha > 0$ is a constant. It is easy to see that for all $x \in (-\infty, \infty)$ one has $1 \le \psi(x) \le 4$ and $\psi'(x) \ge 0$. Since

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \int_{1}^{\infty} \frac{ds}{s} = \infty,$$

the condition (1.2) holds. Taking $\rho(t) = t$, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)q(s) - L_2 r(s)\rho'(s)\left(\frac{2}{s}\right)^\alpha\right] ds = \limsup_{t\to\infty} \int_{1}^{t} \left[s^3 + 1 - 2^{\alpha+2}\right] ds = \infty,$$

so (2.24) holds. Hence, every solution of (3.1) is oscillatory by Theorem 2.5.

Example 3.2. For $t \ge 1$, consider the differential equation

$$\left(\frac{1}{t^2}\frac{2+x^2(t)}{1+x^2(t)}\left|x'(t)\right|^{\alpha-1} x'(t)\right)' + q(t)\left|x(t)\right|^{\alpha-1} x(t)\left(1 + x^2(t) + \left(x'(t)\right)^2\right) = 0, \quad (3.2)$$

where $\alpha = 2$, and q be any continuous function satisfying $q(t) \ge 1$. It is clear that for all $x \in (-\infty, \infty)$ one has $1 \le \psi(x) \le 2$. Then,

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \int_{1}^{\infty} s ds = \infty$$

and

$$\varphi(t) = \left(\int_{t_0}^{t} r^{-1/\alpha}(s) ds\right)^{-1} = \left(\int_{1}^{t} s ds\right)^{-1} = \frac{2}{t^2 - 1}.$$

For $\rho(t) = \frac{t^3}{3} - t^2 + t$, we have

$$\begin{aligned}\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)q(s) - \frac{L_2^2}{L_1}\rho'(s)\varphi^\alpha(s)\right) ds &= \limsup_{t\to\infty} \int_{1}^{t} \left(\left(\frac{s^3}{3} - s^2 + s\right)q(s) - \frac{16}{(t+1)^2}\right) ds \\ &\ge \limsup_{t\to\infty} \int_{1}^{t} \left(\left(\frac{s^3}{3} - s^2 + s\right)q(s) - 4\right) ds = \infty.\end{aligned}$$

Consequently, all conditions of Theorem 2.1 are satisfied. Hence, Eq. (3.2) is oscillatory by Theorem 2.1.

Example 3.3. For $t \ge 1$, consider the differential equation

$$\left(t^5\left(4 + \frac{3^{x(t)} - 3^{-x(t)}}{3^{x(t)} + 3^{-x(t)}}\right)\left|x'(t)\right|^{\alpha-1} x'(t)\right)' + t^{19/2}\left|x(t)\right|^{\alpha-1} x(t)\left(1 + \frac{1}{1 + \left(x'(t)\right)^2}\right) = 0, \quad (3.3)$$

It is easy to see that $L_1 = 3$, $L_2 = 5$ and $\psi'(x) \ge 0$ for all $x \in (-\infty, \infty)$. Let $\alpha = 1/2$. Then,

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \int_{1}^{\infty} \frac{ds}{s^{10}} < \infty,$$

so (1.3) holds. To apply Theorem 2.6, it remains to show that conditions (2.13) and (2.24) hold. To see this, note that if $\rho(t)=\sqrt{t}$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s)-L_2 r(s)\rho'(s)\left(\frac{2}{s}\right)^{\alpha}\right]ds=\limsup_{t\to\infty}\int_{1}^{t}\left[s^{10}-\frac{5\sqrt{2}}{2}s^{4}\right]ds=\infty,$$

which implies that (2.24) holds. Since

$$\int_{t_0}^{\infty}\left(\frac{1}{r(z)}\int_{t_0}^{z}q(s)\chi^{\alpha}(s)ds\right)^{1/\alpha}dz=\int_{1}^{\infty}\left(\frac{1}{z^5}\int_{1}^{z}\frac{s^5}{3}ds\right)^{2}dz=\int_{1}^{\infty}\left(\frac{z^6-1}{18z^5}\right)^{2}dz=\infty,$$

condition (2.13) is satisfied. Hence, equation (3.3) is oscillatory by Theorem 2.6.

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Some properties of meromorphic function and its q-difference *

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Abstract

The purpose of this paper is to investigate the characteristic function of meromorphic function and its q-difference. We also obtain some results on its characteristic function, which may be regarded as q-difference analogues of Valiron-Mohon'ko theorem.

Key words: q-difference; characteristic function; meromorphic function.
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1 Introduction and Main Results

In what follows, the term "meromorphic" will always mean meromorphic in the complex plane C. Considering meromorphic function $f(z)$, we shall assume that reader is familiar with basic notions such as $m(r,f)$, $N(r,f)$, $T(r,f)$, etc. of Nevalinna theory, (see Hayman [11], Yang [19] and Yi and Yang [20]). For a meromorphic function f, we will use $S(r,f)$ to denote any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t} < \infty$, a meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a(z)) = S(r,f)$. In addition, the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}dt.$$

Throughout this paper, the set F of logarithmic density will be not necessarily the same at each occurrence.

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The following theorem is an important result in studying the value distribution of meromorphic function and its polynomial, which is usual regarded as Valiron-Mohon'ko theorem.

Theorem 1.1 *(Valiron-Mohon'ko) ([12, Page 29]). Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies that

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Recently, a number of papers (including [3, 4, 5, 9, 10, 13]) have focused on complex difference equations and difference analogues of Nevanlinna's theory. Correspondingly, there aremany papers focused on the value distribution of q-difference (or q-shift difference) polynomials and equations, such as [6, 8, 14, 15, 16, 17, 18, 23].

A q-difference polynomial of $f(z)$ is an expression of the form

$$P_q(z, f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\chi_\lambda} f(q^{\lambda, j} z)^{\mu_{\lambda, j}}, \tag{1}$$

where $q \in \mathbb{C} \setminus \{0, 1\}$ is complex constant, I is an index set, λ, j are positive integers and $\mu_{\lambda,j}$ are nonnegative integers. In what follows, we assume that the coefficients of q-difference polynomials are small functions. The maximal total degree of $P(z, f)$ in $f(z)$ and the q-difference of $f(z)$ is defined by

$$\deg_f P_q = \max_{\lambda \in I} \sum_{j=1}^{\chi_\lambda} \mu_{\lambda, j}. \tag{2}$$

Our first result of this paper is about the estimate of characteristic function of the following rational function in $f(z)$ and q-difference $f(qz)$ of the form

$$R_1(z, f) = \frac{P_q(z, f)}{d_1(z) f(q^s z) + d_0(z)}, \tag{3}$$

where s is an arbitrary integer, and $d_0(z)$ and $d_1(z)$ are small functions of $f(z)$ with $d_0(z) \not\equiv 0$ or $d_1(z) \not\equiv 0$. The first theorem is listed as follows.

Theorem 1.2 *Let $f(z)$ be a transcendental meromorphic function of zero order such that $N(r, f) = S(r, f)$. Suppose that $P_q(z, f) \not\equiv 0$ is a q-difference polynomial in $f(z)$ and that $R_1(z, f)$ is stated as in (3). Then*

$$T(r, R_1) \leq (\deg_f P_q) T(r, f) + S(r, f), \tag{4}$$

on a set of logarithmic density 1.

The second result is about the estimate of characteristic function of the following rational functions in f and q-difference $f(qz)$ of the form

$$R_2(z,f) = \frac{P_q(z,f)}{f(q^{s_1}z)\cdots f(q^{s_n}z)}, \tag{5}$$

where $s_1, s_2, \ldots, s_n$ are distinct integers.

Theorem 1.3 *Let $f(z)$ be a transcendental meromorphic function of zero order such that $N(r,f) = S(r,f)$. Suppose that $P_q(z,f) \not\equiv 0$ is a q-difference polynomial in $f(z)$ and that $R_2(z,f)$ is stated as in (5). Then*

$$T(r,R_2) \le \max\{\deg_f P_q, n\}T(r,f) + S(r,f), \tag{6}$$

on a set of logarithmic density 1.

As for the general rational function in $f(z)$ and q-difference $f(qz)$ of the form

$$R_3(z,f) = \frac{P_q(z,f)}{Q_q(z,f)}, \tag{7}$$

we obtain the following result

Theorem 1.4 *Let $f(z)$ be a transcendental meromorphic function of zero order such that $N(r,f) = S(r,f)$. Suppose that $P_q(z,f) \not\equiv 0$ is a q-difference polynomial in $f(z)$ and q-difference $f(qz)$, and that $R_3(z,f)$ is stated as in (7).*

(i) If $\deg_f P_q \ge \deg_f Q_q$ and $P_q(z,f)$ contain just one term of maximal total degree, then

$$T(r,R_3) \ge (\deg_f P_q - \deg_f Q_q)T(r,f) + S(r,f), \tag{8}$$

on a set of logarithmic density 1.

(ii) If $\deg_f P_q \le \deg_f Q_q$ and $Q_q(z,f)$ contain just one term of maximal total degree, then

$$T(r,R_3) \ge (\deg_f Q_q - \deg_f P_q)T(r,f) + S(r,f), \tag{9}$$

on a set of logarithmic density 1.

Theorem 1.5 *Let $f(z)$ be a transcendental meromorphic function of zero order such that $N(r,f) + N(r,\frac{1}{f}) = S(r,f)$. Suppose that $P_q(z,f) \not\equiv 0$ and $Q_q(z,f) \not\equiv 0$ are two q-difference polynomials in $f(z)$ and q-difference $f(qz)$, and that $R_3(z,f)$ is stated as in (7). Then*

$$T(r,R_3) \le \max\{\deg_f P_q, \deg_f Q_q\}T(r,f) + S(r,f), \tag{10}$$

on a set of logarithmic density 1.

2 Some Lemmas

The following theorem [1, Theorem 1.1] is an important result in studying the value distribution of q-difference of meromorphic function.

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Lemma 2.1 *([1, Theorem 1.1]). Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f),$$

on a set of logarithmic density 1.

Lemma 2.2 *([7, Page 36] and [21, Theorem 1.1 and Theorem 1.3]) Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero complex constant. Then*

$$T(r, f(qz)) = T(r, f(z)) + S(r, f), \quad N(r, f(qz)) = N(r, f) + S(r, f),$$

on a set of logarithmic density 1.

Lemma 2.3 *([2])*

$$M(r, f(q(z))) = M(|q|r, f), \quad N(r, f(qz)) = N(|q|r, f) + O(1),$$

and

$$T(r, f(qz)) = T(|q|r, f) + O(1)$$

hold for any meromorphic function f and any non-zero constant q.

Similarly, we also get the following result

Lemma 2.4 *Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero complex constant. Then*

$$N\left(r, \frac{1}{f(qz)}\right) \leq N\left(r, \frac{1}{f}\right) + S(r, f),$$

on a set of logarithmic density 1.

By using the same method as in [22], we can get the following lemma.

Lemma 2.5 *Let $f(z)$ be a transcendental meromorphic function of zero order such that $N(r, f) = S(r, f)$. Suppose that $P_q(z, f)$ is a difference polynomial in $f(z)$ and q-difference $f(qz)$, and $P_q(z, f)$ contains just one term of maximal total degree. Then*

$$T(r, P_q) = (\deg_f P_q)T(r, f) + S(r, f),$$

on a set of logarithmic density 1.

3 Proofs of Theorems

3.1 The proof of Theorem 1.2

Let $P_q(z, f)$ be stated as in (1). Set $\deg_f P_q = t$. We can rewrite $P_q(z, f)$ as the following form

$$P_q(z, f) = \sum_{i=0}^{t} h_i(z) f(z)^i, \tag{11}$$

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where for $i = 0, \ldots, t$,

$$h_i(z) = \sum_{\lambda \in I_i} a_\lambda(z) \prod_{j=1}^{\chi_\lambda} \left(\frac{f(q^{\lambda,j} z)}{f(z)} \right)^{\mu_{\lambda,j}}, \quad I_i = \{\lambda \in I | \sum_{j=1}^{\chi_\lambda} \mu_{\lambda,j}\}. \tag{12}$$

Since the coefficients $a_\lambda(z)$ of $P_q(z, f)$ are small functions of $f(z)$, then it follows by Lemma 2.1 that

$$m(r, h_i) \leq \sum_{\lambda \in I_i} m(r, a_\lambda(z)) + S(r, f) = S(r, f), \tag{13}$$

on a set of logarithmic density 1.

W.l.o.g, we may assume that $s = 0$ in (3). Otherwise, substituting $q^{-s}z$ for z, we get

$$R_1(q^{-s}z, f) = \frac{P_q(q^{-s}z, f)}{d_1(q^{-s}z)f(z) + d_0(q^{-s}z)}. \tag{14}$$

Since $f(z)$ is a transcendental meromorphic function of zero order and $q \neq 0$, then it follows from Lemmas 2.2-2.3 and (14) that

$$T(r, R_1(q^{-s}z, f)) = T(r, R_1(z, f)) + S(r, f),$$

on a set of logarithmic density 1.

Thus, we only discuss the following form

$$R_1(z, f) = \frac{P_q(z, f)}{d_1(z)f(z) + d_0(z)}. \tag{15}$$

We will consider two cases as follows.

Case 1. Assume that $d_1(z) \not\equiv 0$. W.o.l.g., we may assume that $d_1(z) = 1$. Thus, from (11), we can rewrite (15) as the following form

$$R_1(z, f) = \frac{h_t(z)f(z)^t + \cdots + h_1(z)f(z) + h_0(z)}{f(z) + d_0(z)}. \tag{16}$$

Then it follows from (16) that

$$\begin{aligned} R_1(z, f) &= h_t(z)f(z)^{t-1} + \frac{h^*_{t-1}(z)f(z)^{t-1} + \cdots + h_1(z)f(z) + h_0(z)}{f(z) + d_0(z)} \\ &= h_t(z)f(z)^{t-1} + h^*_{t-1}(z)f(z)^{t-2} + \frac{h^*_{t-2}(z)f(z)^{t-2} + \cdots + h_1(z)f(z) + h_0(z)}{f(z) + d_0(z)} \\ &\vdots \\ &= h_t(z)f(z)^{t-1} + h^*_{t-1}(z)f(z)^{t-2} + \cdots + h^*_2(z)f(z) + h^*_1(z) + \frac{h^*_0(z)}{f(z) + d_0(z)}, \end{aligned} \tag{17}$$

where $h^*_\iota(z) = h_\iota(z) - h_{\iota+1}(z)d_0(z)$ for $\iota = 0, 1, \ldots, t-1$. From (12) and the definitions of $h^*_\iota(z)$, it follows by Lemma 2.1 that

$$m(r, h^*_\iota(z)) = S(r, f), \quad \iota = 0, 1, \ldots, t-1, \tag{18}$$

on a set of logarithmic density 1.

If $t=1$, then $R_1(z,f)=h_1(z)f(z)+\frac{h_0^*(z)}{f(z)+d_0(z)}$. It follows from (13) and (18) that

$$m(r,R_1)\le m(r,f)+S(r,f), \tag{19}$$

on a set of logarithmic density 1.

If $t\ge 2$, then we can rewrite $R_1(z,f)$ as the form

$$R_1(z,f)=f(z)(h_t(z)f(z)^{t-2}+h_{t-1}^*(z)f(z)^{t-3}+\cdots+h_2^*(z))+h_1^*(z)+\frac{h_0^*(z)}{f(z)+d_0(z)}.$$

Thus, it follows from above equation that

$$\begin{aligned}m(r,R_1)\le m(r,f)&+m\left(r,h_t(z)f(z)^{t-2}+h_{t-1}^*(z)f(z)^{t-3}+\cdots+h_2^*(z)\right)\\&+m(r,h_1^*(z))+m\left(r,\frac{h_0^*(z)}{f(z)+d_0(z)}\right).\end{aligned}$$

By using the inductive argument, from (13) and (18), we have

$$m(r,R_1)\le(t-1)m(r,f)+m\left(r,\frac{1}{f(z)+d_0(z)}\right)+S(r,f), \tag{20}$$

on a set of logarithmic density 1.

Since

$$R_1(z,f)=\frac{P_q(z,f)}{f(z)+d_0(z)}=\frac{\sum_{\lambda\in I_i}a_\lambda(z)\prod_{j=1}^{\chi_\lambda}f(q^{\lambda,j}z)^{\mu_{\lambda,j}}}{f(z)+d_0(z)}. \tag{21}$$

Since $N(r,f)=S(r,f)$ and by Lemma 2.2, we have

$$N(r,R_1)=N\left(r,\frac{1}{f(z)+d_0(z)}\right)+S(r,f), \tag{22}$$

on a set of logarithmic density 1.

Hence, from (20),(22) and $d_0(z)$ is a small function of $f(z)$, we have

$$T(r,R_1)\le(t-1)m(r,f)+T\left(r,\frac{1}{f(z)+d_0(z)}\right)+S(r,f)\le tT(r,f)+S(r,f), \tag{23}$$

on a set of logarithmic density 1.

Case 2. Assume that $d_1(z)=0$. Since $d_0(z)$ and the coefficients of $P_q(z,f)$ are all small functions of $f(z)$, we may assume that $d_0(z)=1$. Thus, it follows by (11) that

$$R_1(z,f)=P_q(z,f)=h_t(z)f(z)^t+h_{t-1}(z)f(z)^{t-1}+\cdots+h_1(z)f(z)+h_0(z). \tag{24}$$

If $t=1$, then $R_1(z,f)=h_1(z)f(z)+h_0(z)$. Then it follows from (13) that

$$m(r,R_1)\le m(r,f)+S(r,f), \tag{25}$$

on a set of logarithmic density 1.

If $t \geq 2$, then we have

$$R_1(z,f) = f(z)(h_t(z)f(z)^{t-1} + h_{t-1}(z)f(z)^{t-2} + \cdots + h_1(z)) + h_0(z). \tag{26}$$

Thus it follows from (13) that

$$\begin{aligned} m(r,R_1) \leq\ & m(r,f) + m(r, h_t(z)f(z)^{t-1} + h_{t-1}(z)f(z)^{t-2} + \cdots + h_1(z)) \\ & + m(r, h_0(z)) + O(1). \end{aligned}$$

By using the inductive argument, we have

$$m(r,R_1) \leq tm(r,f) + S(r,f), \tag{27}$$

on a set of logarithmic density 1.

On the other hand, since $a_\lambda(z)$ are small functions of $f(z)$, then we have by $N(r,f) = S(r,f)$ and Lemma 2.2 that

$$N(r,R_1) \leq \sum_{\lambda \in I} \left(N(r, a_\lambda(z)) + \sum_{j=1}^{\chi_\lambda} \mu_{\lambda,j} N(r, f(q^{\mu_{\lambda,j}} z)) \right) + O(1) \leq S(r,f), \tag{28}$$

on a set of logarithmic density 1.

Thus, it follows from (25) and (28) that

$$T(r,R_1) \leq tT(r,f) + S(r,f),$$

on a set of logarithmic density 1.

Therefore, from Cases 1 and 2, we complete the proof of Theorem 1.2.

3.2 The proof of Theorem 1.3

Let $P_q(z,f)$ be stated as in (1) and $\deg_f P_q = t$. Then we can rewrite $R_2(z,f)$ as the form

$$R_2(z,f) = \frac{P_q(z,f)}{g(z)f(z)^n}, \tag{29}$$

where $g(z) = \frac{f(q^{s_1}z)\cdots f(q^{s_n}z)}{f(z)^n}$. Thus it follows by Lemma 2.1 that

$$m(r, \frac{1}{g}) = S(r,f), \tag{30}$$

on a set of logarithmic density 1.

If $t < n$, then from (11) and (29) we have

$$R_2(z,f) = \frac{\sum_{i=0}^{t} h_i(z)f(z)^i}{g(z)f(z)^n} = \sum_{i=0}^{t} \frac{h_i(z)}{g(z)} \left(\frac{1}{f(z)} \right)^{n-i}. \tag{31}$$

From (13) and (30), by using the same argument as in (25)-(27), we have

$$m(r,R_2) \leq nm(r, \frac{1}{f}) + S(r,f), \tag{32}$$

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on a set of logarithmic density 1.

If $t \geq n$, then from (11) and (29) we have

$$R_2(z,f) = \frac{\sum_{i=0}^{t} h_i(z)f(z)^i}{g(z)f(z)^n} = \sum_{i=n}^{t} \frac{h_i(z)}{g(z)} f(z)^{i-n} + \sum_{j=1}^{n} \frac{h_{n-j}(z)}{g(z)} \left(\frac{1}{f(z)}\right)^j. \tag{33}$$

From (13) and (30), by using the same argument as in (25)-(27), we have

$$m\left(r, \sum_{i=n}^{t} \frac{h_i(z)}{g(z)} f(z)^{i-n}\right) \leq (t-n)m(r,f) + S(r,f), \tag{34}$$

$$m\left(r, \sum_{j=1}^{n} \frac{h_{n-j}(z)}{g(z)} \left(\frac{1}{f(z)}\right)^j\right) \leq nm(r, \frac{1}{f}) + S(r,f), \tag{35}$$

on a set of logarithmic density 1.

On the other hand, since $N(r,f) = S(r,f)$ and $a_\lambda(z)$ are small functions of $f(z)$, from (1), (5) and by Lemma 2.2 and 2.4, we have

$$N(r, R_2) = N\left(r, \frac{1}{f(q^{s_1}z)\cdots f(q^{s_n}z)}\right) + S(r,f) \leq nN\left(r, \frac{1}{f}\right) + S(r,f), \tag{36}$$

on a set of logarithmic density 1.

Thus, it follows from (32)-(36) that

$$T(r, R_2) \leq tT(r,f) + S(r,f),$$

on a set of logarithmic density 1.

Therefore, we complete the proof of Theorem 1.3.

3.3 The proof of Theorem 1.4

(i) Supposed that $\deg_f P_q \geq \deg_f Q_q$ and $P_q(z,f)$ contain just one term of maximal total degree. Set $\deg_f P_q = t, \deg_f Q_q = l$. It follows by Lemma 2.5 that

$$T(r, P_q) = tT(r,f) + S(r,f), \tag{37}$$

on a set of logarithmic density 1. And by Theorem 1.2 we have

$$T(r, Q_q) \leq lT(r,f) + S(r,f), \tag{38}$$

on a set of logarithmic density 1. We can rewrite (7) as the form

$$P_q(z,f) = R_3(z,f)Q_q(z,f). \tag{39}$$

Thus it follows from (37)-(39) that

$$\begin{aligned} tT(r,f) + S(r,f) &= T(r, P_q(z,f)) = T(r, R_3(z,f)Q_q(z,f)) \\ &\leq T(r, R_3(z,f)) + T(r, Q_q(z,f)) \leq T(r, R_3(z,f)) + lT(r,f) + S(r,f), \end{aligned}$$

that is,

$$T(r,R_3) \geq (t-l)T(r,f)+S(r,f), \tag{40}$$

on a set of logarithmic density 1.

(ii) Suppose that $\deg_f P_q \leq \deg_f Q_q$ and $Q_q(z,f)$ contain just one term of maximal total degree. We can discuss $\frac{1}{R_3(z,f)}$. By using the same argument as in (i), we can obtain

$$T(r,R_3)=T(r,\frac{1}{R_3}) \geq (l-t)T(r,f)+S(r,f), \tag{41}$$

on a set of logarithmic density 1.

Thus, from (40) and (41), the proof of Theorem 1.4 is completed.

3.4 The proof of Theorem 1.5

Let $P_q(z,f)$ be stated as in (1) and $\deg_f P_q = t$. Similarly, let

$$Q_q(z,f)=\sum_{\nu\in J} b_\nu(z)\prod_{j=1}^{\sigma_\nu} f(q^{\nu,j}z)^{\mu_{\nu,j}},$$

and $\deg_f Q_q = l$. Similar to (11), we also get that

$$Q_q(z,f)=\sum_{k=0}^{l} g_k(z)f(z)^k, \tag{42}$$

where for $k=0,\ldots,l$,

$$g_k(z)=\sum_{\nu\in J_k} b_\nu(z)\prod_{j=1}^{\sigma_\nu}\left(\frac{f(q^{\nu,j}z)}{f(z)}\right)^{\mu_{\nu,j}}, \quad J_k=\left\{\nu\in J|\sum_{j=1}^{\sigma_\nu}\mu_{\nu,j}\right\}. \tag{43}$$

Thus, it follows from (7), (11) and (42) that

$$R_3(z,f)=\frac{\sum_{i=0}^{t} h_i(z)f(z)^i}{\sum_{k=0}^{l} g_k(z)f(z)^k}. \tag{44}$$

Similar to (13), we get

$$m(r,g_k)=S(r,f), \tag{45}$$

on a set of logarithmic density 1.

On the other hand, since $N(r,f)+N(r,\frac{1}{f})=S(r,f)$ and $a_\lambda(z)$ are small functions of $f(z)$, it follows by Lemma 2.4 that

$$\begin{aligned} N(r,h_i(z)) &\leq \sum_{\lambda\in I_i} N(r,a_\lambda(z))+\sum N\left(r,\frac{f(q^{\lambda,j}z)}{f(z)}\right)+O(1) \\ &\leq \sum_{\lambda\in I_i} N(r,a_\lambda(z))+N(r,f(q^{\lambda,j}z))+N(r,\frac{1}{f(z)})+S(r,f) \\ &\leq S(r,f), \quad for \quad i=0,1,\ldots,t, \end{aligned} \tag{46}$$

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on a set of logarithmic density 1.

Similarly, we have

$$N(r, g_k(z)) = S(r, f), \quad for \quad k = 0, 1, \ldots, l, \tag{47}$$

on a set of logarithmic density 1.

Hence, it follows from (13), (45)-(47) that for $i = 0, 1, \ldots, t; k = 0, 1, \ldots, l$,

$$T(r, h_i(z)) = S(r, f), \quad T(r, g_k(z)) = S(r, f), \tag{48}$$

on a set of logarithmic density 1. From (44), we are not affirm that $R_3(z, f)$ is an irreducible rational function in $f(z)$. Thus, by Theorem 1.1, we obtain

$$T(r, R_3) \leq \max\{t, l\}T(r, f) + S(r, f),$$

on a set of logarithmic density 1.

Therefore, the proof of Theorem 1.5 is completed.

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ISOMETRIC EQUIVALENCE OF WEIGHTED COMPOSITION OPERATORS ON THE BLOCH SPACE

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ABSTRACT. This paper characterized the isometric equivalence of weighted composition operators on the Bloch space $\mathcal{B}$ in the disk $\mathbb{D}$, and also studied isometric equivalence of differentiation composition operators on $\mathcal{B}$ and the space H^∞ of the bounded holomorphic functions in $\mathbb{D}$.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane, and $S(\mathbb{D})$ be the set of holomorphic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

Let $H^\infty(\mathbb{D})$ denote the space of bounded holomorphic functions f on the unit disk with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

We recall that the Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right| < \infty,$$

then $\|\cdot\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$, which is Möbius invariant.

It is well known that $\mathcal{B}$ is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

This space is called the little Bloch space.

For $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, the multiplication operator M_u is defined by

$$(M_u f)(z) = u(z)f(z)$$

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and the weighted composition operator $W_{u,\varphi}$ induced by φ and u is defined by

$$(W_{u,\varphi}f)(z) = u(z)f(\varphi(z))$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. If let $u \equiv 1$, then $W_{u,\varphi} = C_\varphi$, which is called a composition operator. If we let φ equal to the identity function, then $W_{u,\varphi} = M_u$. Thus, the weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator.

Let X and Y be two Banach spaces and T_1 and T_2 are bounded linear operators on X and Y respectively. We say that T_1 and T_2 are *isometrically equivalent* if there exists surjective isometries U_X and U_Y on X and Y respectively such that $U_XT_1 = T_2U_Y$. For $Y = X$, two operators T_1 and T_2 are said to be *similar* if there is a bounded invertible operator S on X such that $ST_2 = T_1S$. If S could be chosen to be an isometry as well, then T_1 and T_2 are said to be *isometrically isomorphic*. If X is a Hilbert space, then an isometric isomorphism on X is referred to as a unitary equivalence.

In [7], R. C. Wright investigated the isometric equivalence of composition operator for $X = Y = H^p(\mathbb{D})$ for $1 \le p < \infty$ and $p \ne 2$, and W. Hornor and J. E. Jamison studied the isometric equivalence composition operators on several important Banach spaces of analytic function spaces on the unit disk $\mathbb{D}$ in [4, 5]. In [6], J.E.Jamison studied isometric equivalence of composition operators on the $X = Y = \mathcal{B}$, where $\mathcal{B}$ is a Bloch space. He obtained that if two composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent on $\mathcal{B}$, then there is an automorphism φ such that $\varphi_1(\varphi(z)) = \varphi(\varphi_2(z))$; he also investigated the isometric equivalence problem of certain operators on some specific types of Banach spaces. In [3], Nadia J. Gal studied isometric equivalence of differentiation composition operator on some analytic function spaces.

Building on these foundations, we characterize isometric equivalence of weighted composition operators on the Bloch space $\mathcal{B}$ in $\mathbb{D}$, and also studied isometric equivalence of differentiation composition operators on $\mathcal{B}$ and the space H^∞ of the bounded holomorphic functions in $\mathbb{D}$.

2. Isometric equivalence of weighted composition operator on the Bloch space $\mathcal{B}$

We state a result on isometry of $\mathcal{B}$, which was obtained by Cima and Wogen in [9].

Lemma 1. *If S: $\mathcal{B} \to \mathcal{B}$ is a surjective isometry, then there is a conformal automorphism τ of $\mathbb{D}$ and a $\lambda \in \Gamma$ such that*

$$Sf = \lambda(f(\tau) - f(\tau(0))).$$

Theorem 2. *Suppose W_{u_1,φ_1} and W_{u_2,φ_2} are isometrically equivalent on $\mathcal{B}$ if and only if there exists a conformal automorphism τ of $\mathbb{D}$ such that $\tau(\varphi_1) = \varphi_2(\tau)$ and $u_1 = u_2 = c$, where c is a constant.*

Proof. Suppose first that W_{u_1,φ_1} and W_{u_2,φ_2} are isometrically equivalent on $\mathcal{B}$. Then there exists an invertible isometry S on $\mathcal{B}$ such that

$$W_{u_1,\varphi_1}Sf = SW_{u_2,\varphi_2}f$$

for any function $f \in \mathcal{B}$. It follows from Lemma 1 that

$$\lambda u_1\{f(\tau(\varphi_1)) - f(\tau(0))\} = \lambda\{u_2(\tau)f(\varphi_2(\tau)) - u_2(\tau(0))f(\varphi_2(\tau(0)))\}, \tag{1}$$

for any function $f \in \mathcal{B}$.

Let us now set special functions in the last displayed equation (1). Choosing $f = 1$ in (1), one has

$$0 = u_2(\tau) - u_2(\tau(0))$$

because τ is a conformal automorphism of the $\mathbb{D}$ so $u_2(z)$ is a constant function c.

Choosing $f = z$ in (1), we have

$$u_1\{\tau(\varphi_1) - \tau(0)\} = c\{\varphi_2(\tau) - \varphi_2(\tau(0))\}. \tag{2}$$

Setting $f = z^2$ in (1), one also has

$$u_1\{\tau(\varphi_1)^2 - \tau(0)^2\} = c\{\varphi_2(\tau)^2 - \varphi_2(\tau(0))^2\},$$

that is,

$$u_1\{\tau(\varphi_1) + \tau(0)\}\{\tau(\varphi_1) - \tau(0)\} = c\{\varphi_2(\tau) + \varphi_2(\tau(0))\}\{\varphi_2(\tau) - \varphi_2(\tau(0))\}. \tag{3}$$

From (2) and (3), we get

$$\tau(\varphi_1) + \tau(0) = \varphi_2(\tau) + \varphi_2(\tau(0)), \tag{4}$$

except the case φ_1 and φ_2 are constant functions.

Letting $f = z^3$ in (1), then

$$u_1\{\tau(\varphi_1)^3 - \tau(0)^3\} = c\{\varphi_2(\tau)^3 - \varphi_2(\tau(0))^3\}$$

that is,

$$\begin{aligned} &u_1\{\tau(\varphi_1) - \tau(0)\}\{\tau(\varphi_1)^2 + \tau(\varphi_1)\tau(0) + \tau(0)^2\} \\ &= c\{\varphi_2(\tau) - \varphi_2(\tau(0))\}\{\varphi_2(\tau)^2 + \varphi_2(\tau)\varphi_2(\tau(0)) + \varphi_2(\tau(0))^2\}. \end{aligned} \tag{5}$$

It follows from (2) and (5) that

$$\tau(\varphi_1)^2 + \tau(\varphi_1)\tau(0) + \tau(0)^2 = \varphi_2(\tau)^2 + \varphi_2(\tau)\varphi_2(\tau(0)) + \varphi_2(\tau(0))^2, \tag{6}$$

From (2),(4) and (6), we get

$$\tau(\varphi_1)\tau(0) = \varphi_2(\tau)\varphi_2(\tau(0)) \tag{7}$$

If $\tau(0) = \varphi_2(\tau(0)) = 0$, then $\tau(z) = \pm z$. The equation (4) implies that $\tau(\varphi_1) = \varphi_2(\tau)$.

If $\tau(0) = 0, \varphi_2(\tau(0)) \neq 0$, then from (7), we can get $\varphi_2(\tau)(z) = 0$ for all $z \in \mathbb{D}$, it is a contraction.

If $\tau(0) \neq 0, \varphi_2(\tau(0)) = 0$, then from (7), we can get $\tau(\varphi_1)(z) = 0$ for all $z \in \mathbb{D}$, it is a contraction.

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If $\tau(\varphi_1(0))\varphi_2(\tau(0)) \neq 0$, from (7) and (4), we can get $\tau(\varphi_1) = \varphi_2(\tau)$ also. From (2), we can get $u_1 = c$.

The converse is immediate. □

The surjective isometry of $\mathcal{B}_0$ has the same form as the surjective isometry of $\mathcal{B}$, so we have the following corollary.

Corollary 1. *Two weighted composition operators W_{u_1,φ_1} and W_{u_2,φ_2} are isometrically equivalent on $\mathcal{B}_0$ if and only if there exists a conformal automorphism τ of $\mathbb{D}$ such that $\tau(\varphi_1) = \varphi_2(\tau)$ and $u_1 = u_2 = c$, where c is a constant.*

3. Isometric equivalence of differentiation composition operator operators on $\mathcal{B}$ and H^∞.

Let D be the differentiation operator and $\varphi \in S(\mathbb{D})$, the product DC_φ is defined by

$$DC_\varphi(f) = (f \circ \varphi)' = f'(\varphi)\varphi'.$$

The following Lemma about the surjective isometry on the space $H^\infty(\mathbb{D})$ was presented by Forelli in [10].

Lemma 3. *An operator T is a linear isometry of $H^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{D})$ if and only if there is an unimodular complex number α and a conformal map τ of the disk such that*

$$Tf = \alpha f(\tau)$$

for every $f \in H^\infty(\mathbb{D})$.

Theorem 4. *Suppose φ_1 and φ_2 are two holomorphic self-maps of the unit disk $\mathbb{D}$. Then $DC_{\varphi_1}S = TDC_{\varphi_2}$, where S and T are surjective isometries on $\mathcal{B}$ and on H^∞ respectively, if and only if there exists a conformal automorphism τ of the the unit disk $\mathbb{D}$ such that*

$$\varphi_2(z) = \tau(\varphi_1(e^{i\theta}z))$$

Proof. Suppose first that $DC_{\varphi_1}S = TDC_{\varphi_2}$, so

$$DC_{\varphi_1}Sf = TDC_{\varphi_2}f$$

for every $f \in \mathcal{B}$.

From Lemma 1 and Lemma 3, it follows that

$$\lambda f'(\tau_1(\varphi_1))\tau_1'(\varphi_1)\varphi_1' = \alpha f'(\varphi_2(\tau_2))\varphi_2'(\tau_2) \tag{8}$$

where $\tau_i, i = 1, 2$ are conformal automorphisms of $\mathbb{D}$, λ and α are unimodular complex numbers.

Setting $f(z) = z$ in (8), then

$$\lambda\tau_1'(\varphi_1)\varphi_1' = \alpha\varphi_2'(\tau_2) \tag{9}$$

Choosing $f(z) = z^2$ in (8), we have

$$\lambda\tau_1'(\varphi_1)\varphi_1' \cdot \tau_1 \circ \varphi_1 = \alpha\varphi_2'(\tau_2) \cdot \varphi_2 \circ \tau_2 \tag{10}$$

From (9) and (10), we can get

$$\tau_1 \circ \varphi_1 = \varphi_2 \circ \tau_2 \tag{11}$$

except for the case φ_1 and φ_2 are constant functions.

It follows from (11) that

$$\tau_1'(\varphi_1)\varphi_1' = \varphi_2'(\tau_2)\tau_2' \tag{12}$$

From (10) and (12), we can get $\tau_2' = \frac{\lambda}{\alpha}$, so that $\tau_2(z) = e^{-i\theta}z$ for some $\theta \in \mathbb{R}$. Then using (10) again, we can get $\varphi_2(z) = \tau(\varphi_1(e^{i\theta}z))$.

The converse is immediate. □

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STABILITY PROBLEMS DERIVING FROM MIXED ADDITIVE AND QUADRATIC FUNCTIONAL EQUATIONS

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ABSTRACT. Let $n \geq 2$ be an integer. We obtain the solution of the following mixed additive and quadratic functional equation

$$(4-n)r^t f(\sum_{j=1}^{n} \frac{x_j}{r}) + \sum_{i=1}^{n} r^t f(\sum_{j=1}^{n} (-1)^{\delta_{i\,j}} \frac{x_j}{r}) = 2^t \sum_{i=1}^{n} f(x_i)\,,$$

where $t \in \{1,2\}$, $r \in \mathbb{R}$ ($r \neq \frac{n}{2}, 2$ and $r^2 \neq n$) and the function δ is a Kronecker delta, and we prove the stability in normed group by using shadowing property and the Hyers-Ulam-Rassias stability in Banach spaces.

1. Introduction

A major objective of the study of dynamical systems is to describe the eventual behavior of the orbits of a map or flow. Especially the investigation of the pseudo orbit is very important connection with the calculation of the orbits by a computer, because a computer can calculate only pseudo orbits. In general, if a dynamical system has the shadowing property(usually abbreviated POTP, pseudo orbit tracing property), then numerically obtained orbits reflect the real behaviour of trajectories of the system. In the case of dynamical systems, the POTP is fundamental property and also plays a role in the field with the useful properties arising from the various applications. Recently, Tabor obtained the some basic results concerning the shadowing property to the case of the stability problem; see [5, 6, 7].

The stability theory of functional equations started with the talk of S.M Ulam held at the Wisconsin University in 1940 as follows: Under what condition does there exist an additive mapping near an approximately additive mapping ? see [8].

In [2], D.H. Hyers gave the first affirmative answer to the question of Ulam's question. Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

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for all $\varepsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\| f(x) - a(x) \| \leq \varepsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

His result was extended and generalized in several directions. In particular, Th. M. Rassias [4] considered a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded.

Recently, Tabor proved the general stability result for functional equations in the case when the target space is a metric group (with some local divisibility condition); see [6].

A function f satisfying the functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

is called a Cauchy additive functional equation, and every solution of the Cauchy additive equation (1.1) is called a Cauchy additive function. Similarly, a function f satisfying the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

is called the quadratic functional equation, and every solution of the quadratic equation (1.2) is called a quadratic function.

In this paper, we develop the Tabor's theorems and interpret the concrete equations. Also, we consider the stability problem concerning functional equations; as follows; a mapping $f : X \to Y$ satisfies the following equation

$$(4-n)r^t f(\sum_{j=1}^{n} \frac{x_j}{r}) + \sum_{i=1}^{n} r^t f(\sum_{j=1}^{n} (-1)^{\delta_{ij}} \frac{x_j}{r}) = 2^t \sum_{i=1}^{n} f(x_i)\,,$$

where $t \in \{1, 2\}$, and the function δ is a Kronecker delta, then the odd mapping $f : X \to Y$ is additive when $t = 1$, and the even mapping $f : X \to Y$ is quadratic when $t = 2$, and we prove the stability in normed group(when $r = 1$ is fixed) by using the shadowing property and the Hyers-Ulam-Rassias stability of the functional equation above in Banach spaces.

Throughout this paper, we assume that $n \geq 2$ is an integer number and $r(r \neq \frac{n}{2}, 2$ and $r^2 \neq n)$ is a real number.

2. General Functional Equations

Theorem 2.1. *Let $t = 1$ be fixed and let X, Y be vector spaces. The given odd mapping $f : X \to Y$ defined by*

$$(4-n)rf(\sum_{j=1}^{n} \frac{x_j}{r}) + \sum_{i=1}^{n} rf(\sum_{j=1}^{n} (-1)^{\delta_{ij}} \frac{x_j}{r}) = 2 \sum_{i=1}^{n} f(x_i)\,,$$

for all $x_1, \cdots, x_n \in X$ *. Then* f *is a Cauchy additive mapping.*

Proof. By letting $x_k = 0\,(k = 1, \cdots, n)$, we have

$$4rf(0) = 2nf(0)\,.$$

Since $r \neq \frac{n}{2}$, $f(0) = 0$.

Letting $x_1 = x$, $x_2 = y$, and $x_k = 0\,(k = 3, \cdots, n)$, we have

$$\begin{aligned}&(4-n)rf(\frac{x+y}{r}) + rf(\frac{-x+y}{r}) + rf(\frac{x-y}{r}) + (n-2)rf(\frac{x+y}{r})\\ =\;& 2f(x) + 2f(y)\,.\end{aligned}$$

Since f is odd, we may write

$$2rf(\frac{x+y}{r}) = 2f(x) + 2f(y)\,,$$

for all $x, y \in X$. Now, by setting $y = 0$, we get

$$rf(\frac{x}{r}) = f(x)\,,$$

for all $x \in X$. Thus

$$f(x+y) = rf(\frac{x+y}{r}) = f(x) + f(y)\,,$$

for all $x, y \in X$, that is, f is Cauchy additive. □

The mapping $f : X \to Y$ as in the Theorem 2.1 is called a *generalized additive mapping of* $r-$*type*.

Theorem 2.2. *Let* $t = 2$ *be fixed and let* X, Y *be vector spaces. The given even mapping* $f : X \to Y$ *defined by*

$$(4-n)r^2 f(\sum_{j=1}^{n} \frac{x_j}{r}) + \sum_{i=1}^{n} r^2 f(\sum_{j=1}^{n} (-1)^{\delta_{ij}} \frac{x_j}{r}) = 4\sum_{i=1}^{n} f(x_i)\,,$$

for all $x_1, \cdots, x_n \in X$ *. Then* f *is a quadratic mapping.*

Proof. By letting $x_k = 0\,(k = 1, \cdots, n)$, we have

$$4r^2 f(0) = 4nf(0)\,.$$

Since $r^2 \neq n$, $f(0) = 0$.

Letting $x_1 = x$, $x_2 = y$, and $x_k = 0\,(k = 3, \cdots, n)$, we have

$$\begin{aligned}&(4-n)r^2 f(\frac{x+y}{r}) + r^2 f(\frac{-x+y}{r}) + r^2 f(\frac{x-y}{r}) + (n-2)r^2 f(\frac{x+y}{r})\\ =\;& 4f(x) + 4f(y)\,.\end{aligned}$$

Since f is even, we may write that

$$4r^2 f(\frac{x+y}{r}) = 4f(x) + 4f(y)\,,$$

for all $x, y \in X$. Now, by setting $y = 0$, we get

$$r^2 f(\frac{x}{r}) = f(x)\,,$$

for all $x \in X$. Thus

$$f(x+y)+f(x-y)=r^2 f(\frac{x+y}{r})+r^2 f(\frac{x-y}{r})=2f(x)+2f(y)\,,$$

for all $x, y \in X$, that is, f is quadratic. $\square$

The mapping $f : X \to Y$ as in the Theorem 2.2 is called a *generalized quadratic mapping of* $r-$*type.*

3. Main Results

In this section, we will investigate the stability of the given functional equation based on the ideas from the dynamical systems and Hyers-Ulam-Rassias stability. Before we proceed, we would like to introduce some basic definitions concerning shadowing and key concepts to establish the stability; see [6]. After then we will investigate the stability of the given functional equation based on the ideas from the dynamical systems.

Let us fix some notations which will be used throughout this section. We will fix $r = 1$, that is, we will investigate the generalized mappings of type one. Also, we denote $\mathbb{N}$ the set of all nonnegative integers, X a complete normed space, $B(x, s)$ the closed ball centered at x with radius s , and let ϕ be given.

Definition 3.1. *Let* $\delta \geq 0$. *We say that a sequence* $(x_k)_{k\in\mathbb{N}}$ *is a* $\delta-$*pseudoorbit* (for ϕ) *if*

$$d(x_{k+1}, \phi(x_k)) \leq \delta \quad \textit{for } k \in \mathbb{N}\,.$$

A $0-$*pseudoorbit is called an orbit.*

Definition 3.2. *Let* $s, R > 0$ *be given. We say that* $\phi : X \to X$ *is locally* $(s, R)-$*invertible at* $x_0 \in X$ *if*

$$\forall y \in B(\phi(x_0), R), \;\; \exists! x \in B(x_0, s) : \;\; \phi(x) = y\,.$$

If ϕ *is locally* $(s, R)-$*invertible at each* $x \in X$, *then we say that* ϕ *is locally* $(s, R)-$*invertible.*

For a locally $(s, R)-$invertible function ϕ , we define a function $\phi^{-1}_{x_0}$: $B(\phi(x_0), R) \to B(x_0, s)$ in such a way that $\phi^{-1}_{x_0}(y)$ denote the unique x from the above definition which satisfies $\phi(x) = y$. Moreover, we put

$$\mathrm{lip}_R \phi^{-1} := \sup_{x_0\in X} \mathrm{lip}(\phi^{-1}_{x_0})\,.$$

Theorem 3.3. [7] *Let* $l \in (0, 1)\,, R \in (0, \infty)$ *be fixed and let* $\phi : X \to X$ *be locally* $(lR, R)-$*invertible. We assume additionally that* $lip_R(\phi^{-1}) \leq l$. *Let* $\delta \leq (1-l)R$ *and let* $(x_k)_{k\in\mathbb{N}}$ *be an arbitrary* $\delta-$*pseudoorbit. Then there exists a unique* $y \in X$ *such that*

$$d(x_k, \phi^k(y)) \leq lR \quad \textit{for } k \in \mathbb{N}\,.$$

Moreover,

$$d(x_k, \phi^k(y)) \leq \frac{l\delta}{1-l} \quad \textit{for } k \in \mathbb{N}\,.$$

Let X be a semigroup. Then the mapping $||\cdot|| : X \to \mathbb{R}$ is called a (*semigroup*) *norm* if it satisfies the following properties:

(1) for all $x \in X$, $||x|| \geq 0$.
(2) for all $x \in X$, $k \in \mathbb{N}$, $||kx|| = k||x||$.
(3) for all $x, y \in X$, $||x|| + ||y|| \geq ||x * y||$ and also the equality holds when $x = y$, where $*$ is the binary operation on X.

Note $||\cdot||$ is called a *group norm* if X is a group with an identity e, and it additionally satisfies that $||x|| = 0$ if and only if $x = e$.

We say that $(X, *, ||\cdot||)$ is a *normed* (*semi*)*group* if X is a (semi)group with a norm $||\cdot||$. Now, given an Abelian group X and $n \in \mathbb{Z}$, we define the mapping $[n_X] : X \to X$ by the formula

$$[n_X](x) := nx \quad \text{for } x \in X\,.$$

Since X is a normed group, it is clear that $[n_X]$ is locally $(\frac{R}{n}, R)-$invertible at 0, and $\mathrm{lip}_R[n_X]^{-1} = 1/n$.

Also, we are going to need the following result. In recent years, Lee et al. [3] showed the next lemma by using Theorem 3.3.

Lemma 3.4. [6] *Let $l \in (0,1), R \in (0,\infty), \delta \in (0,(1-l)R), \varepsilon > 0, m \in \mathbb{N}, n \in \mathbb{Z}$. Let G be a commutative semigroup, X a complete Abelian metric group. We assume that the mapping $[n_X]$ is locally $(lR, R)-$invertible and that $lip_R([n_X]^{-1}) \leq l$. Let $f : G \to X$ satisfy the following two inequalities*

$$\begin{aligned} \Big|\Big|\sum_{i=1}^{N} a_i f(b_{i_1}x_1 + \cdots + b_{i_n}x_n)\Big|\Big| &\leq \varepsilon \quad \textit{for } x_1, \cdots, x_n \in G\,, \\ ||f(mx) - nf(x)|| &\leq \delta \quad \textit{for } x \in G\,, \end{aligned}$$

where all a_i are endomorphisms in X and b_{i_j} are endomorphisms in G. We assume additionally that there exists $K \in \{1, \cdots, N\}$ such that

$$\sum_{i=1}^{K} lip(a_i)\delta \leq (1-l)R\,, \quad \varepsilon + \sum_{i=K+1}^{N} lip(a_i)\frac{l\delta}{1-l} \leq lR\,. \tag{3.1}$$

Then there exists a unique function $F : G \to X$ such that

$$F(mx) = nF(x) \quad \textit{for } x \in G\,,$$

and

$$||f(x) - F(x)|| \leq \frac{l\delta}{1-l} \quad \textit{for } x \in G\,.$$

Moreover, F satisfies

$$\sum_{i=1}^{N} a_i F(b_{i_1}x_1 + \cdots + b_{i_n}x_n) = 0 \quad \textit{for } x_1, \cdots, x_n \in G\,.$$

Now, we are ready to prove our functional equations as follows: we will consider either $t = 1$ or $t = 2$. First, we will start with $t = 1$.

Theorem 3.5. *Let $R > 0$, let $n \geq 3$ be an integer, let G be an Abelian group, and let X be a complete normed Abelian group. Let $\varepsilon \leq \frac{n-2}{6n^2-13n-4}R$ be arbitrary and let $f : G \to X$ be a function such that*

$$||(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) - 2\sum_{i=1}^{n} f(x_i)|| \leq \varepsilon\,, \text{ and} \tag{3.2}$$

$$||f(2x) - 2f(x)|| \leq \frac{2n-5}{2n-4}\varepsilon\,,$$

for all $x_1, \cdots, x_n, x \in G$. Then there exists a unique function $F : G \to X$ such that

$$\begin{aligned} F(2x) &= 2F(x) \\ (4-n)F(\sum_{j=1}^{n} x_j) &+ \sum_{i=1}^{n} F(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 2\sum_{i=1}^{n} F(x_i) \\ ||F(x) - f(x)|| &\leq \frac{2n-5}{2(n-2)}\varepsilon \end{aligned}$$

for all $x \in G$.

Proof. By letting $x_1 = \cdots = x_n = 0$ in the equation (3.2), we have

$$||(2n-4)f(0)|| \leq \varepsilon\,,$$

that is, $||f(0)|| \leq \frac{\varepsilon}{2n-4}$. Now, by putting $x_1 = x_2 = x$, $x_k = 0\,(k = 3, \cdots, n)$ in (3.2),

$$||2f(2x) - 4f(x) - (2n-6)f(0)|| \leq \varepsilon\,.$$

Since $||f(0)|| \leq \frac{\varepsilon}{2n-4}$, we have $||f(2x) - 2f(x)|| \leq \frac{2n-5}{2n-4}\varepsilon$, for all $x \in G$. To apply Lemma 3.4 for the function f, we may let

$l = \frac{1}{2}\,, \delta = \frac{2n-5}{2n-4}\varepsilon\,,$
$a_1 = (4-n)id_X$, $a_2 = \cdots = a_{n+1} = id_X\,, a_{n+2} = \cdots = a_{2n+1} = -2id_X\,,$
$K = 1\,,$ and $N = 2n+1\,.$

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Then we have

$$\delta = \frac{2n-5}{2n-4}\varepsilon \le \frac{2n-5}{2(n-2)} \cdot \frac{n-2}{6n^2-13n-4}R \le \frac{1}{2}R = (1-l)R,$$

$$\mathrm{lip}(a_1)\delta \le (1-l)R\,, \text{ where } n = 3, 4\,,$$

$$\text{if } n \ge 5\,, \mathrm{lip}(a_1)\delta = (n-4)\cdot\frac{2n-5}{2(n-2)}\varepsilon$$

$$\le \frac{(n-4)(2n-5)}{2(n-2)} \cdot \frac{n-2}{6n^2-13n-4}R = \frac{(n-4)(2n-5)}{6n^2-13n-4}\cdot\frac{1}{2}R \le (1-l)R,$$

$$\text{and } \varepsilon + \mathrm{lip}\sum_{i=2}^{2n+1}(a_i)\frac{l\delta}{1-l} = \varepsilon + (1\cdot n + 2\cdot n)\delta \le \Big(1 + 3n\frac{2n-5}{2(n-2)}\Big)\varepsilon$$

$$\le \frac{6n^2-13n-4}{2(n-2)}\cdot\frac{n-2}{6n^2-13n-4}R = \frac{1}{2}R = lR\,.$$

Hence all conditions of Lemma 3.4 are satisfied, and thus we conclude that there exists a unique function $F : G \to X$ such that

$$F(2x) = 2F(x)\,, \tag{3.3}$$

$$(4-n)F(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} F(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 2\sum_{i=1}^{n} F(x_i)\,,$$

and also we have

$$||f(x) - F(x)|| \le \frac{l\delta}{1-l} = \frac{2n-5}{2(n-2)}\varepsilon\,, \quad \text{for all } x_1, \cdots, x_n, x \in G\,.$$

□

Corollary 3.6. *Let $R > 0$, let $n \ge 3$ be an integer, let G be an Abelian group, let X be a complete normed Abelian group, and let $f : G \to X$ be a function. Suppose that $[(2n-4)_X]$ is locally $(\frac{R}{2n-4}, R)-$invertible and $[2_X]$ is locally $(\frac{R}{2}, R)-$invertible. Then f satisfies the following equation*

$$(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 2\sum_{i=1}^{n} f(x_i)\,, \tag{3.4}$$

for all $x_1, \cdots, x_n \in G$ if and only if f is a Cauchy additive odd function.

The following Corollary follows from Theorem 3.5 and Corollary 3.6.

Corollary 3.7. *Let $R > 0$, let $n \ge 3$ be an integer, let G be an Abelian group, and let X be a complete normed Abelian group. Let $\varepsilon \le \frac{n-2}{6n^2-13n-4}R$ be arbitrary and let $f : G \to X$ be a function satisfying equation (3.2). Suppose that $[(2n-4)_X]$ is locally $(\frac{R}{2n-4}, R)-$invertible and $[2_X]$ is locally $(\frac{R}{2}, R)-$invertible. Then there exists a Cauchy additive odd function $F : G \to X$ such that*

$$||F(x) - f(x)|| \le \frac{2n-5}{2(n-2)}\varepsilon\,.$$

Now, we will investigate the stability of the case, where $t = 2$.

Theorem 3.8. *Let $R > 0$, let $n \geq 2$ be an integer, let G be an Abelian group, and let X be a complete normed Abelian group. Let $\varepsilon \leq \frac{3(n-1)}{20n^2-23n-12}R$ be arbitrary and let $f : G \to X$ be a function such that*

$$||(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) - 4\sum_{i=1}^{n} f(x_i)|| \leq \varepsilon\,, \tag{3.5}$$

$$||f(2x) - 4f(x)|| \leq \frac{4n-7}{12(n-1)}\varepsilon\,,$$

for all $x_1, \cdots, x_n, x \in G$. Then there exists a unique function $F : G \to X$ such that

$$F(2x) = 4F(x)\,, \tag{3.6}$$

$$(4-n)F(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} F(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 4\sum_{i=1}^{n} F(x_i)\,,$$

$$||F(x) - f(x)|| \leq \frac{4n-7}{12(n-1)}\varepsilon\,,$$

for all $x_1, \cdots, x_n, x \in G$.

Proof. By letting $x_1 = \cdots = x_n = 0$ in the equation (3.5), we have

$$||(4n-4)f(0)|| \leq \varepsilon\,,$$

that is, $||f(0)|| \leq \frac{\varepsilon}{4n-4}$. Now, by putting $x_1 = x_2 = x$, $x_k = 0\,(k = 3, \cdots, n)$ in (3.5),

$$||2f(2x) + (10-4n)f(0) - 8f(x)|| \leq \varepsilon\,.$$

Since $||f(0)|| \leq \frac{\varepsilon}{4n-4}$, we have $||f(2x) - 4f(x)|| \leq \frac{4n-7}{4n-4}\varepsilon$, for all $x \in G$. To apply Lemma 3.4 for the function f, we may let

$l = \dfrac{1}{4}\,, \delta = \dfrac{4n-7}{4n-4}\varepsilon\,,$

$a_1 = (4-n)id_X$, $a_2 = \cdots = a_{n+1} = id_X\,, a_{n+2} = \cdots = a_{2n+1} = -4id_X\,,$

$K = 1\,,$ and $N = 2n+1\,.$

Then we have

$$\delta \leq (1-l)R\,, \quad \sum_{i=1}^{K} \mathrm{lip}(a_i)\delta \leq (1-l)R\,, \quad \varepsilon + \sum_{i=K+1}^{N} \mathrm{lip}(a_i)\frac{l\delta}{1-l} \leq lR\,. \tag{3.7}$$

Hence all conditions of Lemma 3.4 are satisfied, and thus we conclude that there exists a unique function $F : G \to X$ such that

$$F(2x) = 4F(x)\,, \tag{3.8}$$

$$(4-n)F(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} F(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 4\sum_{i=1}^{n} F(x_i)\,,$$

and also we have

$$||f(x) - F(x)|| \leq \frac{l\delta}{1-l} = \frac{4n-7}{12(n-1)}\varepsilon\,, \quad \text{for all } x_1, \cdots, x_n, x \in G\,.$$

□

Corollary 3.9. *Let $R > 0$, let $n \geq 2$ be an integer, let G be an Abelian group, let X be a complete normed Abelian group, and let $f : G \to X$ be a function. Suppose that $[(4n-4)_X]$ is locally $(\frac{R}{4n-4}, R)-$invertible and $[2_X]$ is locally $(\frac{R}{2}, R)-$invertible. If f satisfies the following equation*

$$(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n}(-1)^{\delta_{ij}} x_j) = 4\sum_{i=1}^{n} f(x_i)\,, \tag{3.9}$$

for all $x_1, \cdots, x_n \in G$, then f is a quadratic even function.

The direct application of Theorem 3.8 and Corollary 3.9 yields the following Corollary.

Corollary 3.10. *Let $R > 0$, let $n \geq 2$ be an integer, let G be an Abelian group, and let X be a complete normed Abelian group. Let $\varepsilon \leq \frac{3(n-1)}{20n^2-23n-12}R$ be arbitrary and let $f : G \to X$ be a function satisfying equation (3.5). Suppose that $[(4n-4)_X]$ is locally $(\frac{R}{4n-4}, R)-$invertible and $[2_X]$ is locally $(\frac{R}{2}, R)-$invertible. Then there exists a quadratic even function $F : G \to X$ such that*

$$||F(x) - f(x)|| \leq \frac{4n-7}{12(n-1)}\varepsilon\,.$$

Now, we will investigate the stability for the given generalized functional equation of $r-$type. To study Hyers-Ulam-Rassias stability, throughout in the rest of this section, let X be a normed vector space with norm $||\cdot||$ and Y a Banach space with norm $\|\cdot\|$.

Let $t \in \{1, 2\}$. For the given mapping $f : X \to Y$, we define

(3.10)

$$D_t f(x_1, \cdots, x_n) := (4-n)r^t f(\sum_{j=1}^{n} \frac{x_j}{r}) + \sum_{i=1}^{n} r^t f(\sum_{j=1}^{n}(-1)^{\delta_{ij}} \frac{x_j}{r}) - 2^t \sum_{i=1}^{n} f(x_i)\,,$$

for all $x_1, \cdots, x_n \in X$.

Theorem 3.11. *Let $t = 1$, and let $f : X \to Y$ be an odd mapping for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} (\frac{r}{2})^j \phi((\frac{2}{r})^j x_1, \cdots, (\frac{2}{r})^j x_n) < \infty\,, \tag{3.11}$$

$$\| D_1 f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n)\,, \tag{3.12}$$

for all $x_1, \cdots, x_n \in X$. *Then there exists a unique generalized additive mapping* $L : X \to Y$ *such that*

$$\| f(x) - L(x) \| \leq \frac{1}{4}\widetilde{\phi}(x, x, 0, \cdots, 0)\,, \tag{3.13}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x$ and $x_k = 0\,(k = 3, \cdots, n)$ in (3.12), since $f(0) = 0$, we have

$$\begin{aligned} &\| (4-n)rf(\frac{2x}{r}) + (n-2)rf(\frac{2x}{r}) - 4f(x) \| \\ = &\| 2rf(\frac{2x}{r}) - 4f(x) \| \leq \phi(x, x, 0, \cdots, 0)\,, \end{aligned}$$

for all $x \in X$. Hence we have

$$\| f(x) - \frac{r}{2}f(\frac{2}{r}x) \| \leq \frac{1}{4}\phi(x, x, 0, \cdots, 0)\,, \tag{3.14}$$

for all $x \in X$. Then

$$\begin{aligned} \| (\frac{r}{2})^d f((\frac{2}{r})^d x) - (\frac{r}{2})^{d+1} f((\frac{2}{r})^{d+1} x) \| &= (\frac{r}{2})^d \| f((\frac{2}{r})^d x) - \frac{r}{2} f((\frac{2}{r})^{d+1} x) \| \\ &\leq \frac{1}{4}(\frac{r}{2})^d \phi((\frac{2}{r})^d x, (\frac{2}{r})^d x, 0, \cdots, 0)\,, \end{aligned}$$

for all $x \in X$ and all positive integer d . Hence we have
(3.15)

$$\| (\frac{r}{2})^s f((\frac{2}{r})^s x) - (\frac{r}{2})^d f((\frac{2}{r})^d x) \| \leq \frac{1}{4}\sum_{j=s}^{d-1}(\frac{r}{2})^j \phi((\frac{2}{r})^j x, (\frac{2}{r})^j x, 0, \cdots, 0)\,,$$

for all $x \in X$ and all positive integers s, d with $s < d$.

Hence we may conclude that the sequence $\{(\frac{r}{2})^s f((\frac{2}{r})^s x)\}$ is a Cauchy sequence. Since Y is complete, the sequence $\{(\frac{r}{2})^s f((\frac{2}{r})^s x)\}$ converges in Y for all $x \in X$. Thus we may define a mapping $L : X \to Y$ via

$$L(x) = \lim_{s\to\infty}(\frac{r}{2})^s f((\frac{2}{r})^s x)\,,$$

for all $x \in X$. Since $f(-x) = -f(x)$, we know that $L(-x) = -L(x)$, for all $x \in X$. Then

$$\begin{aligned} ||D_1 L(x_1, \cdots, x_n)|| &= \lim_{s\to\infty}(\frac{r}{2})^s ||D_1 f((\frac{2}{r})^s x_1, \cdots, (\frac{2}{r})^s x_n)|| \\ &\leq \lim_{s\to\infty}(\frac{r}{2})^s \phi((\frac{2}{r})^s x_1, \cdots, (\frac{2}{r})^s x_n)) = 0\,, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. The Theorem 2.1 induces that L is a generalized additive mapping of $r-$type. Also, by letting $s = 0$, and $d \to \infty$ in the equation (3.15), we have the equation (3.13).

Now, let $L' : X \to Y$ be another additive mapping satisfying the equation (3.13). Then for all $x \in X$

$$\begin{aligned} ||L(x) - L'(x)|| &= (\frac{r}{2})^s ||L((\frac{2}{r})^s x) - L'((\frac{2}{r})^s x)|| \\ \leq (\frac{r}{2})^s \Big(||L((\frac{2}{r})^s x) - f((\frac{2}{r})^s x)|| &+ ||L'((\frac{2}{r})^s x) - f((\frac{2}{r})^s x)|| \Big) \\ &\leq 2 \cdot \frac{1}{4} \cdot (\frac{r}{2})^s \, \widetilde{\phi}((\frac{2}{r})^s x, (\frac{2}{r})^s x, 0 \cdots, 0) \\ &= \frac{1}{2} \sum_{j=0}^{\infty} (\frac{r}{2})^{j+s} \phi((\frac{2}{r})^{j+s} x, (\frac{2}{r})^{j+s} x, 0, \cdots, 0) \\ &= \frac{1}{2} \sum_{j=s}^{\infty} (\frac{r}{2})^{j} \phi((\frac{2}{r})^{j} x, (\frac{2}{r})^{j} x, 0, \cdots, 0) \to 0 \,, \end{aligned}$$

as $s \to \infty$. Thus we may conclude that such a generalized additive mapping L is unique. □

Theorem 3.12. *Let $t = 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} (\frac{r}{2})^{2 \cdot j} \phi((\frac{2}{r})^j x_1, \cdots, (\frac{2}{r})^j x_n) < \infty \,, \tag{3.16}$$

$$\| D_2 f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n) \,, \tag{3.17}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{8} \widetilde{\phi}(x, x, 0, \cdots, 0) \,, \tag{3.18}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.11. □

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Complete Asymptotic Expansions for some Summation-Integral Type Operators with Different Weights[1]

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Abstract In this paper, we introduce a sequence of linear positive operators of summation-integral type with different weights and present a pointwise complete asymptotic expansion formula.

Key words Baskakov operators, Szász-Mirakjian operators, Bernstein-Durrmeyer operators, Stirling numbers, complete asymptotic expansion.

1. INTRODUCTION

In [1], a sequence of so-called summation-integral type operators which are linear and positive, is introduced as follows

$$\widehat{V}_{n,\alpha,\beta}(f;x) = n \sum_{\nu=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,\nu}(x) \int_0^{\infty} s_{n,\nu+\beta}(t) f(t)dt, \qquad x \in [0,\infty),$$

with parameters $\alpha \in R$, $\beta \in Z$ and

$$b_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-n-\nu},$$

$$s_{n,\nu}(t) = e^{-nt}\frac{(nt)^{\nu}}{\Gamma(\nu+1)}$$

are named Baskakov and Szász-Mirakjian basis functions, respectively. In particular, when the parameters $\alpha = 0, \beta = 0$, those operators degenerate into

$$\widehat{V}_n(f;x) = n \sum_{\nu=0}^{\infty} b_{n,\nu}(x) \int_0^{\infty} s_{n,\nu}(t) f(t)dt, \qquad x \in [0,\infty),$$

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which are called Baskakov-Szász-Durrmeyer operators (abbreviated as BSD operators).

Now we interchange $b_{n,\nu}$ and $s_{n,\nu}$, and modify $b_{n,\nu}$ into

$$v^*_{n,\nu}(x) = b_{n+1,\nu}(x) = \binom{n+\nu}{\nu} x^\nu (1+x)^{-n-\nu-1}$$

in order to make the operators $\widehat{V}_n(f;x)$ be well defined when $n = 1$. So we achieve a new sequence of summation-integral type operators, below

$$S^*_n(f;x) = \sum_{k=0}^{\infty} n \int_0^\infty f(t) v^*_{n,k}(t) dt \; s_{n,k}(x), \qquad x \in [0, \infty),$$

which are named Szász-Baskakov-Durrmeyer operators (abbreviated as SBD operators). This paper is dedicated to establish a complete asymptotic expansion formula in the following form for the SBD operators

$$S^*_n(f;x) \sim \sum_{k=0}^{\infty} c_k(f;x) n^{-k} \qquad (n \to \infty),$$

where all coefficients $c_k(f;x)$ of $n^{-k}(k = 0, 1, \cdots)$ are independent of n and calculated explicitly. For many sequences of positive linear operators, complete asymptotic expansions have been considered [2−12].

2. AUXILIARY CONCLUSIONS

Let

$$e_m(x) = x^m \ (m = 0, 1, \cdots); \quad \psi_x(t) = t - x$$

and define the falling factorials of x by

$$x^{\underline{j}} = x(x-1)\cdots(x-j+1) \ \ j \in N; \quad x^{\underline{0}} = 1.$$

Lemma 1 Suppose $n > m, x \in [0, \infty)$, then $S^*_n(e_0; x) = 1$ and

$$S^*_n(e_m; x) = \frac{1}{(n-1)^{\underline{m}}} \sum_{k=0}^{m} \binom{m}{k} m^{\underline{m-k}} (nx)^k, \quad m \in N.$$

Proof Obviously when $m = 0, S^*_n(e_m; x) = 1$. For $m \in N$, by

$$n \int_0^\infty v^*_{n,k}(t) t^m dt = \frac{(k+m)^{\underline{m}}}{(n-1)^{\underline{m}}}$$

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we have

$$\begin{aligned}
S_n^*(e_m;x) &= \sum_{k=0}^{\infty} \frac{(k+m)^{\underline{m}}}{(n-1)^{\underline{m}}} s_{n,k}(x) \\
&= \frac{e^{-nx}}{(n-1)^{\underline{m}})} \sum_{k=0}^{\infty} \frac{(k+m)^{\underline{m}}}{k!}(nx)^k \\
&= \frac{e^{-nx}}{(n-1)^{\underline{m}}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^m}{du^m} u^{k+m}|_{u=nx} \\
&= \frac{e^{-nx}}{(n-1)^{\underline{m}}} \frac{d^m}{du^m} \left(\sum_{k=0}^{\infty} \frac{1}{k!} u^{k+m} \right)_{u=nx} \\
&= \frac{e^{-nx}}{(n-1)^{\underline{m}}} \frac{d^m}{du^m} \left(u^m e^u\right)_{u=nx} \\
&= \frac{e^{-nx}}{(n-1)^{\underline{m}}} \sum_{k=0}^{m} \binom{m}{k} e^u \frac{d^{m-k}}{du^{m-k}} u^m|_{u=nx} \\
&= \frac{1}{(n-1)^{\underline{m}}} \sum_{k=0}^{m} \binom{m}{k} m^{\underline{m-k}} (nx)^k.
\end{aligned}$$

The proof is completed.

Let $S(k,j)$ denote the Stirling numbers of second kind defined by

$$x^k = \sum_{j=0}^{k} S(k,j) x^{\underline{j}} \ .$$

For $S(k,j)$, the following identity can be found in [13]

$$(1) \quad S(k,j) = \begin{cases} \dfrac{1}{j!} \displaystyle\sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k, & 0 \le j \le k \\ 0, & j > k. \end{cases}$$

Thus it follows that

$$(2) \qquad \sum_{j=0}^{n} (-1)^j \binom{n}{j} p_i(j) = 0 \quad (i = 0, 1, \cdots, n-1)$$

where $p_i(j)$ denotes a polynomial of degree at most i. In this paper, we will also need the following two identities

$$(3) \qquad \frac{1}{a(a+1)\cdots(a+n)} = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{a+j}, \quad a \in R \backslash \{-n, \cdots, -1, 0\}$$

$$(4) \qquad \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^k \right) \frac{x^k}{k!} = (e^x - 1)^m.$$

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Denoting

$$J(r,i,k)=\sum_{m=0}^{k}\binom{k}{m}\frac{1}{m!}\sum_{j=0}^{r+m-k}(-1)^j\binom{r+m-k}{j}j^{i+m},$$

we have the following results.

Lemma 2 $J(r,i,k)=0$, for $r\in N, i=0,1,\cdots[\frac{r-1}{2}]$.

Proof If $k<r-i$, then $i+m<r-k+m$, from (1) we can see that $J(r,i,k)=0$ holds obviously , therefore we only need to consider the case when $k\geq r-i$. By (4) we get

$$\sum_{j=0}^{r+m-k}(-1)^{r+m-k-j}\binom{r+m-k}{j}j^{i+m}=D_{i+m}(e^x-1)^{r+m-k}|_{x=0}$$

where

$$D_kf(x)=f^{(k)}(x),\quad k\in N;\qquad D_0f(x)=f(x).$$

Therefore $J(r,i,k)$ may be rewritten into

$$J(r,i,k)=\sum_{m=0}^{k}(-1)^{r+m-k}\binom{k}{m}\frac{1}{m!}D_{i+m}(e^x-1)^{r+m-k}|_{x=0}.$$

If writing

$$(e^x-1)^{r+m-k}=x^{r+m-k}\varphi^{r+m-k}(x)$$

wit $\varphi(x)=\sum_{l=0}^{\infty}\frac{x^l}{(l+1)!}$, then for $i=0,1,\cdots[\frac{r-1}{2}]$ and $k\geq r-i$, i.e. $i+m\geq r+m-k$, we have

$$D_{i+m}(e^x-1)^{r+m-k}|_{x=0}=\binom{i+m}{r+m-k}(r+m-k)!D_{i+k-r}\varphi^{r+m-k}(x)|_{x=0}.$$

Hence it follows that

$$J(r,i,k)=\frac{(-1)^r}{(i+k-r)!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}(i+m)^{\underline{i}}D_{i+k-r}\varphi^{r+m-k}(x)|_{x=0}.$$

By induction, we can prove that $D_{i+k-r}\varphi^{r+m-k}(x)|_{x=0}$ is a polynomial in m of degree $i+k-r$. In fact, by the formula of Faà di Bruno:

$$D_i\varphi^m(x)=\sum_{k=0}^{i}m^{\underline{k}}\varphi^{m-k}(x)\times\Big(Product\ of\ derivatives\ of\varphi(x)\Big),$$

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and $\varphi(0) = 1$, we have

$$D_{i+k-r}\varphi^{r+m-k}(x)|_{x=0} = \sum_{j=0}^{i+k-r} c_j (r+m-k)^{\underline{j}}$$

where c_j is independent of m which implies that

$$(i+m)^{\underline{i}} D_{i+k-r}\varphi^{r+m-k}(x)|_{x=0} = p_{2i+k-r}(m).$$

When $i < \frac{r}{2}$, i.e. $2i+k-r < k$, from (2) we can see $J(r,i,k) = 0$. The proof is completed.

Lemma 3 For arbitrary $r \in N$, $x \in [0,\infty)$, $n > r$, there holds

$$S_n^*(\psi_x^r; x) = (-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k) x^k$$

Proof As

$$S_n^*(\psi_x^r; x) = \sum_{m=0}^{r} \binom{r}{m} (-x)^{r-m} S_n^*(e_m; x),$$

by Lemma 1 we have

$$\begin{aligned} S_n^*(\psi_x^r; x) &= \sum_{m=0}^{r} \binom{r}{m} (-x)^{r-m} \frac{1}{(n-1)^{\underline{m}}} \sum_{k=0}^{m} \binom{m}{k} m^{\underline{m-k}} (nx)^k \\ &= \sum_{m=0}^{r} \binom{r}{m} (-x)^{r-m} \frac{(-1)^{m+1} m! n}{(-n+m)\cdots(-n+1)(-n)} \sum_{k=0}^{m} \binom{m}{k} \frac{(nx)^k}{k!}, \end{aligned}$$

as well as by using the identity (3) in the last formula above, we have

$$\begin{aligned} S_n^*(\psi_x^r; x) &= \sum_{m=0}^{r} \binom{r}{m} (-x)^{r-m} (-1)^{m+1} n \sum_{j=0}^{m} \frac{(-1)^j}{-n+j} \binom{m}{j} \sum_{k=0}^{m} \binom{m}{k} \frac{(nx)^k}{k!} \\ &= (-1)^r \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{i=0}^{\infty} (\frac{j}{n})^i \sum_{k=0}^{m} \binom{m}{k} \frac{(nx)^k}{k!} \\ &= (-1)^r \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{k=0}^{m} \binom{m}{k} \frac{(nx)^k}{k!} \sum_{i=0}^{\infty} \frac{1}{n^i} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^i. \end{aligned}$$

Using (2) we derive

$$S_n^*(\psi_x^r; x) \quad = (-1)^r \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{k=0}^{m} \binom{m}{k} \frac{(nx)^k}{k!} \sum_{i=m}^{\infty} \frac{1}{n^i} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^i$$

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$$
\begin{aligned}
&= (-1)^r \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{k=0}^{m} \binom{m}{k} \frac{x^k}{k!} \sum_{i=m-k}^{\infty} \frac{1}{n^i} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{i+k} \\
&= (-1)^r \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{i=0}^{\infty} \frac{1}{n^i} \sum_{k=max(0,m-i)}^{m} \binom{m}{k} \frac{x^k}{k!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{i+k} \\
&= (-1)^r \sum_{i=0}^{r} \frac{1}{n^i} \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{k=max(0,m-i)}^{m} \binom{m}{k} \frac{x^k}{k!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{i+k} \\
&= (-x)^r \sum_{m=0}^{r} \binom{r}{m} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^m \\
&\quad +(-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{m=0}^{r} \binom{r}{m} x^{r-m} \sum_{k=max(0,m-i)}^{m} \binom{m}{k} \frac{x^k}{k!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{i+k} \\
&=: I_1 + I_2
\end{aligned}
$$

From (4) we have

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} j^m = \frac{d^m}{dx^m} (1-e^x)^m |_{x=0} = (-1)^m m!.
$$

Thus

$$
I_1 = (-x)^r \sum_{m=0}^{r} \binom{r}{m} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^m = (-x)^r \sum_{m=0}^{r} \binom{r}{m} (-1)^m = 0.
$$

Hence we obtain that

$$
\begin{aligned}
S_n^*(\psi_x^r; x) &= (-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{m=0}^{r} \binom{r}{m} \sum_{k=0}^{min(i,m)} \binom{m}{m-k} \frac{x^{r-k}}{(m-k)!} \\
&\quad \times \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{i+m-k} \\
&= (-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{k=0}^{min(i,r)} \binom{r}{k} x^{r-k} \sum_{m=0}^{r-k} \binom{r-k}{m} \frac{1}{m!} \\
&\quad \times \sum_{j=0}^{m+k} (-1)^j \binom{m+k}{j} j^{i+m} \\
&= (-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{max(0,r-i)}^{r} \binom{r}{k} x^k \sum_{m=0}^{k} \binom{k}{m} \frac{1}{m!} \\
&\quad \times \sum_{j=0}^{r+m-k} (-1)^j \binom{r+m-k}{j} j^{i+m} \\
&= (-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} x^k J(r,i,k).
\end{aligned}
$$

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That is the proof.

Remark By Lemma 2 we easily get that

$$S_n^*(\psi_x^r;x) = O(n^{-[\frac{r+1}{2}]}).$$

For $q \in N, x \in (0,\infty)$, let $f \in K[q;x]$ be the class of functions which are from $W_\gamma(0,\infty)$ (locally integrable and $f(t) = O(t^\gamma)(t \to +\infty)$)which q times differentiable at x, then we have the following theorems of approximation.

Lemma 4 [14] Let $q \in N, x \in I, A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators such that

$$A_n(\psi_x^s;x) = O(n^{-[\frac{s+1}{2}]}) \quad (n \to \infty) \quad (s = 0,1,\cdots,2q+2),$$

then for arbitrary $f \in K[2q;x]$, there holds

$$A_n(f;x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n(\psi_x^s;x) + o(n^{-q}) \quad (n \to \infty).$$

In particular, if $f^{(2q+2)}(x)$ exists, then the $o(n^{-q})$ can be replaced by $O(n^{-q-1})$.

With similar argument as in [1], we have the following theorem of localization.

Lemma 5 (Localization) For $f \in W_\gamma(0,\infty)$ and vanish in a neighborhood of x, then for arbitrary $q \in N$, there holds that

$$S_n^*(f;x) = O(n^{-q}) \quad (n \to \infty).$$

MAIN RESULTS

For the SBD operators, we establish a pointwise complete asymptotic expansion formula as follows.

Theorem If $q \in N, x \in (0,\infty)$, then for $f \in K[2q;x]$ there holds

$$S_n^*(f;x) = f(x) + \sum_{i=1}^{q} c_i(f;x)n^{-i} + o(n^{-q}) \quad (n \to \infty)$$

where

$$c_i(f;x) = \sum_{r=1}^{2i} (-1)^r \frac{f^{(r)}(x)}{r!} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k)x^k,$$

and $J(r,i,k)$ defined as before Lemma 2.

Proof By Lemma 5, without loss of generality we may assume that $f \in K[2q;x]$ and be bounded on $(0,\infty)$. From Lemma 2, 3 and 4, we have

$$\begin{aligned} S_n^*(f;x) &= \sum_{r=0}^{2q} \frac{f^{(r)}(x)}{r!} S_n^*(\psi_x^r;x) + o(n^{-q}) \quad (n\to\infty) \\ &= f(x) + \sum_{r=1}^{2q} \frac{f^{(r)}(x)}{r!}(-1)^r \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k)x^k \\ &\quad +o(n^{-q}) \quad (n\to\infty) \\ &= f(x) + \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{r=1}^{2q} \frac{f^{(r)}(x)}{r!}(-1)^r \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k)x^k \\ &\quad +o(n^{-q}) \quad (n\to\infty) \\ &= f(x) + \sum_{i=1}^{q} \frac{1}{n^i} \sum_{r=1}^{2q} (-1)^r \frac{f^{(r)}(x)}{r!} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k)x^k \\ &\quad +o(n^{-q}) \quad (n\to\infty), \end{aligned}$$

denoting

$$c_i(f;x) = \sum_{r=1}^{2q} (-1)^r \frac{f^{(r)}(x)}{r!} \sum_{k=max(0,r-i)}^{r} \binom{r}{k} J(r,i,k)x^k.$$

That is the proof.

Remark 1 If $f \in \bigcap_{q=1}^{\infty} K[q;x]$, then the theorem yields the asymptotic expansion

$$S_n^*(f;x) \sim f(x) + \sum_{i=1}^{\infty} c_i(f;x)n^{-i} \quad (n\to\infty)$$

Remark 2 We present the $c_i(f;x)$ as $i=1,2$ below

$$\begin{aligned} c_1(f;x) &= f'(x) + \frac{x(x+2)}{2!} f''(x) \\ c_2(f;x) &= \frac{x+1}{1\cdot 1!} f'(x) + \frac{5x^2+10x+2}{2!} f''(x) \\ &\quad + \frac{23x^3+63x^2+36x}{3\cdot 3!} f'''(x) + \frac{3x^4+174x^3+12x^2}{4!} f^{(4)}(x). \end{aligned}$$

Let $q=1$ in Theorem above, we immediately get the following asymptotic expansion formula of Voronovskaja's type.

Corollary For $x \in (0,\infty)$ if $f \in K[2;x]$, then

$$\lim_{n\to\infty} n(S_n^*(f;x) - f(x)) = c_1(f;x)$$

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ON FOURTH-ORDER ITERATIVE METHODS FOR MULTIPLE ROOTS OF NONLINEAR EQUATIONS WITH HIGH EFFICIENCY

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ABSTRACT. Focusing on the order properties of convergent sequences, we construct two models of multi-step iterative methods for multiple roots. We prove that it is impossible for one to achieve the highest-existing efficiency index of fourth order and three function evaluations per step. Using the other model, which has been suggested by Zhou et al., we derive 53 unique methods of rational forms with the highest efficiency index, those including all known methods of the same efficiency index. Numerical comparisons are done to compare the performances of the methods, and for those with the best results, explicit formulae are given. The methods thus presented are superior in efficiency when compared with any existing methods.

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1. INTRODUCTION

Solving nonlinear equations numerically is an important topic in numerical analysis. Developed from the well-known Newton's method, there have been numerous efforts to accelerate the order of convergence and thus improve the computational efficiency of iterative methods. An iterative method is a recursive relation of a sequence converging to the root.

Equations with multiple roots need to be treated with different methods. Newton's method modified for multiple roots to obtain the root α with multiplicity m of a nonlinear equation $f(x) = 0$ is

$$x_{n+1} = x_n - mu_n, \tag{1}$$

with $u_n = f(x_n)/f'(x_n)$. The method (1) is of quadratic order of convergence, and requires two function evaluations, for $f(x_n)$ and $f'(x_n)$, each step.

An efficiency index of an iterative method is defined by $p^{1/d}$, where p is the order of convergence and d is the number of function evaluations per step. (See [1]) Considering this, Newton's method (1) acquires an efficiency index of $2^{1/2} = 1.414$.

As is Newton's method (1), a majority of iterative methods is of form

$$x_{n+1} = x_n - g(x_n), \tag{2}$$

with g an iterative function that varies with f. When the iterative function $g(x_n)$ consists only of $f(x_n)$ and their derivatives, then the method is referred to as a single-step iterative method.

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Single-step iterative methods of cubic convergence, requiring three function evaluations per step, were actively developed in [2] through [6]. Their efficiency index, $3^{1/3} = 1.442$, is higher than that of Newton's method (1), which proves that these methods are more efficient than Newton's method (1).

Some iterative methods are referred to as multi-step methods. Recently, a multi-step method that achieves a fourth order of convergence, yet requires only three function evaluations per step, has been developed by Li and Cheng in [7] as follows:

$$\begin{cases} y_n = x_n - \frac{2m}{m+2} u_n, \\ x_{n+1} = x_n - \frac{\frac{1}{2}m(m-2)(\frac{m}{m+2})^{-m} f'(y_n) - \frac{m^2}{2} f'(x_n)}{f'(x_n) - (\frac{m}{m+2})^{-m} f'(y_n)} u_n. \end{cases} \tag{3}$$

This method achieves an efficiency index of $4^{1/3} = 1.587$, exceeding those of any previous methods. Different methods of the same efficiency index were followed by Li et al. in [8], and by Sharma et al. in [9],

$$\begin{cases} y_n = x_n - \frac{2m}{m+2} u_n, \\ x_{n+1} = x_n - a_3 \frac{f(x_n)}{f'(y_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{cases} \tag{4}$$

where

$$a_3 = -\frac{1}{2} \frac{(\frac{m}{m+2})^m m(m^4 + 4m^3 - 16m - 16)}{m^3 - 4m + 8}, \tag{5}$$

$$b_1 = -\frac{(m^3 - 4m + 8)^2}{m(m^4 + 4m^3 - 4m^2 - 16m + 16)(m^2 + 2m - 4)}, \tag{6}$$

$$b_2 = \frac{m^2(m^3 - 4m + 8)}{(\frac{m}{m+2})^m (m^4 + 4m^3 - 4m^2 - 16m + 16)(m^2 + 2m - 4)}, \tag{7}$$

and

$$\begin{cases} y_n = x_n - \frac{2m}{m+2} u_n, \\ x_{n+1} = x_n - \big(a_1 + a_2 \frac{f'(x_n)}{f'(y_n)} + a_3 \big(\frac{f'(x_n)}{f'(y_n)}\big)^2\big) u_n, \end{cases} \tag{8}$$

where

$$a_1 = \frac{m}{8}(m^3 - 4m + 8), \tag{9}$$

$$a_2 = -\frac{m}{4}(m-1)(m+2)^2 \Big(\frac{m}{m+2}\Big)^m, \tag{10}$$

$$a_3 = \frac{m}{8}(m+2)^3 \Big(\frac{m}{m+2}\Big)^{2m}, \tag{11}$$

respectively.

Zhou et al. in [10], suggested to construct a multi-step iterative method as

$$\begin{cases} y_n = x_n - t u_n, \\ x_{n+1} = x_n - Q\big(\frac{f'(y_n)}{f'(x_n)}\big) u_n, \end{cases} \tag{12}$$

with t as a parameter and $Q \in C^2(\mathbb{R})$. The condition for (12) to achieve the fourth order of convergence is, according to [10],

$$t = \frac{2m}{m+2}, \tag{13}$$

and

$$\begin{cases} Q(u) = m, \\ Q'(u) = -\frac{1}{4}m^{3-m}(m+2)^m, \\ Q''(u) = \frac{1}{4}m^4(\frac{m}{m+2})^{-2m}, \end{cases} \tag{14}$$

where $u = (\frac{m}{m+2})^{m-1}$.

Five methods were derived, each of which corresponded to the five simple types of Q. Three of them were identical to (3), (4), (8), respectively, and the other two were new,

$$\begin{cases} y_n &= x_n - \frac{2m}{m+2}u_n, \\ x_{n+1} &= x_n - \frac{m}{8}\Big(m^3\big(\frac{m+2}{m}\big)^{2m}\big(\frac{f'(y_n)}{f'(x_n)}\big)^2 - 2m^2(m+3)\big(\frac{m+2}{m}\big)^m\frac{f'(y_n)}{f'(x_n)} \\ & \quad +(m^3+6m^2+8m+8)\Big)u_n, \end{cases} \tag{15}$$

and

$$\begin{cases} y_n &= x_n - \frac{2m}{2+m}u_n, \\ x_{n+1} &= x_n - \Big(\frac{m^4}{8}\big(\frac{m+2}{m}\big)^m\frac{f'(y_n)}{f'(x_n)} - \frac{m(m+2)^3}{8}\big(\frac{m}{m+2}\big)^m\frac{f'(x_n)}{f'(y_n)} \\ & \quad +\frac{1}{4}m(m^3+3m^2+2m-4)\Big)u_n. \end{cases} \tag{16}$$

Recently, Sharma et al. in [11] constructed a one-parameter family of the fourth-order methods.

$$\begin{cases} y_n &= x_n - \theta u_n, \\ x_{n+1} &= x_n - [\delta + \frac{M_n}{2\theta}(\gamma + \frac{\beta M_n}{2\theta - \alpha M_n})]u_n, \end{cases} \tag{17}$$

with $M_n = 1 - f'(y_n)/f'(x_n)$, $\theta = 2m/(m+2)$, and α, β, γ expressed with free parameters δ and m.

It is interesting that, all existing iterative methods for multiple roots with three function evaluations and fourth order of convergence, even family (17), follow Zhou's model (12). In Section 2, we consider other possibilities of multi-step iterative methods, and construct a model similar to (12). We prove that achieving an efficiency as high as the one obtained in (12) is impossible. Then in Section 3, we employ (12) with a more general rational function Q to derive 53 different iterative methods, 48 of them newly derived. In Section 4, we make numerical comparisons between the methods to figure out the best and the most efficient iterative methods among the methods that exist in the present. We give the explicit formulae for the best methods.

2. Exploration for another model

Let α be a root of $f(x) = 0$ with multiplicity m, in other words, $f^{(i)}(\alpha) = 0$ for a non-negative integer $i < m$, and $f^{(m)}(\alpha) \neq 0$. Define $\{e_n\}$ as $e_n = x_n - \alpha$, so that

$$x_n = \alpha + e_n. \tag{18}$$

The main characteristic that distinguishes multi-step methods from single-step methods is the use of other sequences such as $\{y_n\}$ or $\{\eta_n\}$, producing two or more sequences that depend on each other to converge into the desired root. However, if $\{y_n\}$ is to be used along with $\{x_n\}$ for a multi-step method,

$$y_n = \alpha + O(e_n) \tag{19}$$

should be satisfied with the big O notation.

Now it is straightforward to assume that

$$\begin{cases} y_n = x_n - tu_n \\ x_{n+1} = x_n - su_n \end{cases} \tag{20}$$

with t, s constants.

What makes the multi-step scheme so complicated is that the constant s may include either $f(y_n)/f(x_n)$ or $f'(y_n)/f'(x_n)$. The reason for this complication comes from Taylor's expansions for f and f' about α, giving

$$\frac{f(y_n)}{f(x_n)} = O(e_n^0), \quad \frac{f'(y_n)}{f'(x_n)} = O(e_n^0), \tag{21}$$

allowing us to treat them in iterative methods as if they are constants, just as m in front of u_n is in Newton's method (1). This justifies Zhou's model (12), where s in (20) is designed to freely include $f'(y_n)/f'(x_n)$.

Thus we construct two models for the multi-step methods, namely,

$$\begin{cases} y_n = x_n - tu_n, \\ x_{n+1} = x_n - Q\left(\frac{f(y_n)}{f(x_n)}\right)u_n, \end{cases} \tag{22}$$

and

$$\begin{cases} y_n = x_n - tu_n, \\ x_{n+1} = x_n - Q\left(\frac{f'(y_n)}{f'(x_n)}\right)u_n, \end{cases} \tag{23}$$

with t as a parameter and $Q \in C^2(\mathbb{R})$, the latter of which is the same as (12).

Remark 1. *Usage of a third sequence η_n will require a fourth function evaluation for each step. Therefore, for the highest efficiency index of $4^{1/3} = 1.587$, we consider those including only y_n besides x_n.*

Theorem 1. *Multi-step methods of form (22) cannot achieve a fourth order of convergence. That is, no combination of t and Q makes an error equation of (22) be $e_{n+1} = O(e_n^4)$.*

Proof. The proof follows the procedure represented in [10] quite much. Let

$$c_n = \frac{1}{(m+n)!}\frac{f^{(m+n)}(\alpha)}{f^{(m)}(\alpha)}, \tag{24}$$

and

$$e_n = x_n - \alpha, \quad \epsilon_n = y_n - \alpha = x_n - tu_n - \alpha = e_n - tu_n. \tag{25}$$

Taylor's expansions of f and f' about α,

$$f(x_n) = f^{(m)}(\alpha)(c_0e_n^m + c_1e_n^{m+1} + c_2e_n^{m+2} + \cdots), \tag{26}$$

$$f'(x_n) = f^{(m)}(\alpha)\{mc_0e_n^{m-1} + (m+1)c_1e_n^m + (m+2)c_2e_n^{m+1} + \cdots\}, \tag{27}$$

yield

$$\epsilon_n = (1-\frac{t}{m})e_n + \frac{t}{m^2}\frac{c_1}{c_0}e_n^2 + (\frac{2t}{m^2}\frac{c_2}{c_0} - \frac{(m+1)t}{m^3}\frac{c_1^2}{c_0^2})e_n^3 + O(e_n^4), \tag{28}$$

Then, by

$$f(y_n) = f^{(m)}(\alpha)(c_0\epsilon_n^m + c_1\epsilon_n^{m+1} + c_2\epsilon_n^{m+2} + \cdots), \tag{29}$$

we have,

$$\frac{f(y_n)}{f(x_n)} = (1-\frac{t}{m})^m + \frac{t^2}{m^2}(1-\frac{t}{m})^{m-1}\frac{c_1}{c_0}e_n + \Big(h_1\frac{c_2}{c_0} + h_2\frac{c_1^2}{c_0^2}\Big)t^2(1-\frac{t}{m})^m e_n^2 + O(e_n^3), \tag{30}$$

with

$$\begin{cases} h_1 = \dfrac{3m-t}{m^2(m-t)} \\ h_2 = \dfrac{-3m^2+2t+m(-3+4t)}{2m^2(m-t)^2}. \end{cases} \tag{31}$$

Letting

$$\frac{f(y_n)}{f(x_n)} = u + v, \quad u = (1 - \frac{t}{m})^m, \tag{32}$$

we have

$$Q\left(\frac{f(y_n)}{f(x_n)}\right) = Q(u) + Q'(u)v + \frac{Q''(u)v^2}{2!} + O(e_n^3), \tag{33}$$

and thus from (22),

$$e_{n+1} = e_n - Q\left(\frac{f(y_n)}{f(x_n)}\right)u_n, \tag{34}$$

the error equation is derived,

$$\begin{aligned} e_{n+1} = (1 - \frac{Q(u)}{m})e_n &+ \left(\frac{1}{m^2}Q(u) - k_1Q'(u)\right)\frac{c_1}{c_0}e_n^2 \\ &+ \left(\left(\frac{2}{m^2}Q(u) - k_2Q'(u)\right)\frac{c_2}{c_0} + \left(\frac{m+1}{m^3}Q(u) + k_3Q'(u) + k_4Q''(u)\right)\frac{c_1^2}{c_0^2}\right)e_n^3 \\ &+ O(e_n^4), \end{aligned} \tag{35}$$

with

$$\begin{cases} k_1 = \frac{t^2}{m^3}(1-\frac{t}{m})^{m-1} \\ k_2 = \frac{t^2}{m^5}(3m^2-4mt+t^2)(1-\frac{t}{m})^{m-2} \\ k_3 = \frac{t^2}{2m^5}(-5m-3m^2+4t+4mt)(1-\frac{t}{m})^{m-2} \\ k_4 = \frac{t^4}{2m^5}(1-\frac{t}{m})^{2m-2}. \end{cases} \tag{36}$$

If this error equation is to be $e_{n+1} = O(e_n^4)$ regardless of c_i's as assumed, it is necessary from the first three terms, though not sufficient, that

$$\begin{cases} Q(u) = m \\ k_1Q'(u) = \frac{Q(u)}{m^2} \\ k_2Q'(u) = \frac{2Q(u)}{m^2}. \end{cases} \tag{37}$$

It can be assumed that $m \neq t$, if so, $\frac{f(y_n)}{f(x_n)}$ is identically zero in (30). However, substituting the first equation into the other two in (37), and equating about $Q'(u)$ gives $(m-t)^2 = 0$, which is apparently contradictory. This completes the proof. □

3. General results on the existing model

Given the impossibility for (22) to achieve the fourth order of convergence, we focus on the other model (23) which uses $f'(y_n)/f'(x_n)$ rather than $f(y_n)/f(x_n)$. We constitute Q as a quotient of two polynomials, the degree of which we limit to three,

$$Q(v) = \frac{A + Bv + Cv^2 + Dv^3}{E + Fv + Gv^2 + Hv^3}. \tag{38}$$

Note that this form of Q is the most general and ideal one in the sense of computational efficiency so that it does not include any transcendental or irrational functions.

To achieve the fourth order of convergence, the parameter t must satisfy (13) according to [10], whereas the function Q may be chosen arbitrarily within the conditions (14). However, (14) with (38) leads to a very complicated non-linear system of the parameters, making it impossible to be solved. The conditions for (23) and (38) to be of fourth order of convergence are presented in following theorem.

Theorem 2. *A multi-step iterative method defined by* (23), (38), *and* (13) *converges with at least fourth order into the root α of $f(x) = 0$ with multiplicity m, if and only if,*

$$A + \mu^{m-1}B + \mu^{2m-2}C + \mu^{3m-3}D - mE - m\mu^{m-1}F - m\mu^{2m-2}G - m\mu^{3m-3}H, \tag{39}$$

$$\begin{aligned} m^2A + \mu^m(m^2+2m+4)B + \mu^{2m-1}(m^2+2m+8)C + \mu^{3m-2}(m^2+2m+12)D \\ - 4\mu^m mF - 8\mu^{2m-1}mG - 12\mu^{3m-2}mH, \end{aligned} \tag{40}$$

and

$$\begin{aligned} m^4(m+1)A + \mu^m(m^5+3m^4+6m^3+4m^2+8m)B \\ + \mu^{2m}(m^5+5m^4+16m^3+28m^2+32m+48)C \\ + \mu^{3m-1}(m^5+5m^4+20m^3+40m^2+48m+96)D - 4\mu^m(m^4+2m^2)F \\ - 8\mu^{2m}(m^4+2m^3+2m^2+6m)G - 12\mu^{3m-1}(m^4+2m^3+2m^2+8m)H, \end{aligned} \tag{41}$$

all equal to zero, $\mu = m/(m+2)$ is satisfied, and both sides of the fraction in (38) *are not zero.*

Proof. Refer to (24) through (28), and (13). Then, by

$$f'(y_n) = f^{(m)}(\alpha)\{mc_0\epsilon_n^{m-1} + (m+1)c_1\epsilon_n^m + (m+2)c_2\epsilon_n^{m+1} + \cdots\}, \tag{42}$$

we have

$$v = \frac{f'(y_n)}{f'(x_n)} = \mu^{m-1} - \frac{4}{m^3}\mu^m\frac{c_1}{c_0}e_n + \Big(\frac{4(m^2+2)}{m^5}\mu^m\frac{c_1^2}{c_0^2} - \frac{8}{m^3}\mu^m\frac{c_2}{c_0}\Big)e_n^2 + \cdots, \tag{43}$$

$$\begin{aligned} v^2 = \mu^{2m-2} - \frac{8}{m^3}\mu^{2m-1}\frac{c_1}{c_0}e_n + \\ \Big(\frac{8(m^3+2m^2+2m+6)}{m^6}\mu^{2m}\frac{c_1^2}{c_0^2} - \frac{16}{m^3}\mu^{2m-1}\frac{c_2}{c_0}\Big)e_n^2 + \cdots, \end{aligned} \tag{44}$$

and

$$\begin{aligned} v^3 = \mu^{3m-3} - \frac{12}{m^3}\mu^{3m-2}\frac{c_1}{c_0}e_n + \\ \Big(\frac{12(m^3+2m^2+2m+8)}{m^6}\mu^{3m-1}\frac{c_1^2}{c_0^2} - \frac{24}{m^3}\mu^{3m-2}\frac{c_2}{c_0}\Big)e_n^2 + \cdots. \end{aligned} \tag{45}$$

For the fourth order of convergence,

$$Q\Big(\frac{f'(y_n)}{f'(x_n)}\Big)u_n = \frac{A + Bv + Cv^2 + Dv^3}{E + Fv + Gv^2 + Hv^3}u_n = e_n + O(e_n^4), \tag{46}$$

which is equivalent to (39), (40), and (41).

□

Note that (39), (40), and (41) are equivalent to some 3×8 linear system. For the simplest non-trivial solutions, we assume four parameters to be non-zero, and obtain a one-dimensional solution space, which indicates a unique multi-step method. By choosing different combinations of non-zero parameters, 53 methods in total are derived, and their explicit formulae can be easily derived by solving the system. However, because we thought that it would not be wise to list them all, they are only denoted as abbreviations M1 through M53. In Section 4, explicit formulae for the best methods will be presented.

Since E, F, G, H cannot all be zero, we classify the methods according to the first term of denominator. Those with a non-zero E are displayed in Table 1, according to non-zero parameters other than E. In Table 2 are those with $E = 0$ but a non-zero F. Similarly, Table 3 and Table 4 show those with $E = F = 0, G \neq 0$ and $E = F = G = 0, H \neq 0$.

Parameters	Methods	Parameters	Methods	Parameters	Methods
A,B,C	M1 (15)	A,F,G	M13	B,G,H	M25
A,B,D	M2	A,F,H	M14	C,D,F	M26
A,B,F	M3 (3)	A,G,H	M15	C,D,G	M27
A,B,G	M4	B,C,D	M16	C,D,H	M28
A,B,H	M5	B,C,F	M17	C,F,G	M29
A,C,D	M6	B,C,G	M18	C,F,H	M30
A,C,F	M7	B,C,H	M19	C,G,H	M31
A,C,G	M8	B,D,F	M20	D,F,G	M32
A,C,H	M9	B,D,G	M21	D,F,H	M33
A,D,F	M10	B,D,H	M22	D,G,H	M34
A,D,G	M11	B,F,G	M23		
A,D,H	M12	B,F,H	M24		

Table 1. Non-zero parameters besides E and corresponding iterative methods.

Parameters	Methods	Parameters	Methods	Parameters	Methods
A,B,C	M35 (16)	A,C,D	M39	A,D,G	M42
A,B,D	M36	A,C,G	M40	A,D,H	M43
A,B,G	M37 (4)	A,C,H	M41	A,G,H	M44
A,B,H	M38				

Table 2. Non-zero parameters besides E, F and corresponding iterative methods.

Parameters	Methods	Parameters	Methods	Parameters	Methods
A,B,C	M45 (8)	A,B,H	M47	A,C,H	M49
A,B,D	M46	A,C,D	M48	A,D,H	M50

Table 3. Non-zero parameters besides E, F, G and corresponding iterative methods.

Parameters	Methods	Parameters	Methods	Parameters	Methods
A,B,C	M51	A,B,D	M52	A,C,D	M53

Table 4. Non-zero parameters besides E, F, G, H and corresponding iterative methods.

4. Numerical comparisons

As a result from the previous section, we have obtained 53 iterative methods among the family (23), (38), and (13), five of which were introduced previously. They are proven to achieve the highest existing efficiency index of $4^{1/3} = 1.587$.

Now, we conduct numerical comparisons between all 53 methods to confirm their quality, and also figure out the best methods among the family. Displayed in Table 5 are the test functions used for root-finding,

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their approximate roots with their multiplicities, and the initial values used for each function. $f_1(x)$ through $f_5(x)$ are chosen from [6], the others from [10] and [7].

Since all methods perform quite similarly with the same fourth order, the usual criterion for numerical comparisons between iterative methods, in which the number of iterations required for a certain level of tolerance are observed, does not reveal the quality of iterative methods significantly. Instead, we observe what level of convergence each method gains after a certain number of iterations.

test function	approx. root	multiplicity	initial	value
$f_1(x) = (x^3 + 4x^2 - 10)^3$	1.36523	m=3	1	2
$f_2(x) = (\sin^2 x - x^2 + 1)^2$	1.40449	m=2	2.5	1
$f_3(x) = (x^2 - e^x - 3x + 2)^5$	0.25753	m=5	1.5	3
$f_4(x) = (\cos x - x)^3$	0.73909	m=3	1.7	0.1
$f_5(x) = ((x-1)^3 - 1)^6$	2.0	m=6	1.6	4
$f_6(x) = (\ln x + \sqrt{x} - 5)^4$	8.30943	m=4	5	10
$f_7(x) = (e^x + x - 20)^2$	2.84244	m=2	3.5	2.5

Table 5. Test functions, approximate roots, their multiplicities, and initial values used.

Displayed in Table 6 are the minuses of the common logarithms of $|f(x_n)|$ after n iterations from the initial value. n's are chosen to enable fair comparisons using various functions, denoted on top of each Table in parenthesis. For each of the 53 methods, 14 different combinations of test functions and initial values as shown in Table 5 are applied, and they are averaged to obtain the overall quality. All computations were done using Mathematica. Note that some do not perform well with $m = 2$, the non-convergence denoted with *.

	$f_1(x)$	(4)	$f_2(x)$	(5)	$f_3(x)$	(4)	$f_4(x)$	(4)
methods	$x_0 = 1$	$x_0 = 2$	2.5	1	1.5	3	1.7	0.1
M1 (15)	397	385	545	389	568	268	528	341
M2	395	383	539	385	569	268	528	340
M3 (3)	412	406	597	420	564	285	531	347
M4	411	404	597	420	564	282	531	347
M5	410	402	610	428	565	279	531	346
M6	393	380	530	380	569	267	528	339
M7	413	407	597	420	563	288	531	347
M8	412	405	593	417	564	284	531	347
M9	410	403	595	418	564	281	531	346
M10	414	409	597	420	563	292	532	348
M11	412	406	589	415	563	287	531	347
M12	411	403	*	*	564	283	531	346
M13	409	399	597	420	576	210	530	346
M14	410	401	597	420	575	224	531	346
M15	410	402	592	416	575	238	531	346
M16	389	375	514	371	570	267	527	337
M17	415	411	609	427	563	292	532	348
M18	413	408	601	422	563	288	531	347
M19	412	405	599	421	564	283	531	347
M20	417	414	615	431	562	298	532	349

Table 6. $-\log|\ f(x_n)\ |$ for each method, test function & initial value.

	$f_1(x)$	(4)	$f_2(x)$	(5)	$f_3(x)$	(4)	$f_4(x)$	(4)
methods	$x_0 = 1$	$x_0 = 2$	2.5	1	1.5	3	1.7	0.1
M21	414	410	602	423	563	291	532	348
M22	412	406	595	418	564	286	531	347
M23	406	394	*	*	579	178	530	345
M24	408	397	603	423	578	189	530	345
M25	408	399	600	422	577	199	530	345
M26	420	419	639	446	562	305	533	350
M27	417	414	618	433	562	297	532	349
M28	414	410	605	425	563	291	532	348
M29	402	384	579	409	583	154	529	343
M30	404	389	594	418	581	162	529	344
M31	405	392	*	*	580	170	530	344
M32	396	370	539	387	587	138	527	340
M33	399	377	562	400	586	142	528	341
M34	400	381	573	406	584	148	529	342
M35 (16)	403	392	568	402	567	270	529	343
M36	402	391	564	400	567	270	529	343
M37 (4)	411	403	597	420	564	280	531	346
M38	410	402	619	433	565	278	531	346
M39	400	389	558	396	568	269	529	342
M40	411	404	591	416	564	282	531	346
M41	410	402	595	418	565	280	531	346
M42	411	404	587	414	564	284	531	347
M43	410	403	*	*	564	281	531	346
M44	410	402	589	415	574	254	531	346
M45 (8)	407	397	581	410	566	273	530	345
M46	406	396	579	409	566	272	530	345
M47	410	402	230	178	565	278	531	346
M48	405	395	576	407	567	271	530	344
M49	410	402	597	420	565	279	531	346
M50	410	402	*	*	564	280	531	346
M51	409	400	588	414	566	275	531	346
M52	408	400	586	413	566	274	530	345
M53	408	399	586	413	566	274	530	345
Average	407.9	398.7	579.4	408.8	568.5	257.3	530.3	345.2
Stdev.	6.25	10.4	56.7	36.9	6.98	46.0	1.26	2.60

Table 6. (Continued)

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methods	$f_5(x)$ $x_0 = 1.6$	(5) $x_0 = 4$	$f_6(x)$ 5	(3) 10	$f_7(x)$ 3.5	(4) 4.5	Average
M1 (15)	305	518	251	361	244	343	389
M2	306	517	246	356	239	336	386
M3 (3)	302	538	294	398	294	397	413
M4	302	536	314	413	294	397	415
M5	303	534	361	437	315	408	423
M6	306	515	241	350	232	327	382
M7	301	540	283	389	294	397	412
M8	302	537	299	402	288	393	412
M9	302	535	327	421	291	395	416
M10	301	542	274	381	294	397	412
M11	301	539	288	393	284	389	410
M12	302	537	310	410	*	*	410
M13	348	441	244	363	294	397	398
M14	344	446	250	369	294	397	400
M15	341	449	255	374	287	391	400
M16	306	513	235	343	222	312	377
M17	301	542	271	379	311	410	415
M18	301	539	284	390	299	400	413
M19	302	537	303	405	296	398	414
M20	300	544	262	371	321	417	417
M21	301	541	274	381	301	402	413
M22	302	539	290	395	291	395	412
M23	359	428	230	350	*	*	380
M24	355	433	236	355	302	403	397
M25	351	437	240	360	298	400	398
M26	299	547	253	363	375	450	426
M27	300	544	263	372	327	420	418
M28	301	541	276	383	306	405	414
M29	373	413	217	337	271	383	384
M30	368	419	222	342	289	394	390
M31	363	424	227	346	*	*	378
M32	391	397	205	326	230	357	371
M33	384	403	210	330	252	371	377
M34	378	408	214	334	264	379	381
M35 (16)	305	523	270	384	262	367	399
M36	305	522	265	377	259	362	397
M37 (4)	303	535	330	423	294	397	417
M38	303	533	401	445	337	414	430
M39	305	520	259	370	254	356	394
M40	302	536	312	411	286	391	413

Table 6. (Continued)

methods	$f_5(x)$ $x_0 = 1.6$	(5) $x_0 = 4$	$f_6(x)$ 5	(3) 10	$f_7(x)$ 3.5	(4) 4.5	Average
M41	303	534	343	429	291	395	417
M42	302	538	299	401	282	387	411
M43	302	536	321	417	*	*	411
M44	339	453	258	378	283	388	401
M45 (8)	304	526	295	414	275	381	407
M46	304	525	288	406	272	378	406
M47	303	532	387	457	118	209	353
M48	305	524	281	396	270	376	403
M49	303	533	372	440	294	397	420
M50	303	535	338	427	*	*	413
M51	304	528	325	467	283	389	416
M52	304	528	319	453	280	386	414
M53	304	527	309	435	280	386	412
Average	317.0	506.7	281.5	388.8	281.5	383.7	403.5
Stdev.	26.6	47.3	45.2	35.0	36.7	35.3	15.8

Table 6. (Continued)

As a result, we obtain the best methods among the family.

Rank.	Methods	Average	Rank.	Methods	Average
1	M38	429.7	6	M41	417.2
2	M26	425.7	7	M20	416.6
3	M5	423.4	8	M37 (4)	416.6
4	M49	420.5	9	M51	415.9
5	M27	417.7	10	M9	415.8

Table 7. The best methods

From these results, we conclude M38 to be the best method among the family, followed by M26, M5, M49, and M27. Although all 53 methods perform fairly well with fourth order of convergence, the best method performs up to a few percents better than the average methods of the same order, the difference of which is statistically significant. The five selected methods are given as follows:

$$x_{n+1} = x_n + \frac{m^2\mu^{m-2}(m^2-6) - (24-20m-10m^2+2m^3+m^4)v_n}{(24-8m-2m^2+m^3)\left(1 - \dfrac{m^4\mu^{-2m}v_n^2}{48+8m-12m^2+m^4}\right)v_n} mu_n \tag{47}$$

$$x_{n+1} = x_n + mu_nv_n^2 \times \frac{\mu^{m-1}(48+16m+8m^2+2m^3+m^4) - (16+8m+4m^2+2m^3+m^4)v_n}{2\mu^{3m-3}(8+8m+6m^2+m^3) - 2\mu^{2m-2}(24+12m+8m^2+m^3)v_n} \tag{48}$$

$$x_{n+1} = x_n + \frac{(-16-8m-4m^2+2m^3+m^4) - m^2\mu^{-m}(-8+m^2)v_n}{2(24-8m-2m^2+m^3)\left(1 - \dfrac{m^4\mu^{-3m}v_n^3}{48+8m-12m^2+m^4}\right)} 3mu_n \tag{49}$$

$$x_{n+1} = x_n + \frac{\mu^{3m-4}m^3(m-2) - \mu^m\left(48-24m-12m^2+2m^3+m^4\right)v_n^2}{4\mu^m(12-6m-m^2+m^3)v_n^2 - 4(m-1)m^2v_n^3} mu_n \tag{50}$$

$$x_{n+1} = x_n + \frac{\mu^m(24+12m+6m^2+2m^3+m^4)v_n - m^2(2+m^2)v_n^3}{\mu^{3m-4}m^3 - \mu^m\left(24+12m+8m^2+m^3\right)v_n^2} m u_n \tag{51}$$

where $\mu = m/(m+2)$, $u_n = f(x_n)/f'(x_n)$, $y_n = x_n - 2\mu u_n$, and $v_n = f'(y_n)/f'(x_n)$.

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The representations for the Drazin inverse of a sum of two matrices involving an idempotent matrix and applications

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Abstract

In this paper, we study the Drazin inverse of $P+Q$, where P is idempotent, and derive additive formulas under condition $PQ^2 = 0$ or $Q^2P = 0$. As its applications we establish some representations for the Drazin inverse of a class of block matrices with an idempotent subblock and under some conditions expressed in terms of the individual blocks. The results extend earlier work obtained by several authors.

Keywords: Drazin inverse; Binomial coefficient; Block matrix; Idempotent matrix

AMS(2000) Subject Classification: 15A09

1 Introduction

Throughout this paper, let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$. As we know, the Drazin inverse [1] of $A \in \mathbb{C}^{m\times m}$, denoted by A^D, is the unique matrix satisfying the following three equations:

$$A^kA^DA = A^k, \quad A^DAA^D = A^D, \quad AA^D = A^DA,$$

where $k = \text{ind}(A)$ is the index of A. If $\text{ind}(A) = 1$, then the Drazin inverse of A is reduced to the group inverse, denoted by $A^{\#}$. If $\text{ind}(A) = 0$, then $A^D = A^{-1}$. In addition, we denote $A^\pi = I - AA^D$, and define $A^0 = I$, where I is the identity matrix with proper sizes.

The Drazin inverse is very useful, and the applications in singular differential or difference equations, Markov chains, cryptography, iterative method and numerical analysis can be found in [1, 2], respectively.

Suppose P, $Q \in \mathbb{C}^{m\times m}$, such that $PQ = QP = 0$, then $(P + Q)^D = P^D + Q^D$. This result was firstly proved by Drazin [7]. Hartwig, Wang and Wei [5] gave a formula for $(P + Q)^D$ under the one side condition $PQ = 0$. Castro-González [8] derived a result under the conditions $P^DQ = 0$, $PQ^D = 0$ and $Q^\pi PQP^\pi = 0$, and Mosić and Djordjević [9] extended these results to the W-weight Drazin inverse on a

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Banach space. In [10], Castro-González et al. extended these results to the case $P^2Q = 0$ and $PQ^2 = 0$, and some similar results were extended to a Banach algebra by Castro-González and Martínez-Serrano [11]. For idempotent matrices P and Q on a Hilbert space, their Drazin inverses of sum and difference were established by Deng [12], with $PQP = 0$ or $PQP = PQ$ or $PQP = P$ satisfied. Recently, Liu et al. [13] gave the representations of the Drazin inverse of $(P \pm Q)^D$ with $P^3Q = QP$ and $Q^3P = PQ$ satisfied. Recently, Yang and Liu [21] offered a formula for $(P+Q)^D$ under conditions $PQ^2 = 0$ and $PQP = 0$. This result was generalized by Ljubisavljević and Cvetković-llić [28]. Newly, Bu and Zhang [29] presented some improved results on this topic and also considered its applications in the Drazin inverse of block matrix. In addition, Rabanovich [27] showed that every square matrix is a linear combination of three idempotent matrices. Establishing the formula for the Drazin inverse of a linear combination of three idempotent matrices is very complicated. Simply, we can split each matrix as the sum of an idempotent matrix with another matrix which is a linear combination of two idempotent matrices.

Related topic is to establish a representation of the Drazin inverse of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square complex matrices but need not to be the same size. This problem was first proposed by Campbell and Meyer [2], and is quite complicated. To the best of our knowledge, there was no explicit formula for the Drazin inverse of M. However, some special cases have been considered, which can be found in [3, 6, 10, 14-26, 30]. Since the square matrix A can be written as a linear combination of three idempotent matrices, therefore, some authors focused their attention on the case $A^2 = A$, for example,

(1) $A = B = I$, $D = 0$ (see [6]);

(2) $A = I$, $D = 0$ (see [3]);

(3) $B = A = A^2$, $D = 0$ (see [22]);

(4) $A^2 = A$, $D = 0$ and $AB = B$ (see [22]);

(5) $A = I$, $D = UV$ and $BU = 0$ (or $VC = 0$) (see [3]).

The results referred above were deduced by a series complicated operations involving block matrices. In this paper, based on the existed results, we adopt another method to derive some more general conclusions. It is clear that the case (5) in above implies that $BD = 0$ (or $DC = 0$). In fact, the block matrix M can be split as the sum of two matrices involving an idempotent matrix under the cases (1)-(5). For example, if $A = I$ and $BD = 0$, we can split M as $M = P + Q$, where $P = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$. And we have $P^2 = P$, $PQ^2 = 0$. Observe that the condition $BD = 0$ is just sufficient, not necessary, and can be replaced by $BDC = 0$ and $BD^2 = 0$.

Hence, according to the above analysis, when P is idempotent, investigating the representation for the Drazin inverse of $P+Q$ under the assumption $PQ^2=0$ is very significative.

Next we recall some properties of the Drazin inverse of a square matrix A. Suppose that A has the Jordan decomposition

$$A = P\begin{pmatrix} \sum & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

where P and $\sum$ are nonsingular matrices, and N is nilpotent matrix. Then by definition we have

$$A^D = P\begin{pmatrix} \sum^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}, \ and\ \operatorname{ind}(N) = \operatorname{ind}(A) = k.$$

For convenience, throughout this paper, for integers n and k we denote $C(n,k)$ the binomial coefficient $\binom{n}{k}$. We will make use of the following well known identities involving binomial coefficients: $C(n,k) = C(n-1,k) + C(n-1,k-1)$, $C(n,k) = C(n,n-k)$, $C(-l,k) = (-1)^k C(l+k-1,k)$.

Let $E \in \mathbb{C}^{m\times m}$, $\operatorname{ind}(E) = r$, the following sequences of matrices will be used in this article:

$$Y_E(k) = \sum_{i=0}^{r-1} (-1)^i C(k+2i,i) E^i E^\pi, k \geqslant -1; \tag{1.1}$$

$$X_E(k) = \sum_{i=0}^{s(k)} (-1)^k C(k-i,i) (E^D)^{k-i}, k \geqslant 0 \ and\ X_E(-1) = 0; \tag{1.2}$$

$$Z_E(k) = X_E(k) E^D + Y_E(k+1), k \geqslant -1; \tag{1.3}$$

$$U_E(k) = \sum_{i=0}^{s(k)} C(k-i,i) E^i, k \geqslant 0 \ and\ U_E(-1) = 0; \tag{1.4}$$

$$\Gamma_E(k) = Y_E(k) E^k, k \geqslant 0; \tag{1.5}$$

where $s(k)$ is the integer part of $k/2$.

This paper is organized as follows. In section 2, we will introduce some lemmas and deduce some auxiliary conclusions. In section 3, we derive representations for $(P+Q)^D$ under condition $PQ^2 = 0$ or $Q^2P = 0$, when P is idempotent. Since the formula of $(P+Q)^D$ is very meaningful in establishing the expressions of the Drazin inverse of a 2×2 block matrix. Therefore, in section 4, we apply our results to establish the representations of the Drazin inverse of a 2×2 block matrix.

2 Some lemmas

In order to establish the expression of the Drazin inverse of $(P+Q)^D$, in this section, we will make preparations for the further development. To this end, we will present some known results and prove some auxiliary conclusions.

Proposition 2.1. Let $E \in \mathbb{C}^{m\times m}$ with $\operatorname{ind}(E) = r$.

(1) The sequence (1.1) has the property

$$Y_E(k-1) - Y_E(k) = Y_E(k+1)E,\ k \geqslant 0.$$

(2) The sequence (1.3) has the property

$$Z_E(k) + Z_E(k+1)E = Z_E(k-1),\ k \geqslant 0.$$

(3) The sequence (1.4) has the property

$$U_E(k) - U_E(k-1) = U_E(k-2)E,\ k \geqslant 1.$$

(4) The sequence (1.5) has the property

$$\Gamma_E(k-1)E - \Gamma_E(k) = \Gamma_E(k+1),\ k \geqslant 1.$$

Proof. Since EE^π is r-nilpotent, it follows from [3, Lemma 3.1] that the statement (1) is evident. The statement (3) is given by [3, Lemma 3.8]. The statement (4) follows from (1). And by [3, Lemma 3.3] it follows that

$$X_E(k-1)E^D - X_E(k)E^D = X_E(k+1),\ k \geqslant 0.$$

Then according to (1) we get

$$\begin{aligned} Z_E(k) + Z_E(k+1)E &= X_E(k)E^D + Y_E(k+1) + X_E(k+1)E^D E + Y_E(k+2)E \\ &= [X_E(k)E^D + X_E(k+1)] + [Y_E(k+1) + Y_E(k+2)E] \\ &= X_E(k-1)E^D + Y_E(k) \\ &= Z_E(k-1). \end{aligned}$$

Hence, the proof is complete. □

Lemma 2.1. Let $F = \begin{pmatrix} I & I \\ E & 0 \end{pmatrix}$, where $E \in \mathbb{C}^{m\times m}$ with $\mathrm{ind}(E) = r$. Then, for all $k \geqslant 1$

$$(F^D)^k = \begin{pmatrix} Z_E(k-2) & Z_E(k-1) \\ Z_E(k-1)E & Z_E(k)E \end{pmatrix}, \tag{2.1}$$

where $Z_E(k)$ is given by (1.3).

Proof. According to [6, Theorem 3.3], it is evident that

$$\begin{aligned} F^D &= \begin{pmatrix} Y_E(0) & E^D + Y_E(1) \\ EE^D + EY_E(1) & Y_E(2)E - E^D \end{pmatrix} \\ &= \begin{pmatrix} Z_E(-1) & Z_E(0) \\ Z_E(0)E & Z_E(1)E \end{pmatrix}. \end{aligned}$$

On the other hand, for all $k \geqslant 2$, by [3, Theorem 3.5] it follows that

$$\begin{aligned}(F^D)^k &= \begin{pmatrix} X_E(k-2)E^D & X_E(k-1)E^D \\ X_E(k-1) & X_E(k) \end{pmatrix} + \begin{pmatrix} Y_E(k-1) & Y_E(k) \\ Y_E(k)E & Y_E(k+1)E \end{pmatrix} \\ &= \begin{pmatrix} Z_E(k-2) & Z_E(k-1) \\ Z_E(k-1)E & Z_E(k)E \end{pmatrix}.\end{aligned}$$

Hence, (2.1) holds for any $k \geqslant 1$. $\square$

Similarly, we can deduce the following result.

Lemma 2.2. Let $F = \begin{pmatrix} I & E \\ I & 0 \end{pmatrix}$, where $E \in \mathbb{C}^{m\times m}$ with $\mathrm{ind}(E) = r$. Then, for all $k \geqslant 1$

$$(F^D)^k = \begin{pmatrix} Z_E(k-2) & Z_E(k-1)E \\ Z_E(k-1) & Z_E(k)E \end{pmatrix}, \tag{2.2}$$

where $Z_E(k)$ is given by (1.3).

Lemma 2.3. ([4]) Let $M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$, $M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix}$, where A and B are square matrices with $\mathrm{ind}(A) = r$ and $\mathrm{ind}(B) = s$, respectively. Then $\max\{r, s\} \leqslant \mathrm{ind}(M_i) \leqslant r + s$, $i = 1, 2$, and

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix}, \quad M_2^D = \begin{pmatrix} B^D & X \\ 0 & A^D \end{pmatrix},$$

where $X = (B^D)^2[\sum_{i=0}^{r-1} (B^D)^i C A^i]A^\pi + B^\pi[\sum_{i=0}^{s-1} B^i C (A^D)^i](A^D)^2 - B^D C A^D$.

Lemma 2.4. ([5]) Let P, $Q \in \mathbb{C}^{m\times m}$, with $\mathrm{ind}(P) = r$ and $\mathrm{ind}(Q) = s$. If $PQ = 0$, then

$$(P + Q)^D = Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi.$$

Lemma 2.5. Let $M = \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}$, where P, $Q \in \mathbb{C}^{m\times m}$, with $P^2 = P$ and $\mathrm{ind}(PQP) = r$. Then, for all $k \geqslant 1$,

$$\begin{aligned}(M^D)^k &= \begin{pmatrix} Z_{PQP}(k-2) & Z_{PQP}(k-1)PQP \\ Z_{PQP}(k-1) & Z_{PQP}(k)PQP \end{pmatrix} \\ &\quad + \begin{pmatrix} Z_{PQP}(k) & Z_{PQP}(k-1) \\ Z_{PQP}(k+1) & Z_{PQP}(k) \end{pmatrix} \begin{pmatrix} PQP^\pi & 0 \\ 0 & PQP^\pi \end{pmatrix}.\end{aligned} \tag{2.3}$$

Moreover,

$$\begin{aligned}M^\pi &= \begin{pmatrix} \Gamma_{PQP}(1) & -\Gamma_{PQP}(0)PQ \\ -\Gamma_{PQP}(0)P & Y_{PQP}(-1) \end{pmatrix} \\ &\quad - \begin{pmatrix} Z_{PQP}(0)PQP^\pi - P^\pi & 0 \\ Z_{PQP}(1)PQP^\pi & Z_{PQP}(0)PQP^\pi \end{pmatrix},\end{aligned} \tag{2.4}$$

where $Y_{PQP}(k)$ and $Z_{PQP}(k)$ are given by (1.2) and (1.3) with $E = PQP$ respectively.

Proof. Since P is idempotent, without loss of generality, suppose P can be partitioned as

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix},$$

be partitioned conformably with P. Hence,

$$M = \begin{pmatrix} I & 0 & Q_1 & Q_2 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tilde{M} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{2.5}$$

and, for all $k \geqslant 1$,

$$(M^D)^k = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} (\tilde{M}^D)^k \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{2.6}$$

where

$$\tilde{M} = \begin{pmatrix} I & Q_1 & 0 & Q_2 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

$$A = \begin{pmatrix} I & Q_1 \\ I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & Q_2 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

By Lemma 2.3, we get

$$(\tilde{M}^D)^k = \begin{pmatrix} (A^D)^k & (A^D)^{k+1}B + (A^D)^{k+2}BD \\ 0 & 0 \end{pmatrix}, \quad k \geqslant 1.$$

After applying Lemma 2.2, we have

$$(\tilde{M}^D)^k = \begin{pmatrix} Z_{Q_1}(k-2) & Z_{Q_1}(k-1)Q_1 & Z_{Q_1}(k)Q_2 & Z_{Q_1}(k-1)Q_2 \\ Z_{Q_1}(k-1) & Z_{Q_1}(k)Q_1 & Z_{Q_1}(k+1)Q_2 & Z_{Q_1}(k)Q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Substituting (2.7) in (2.6) produces (2.3).

A simple computation shows that

$$\begin{aligned} M^\pi &= I - M^D M \\ &= I - \begin{pmatrix} Z_{PQP}(-1)P + Z_{PQP}(0)PQP & Z_{PQP}(-1)PQ \\ Z_{PQP}(0)P + Z_{PQP}(1)PQP & Z_{PQP}(0)PQ \end{pmatrix} - \begin{pmatrix} Z_{PQP}(0)PQP^\pi & 0 \\ Z_{PQP}(1)PQP^\pi & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_{PQP}(1) & -\Gamma_{PQP}(0)PQ \\ -\Gamma_{PQP}(0)P & Y_{PQP}(-1) \end{pmatrix} - \begin{pmatrix} Z_{PQP}(0)PQP^\pi - P^\pi & 0 \\ Z_{PQP}(1)PQP^\pi & Z_{PQP}(0)PQP^\pi \end{pmatrix}. \end{aligned}$$

So, from the above computations, we get the statements of this lemma. □

Lemma 2.6. let $M = \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}$, where P, $Q \in \mathbb{C}^{m\times m}$, with $P^2 = P$ and $\mathrm{ind}(PQP) = r$. Then, for all $k \geqslant 2$,

$$M^k = \begin{pmatrix} U_{PQP}(k-1)P + U_{PQP}(k-2)PQ & U_{PQP}(k-1)PQ \\ U_{PQP}(k-2)P + U_{PQP}(k-3)PQ & U_{PQP}(k-2)PQ \end{pmatrix}, \tag{2.8}$$

and consequently,

$$M^k M^\pi = (-1)^k \begin{pmatrix} -\Gamma_{PQP}(k) + \Gamma_{PQP}(k-1)PQ & -\Gamma_{PQP}(k)PQ \\ \Gamma_{PQP}(k-1) - \Gamma_{PQP}(k-2)PQ & \Gamma_{PQP}(k-1)PQ \end{pmatrix}, \tag{2.9}$$

where $U_{PQP}(k)$ and $\Gamma_{PQP}(k)$ are given by (1.4) and (1.5) with $E = PQP$ respectively.

Proof. Trivially (2.8) is valid for $k = 2$. Assume that the statement (2.8) is true for k ($k \geqslant 2$), then by Proposition 2.1 (3), we compute

$$\begin{aligned} M^{k+1} &= \begin{pmatrix} U_{PQP}(k-1)P + U_{PQP}(k-2)PQ & U_{PQP}(k-1)PQ \\ U_{PQP}(k-2)P + U_{PQP}(k-3)PQ & U_{PQP}(k-2)PQ \end{pmatrix} \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} U_{PQP}(k)P + U_{PQP}(k-1)PQ & U_{PQP}(k)PQ \\ U_{PQP}(k-1)P + U_{PQP}(k-2)PQ & U_{PQP}(k-1)PQ \end{pmatrix}. \end{aligned} \tag{2.10}$$

Namely (2.8) holds for $k + 1$. By mathematical induction (2.8) is true for all $k \geqslant 2$.

On the other hand,

$$\begin{aligned} M^k M^\pi &= M^k - M^D M^{k+1} \\ &= M^k - \begin{pmatrix} Z_{PQP}(-1) & Z_{PQP}(0)PQP \\ Z_{PQP}(0) & Z_{PQP}(1)PQP \end{pmatrix} M^{k+1} \\ &= M^k - \begin{pmatrix} Y_{PQP}(0) & PQP(PQP)^D + Y_{PQP}(1)PQP \\ (PQP)^D + Y_{PQP}(1) & Y_{PQP}(2)PQP - (PQP)^D \end{pmatrix} M^{k+1} \\ &= M^k - (S+T)M^{k+1}, \end{aligned} \tag{2.11}$$

where

$$S = \begin{pmatrix} 0 & PQP(PQP)^D \\ (PQP)^D & -(PQP)^D \end{pmatrix}, \quad T = \begin{pmatrix} Y_{PQP}(0) & Y_{PQP}(1)PQP \\ Y_{PQP}(1) & Y_{PQP}(2)PQP \end{pmatrix}.$$

It is easy to show that

$$SM^{k+1} = \begin{pmatrix} \begin{array}{c} U_{PQP}(k-1)(PQP)^D PQP \\ +U_{PQP}(k-2)(PQP)^D PQPQ \end{array} & U_{PQP}(k-1)(PQP)^D PQPQ \\ \begin{array}{c} U_{PQP}(k-2)(PQP)^D PQP \\ +U_{PQP}(k-3)(PQP)^D PQPQ \end{array} & U_{PQP}(k-2)(PQP)^D PQPQ \end{pmatrix}, \tag{2.12}$$

and

$$TM^{k+1} = \begin{pmatrix} \Phi_1(k-1) + \Phi_1(k-2)PQ & \Phi_1(k-1)PQ \\ \Phi_2(k-1) + \Phi_2(k-2)PQ & \Phi_2(k-1)PQ \end{pmatrix},$$

where

$$\begin{aligned} \Phi_1(k) &= Y_{PQP}(0)U_{PQP}(k+1) + Y_{PQP}(1)U_{PQP}(k)PQP, \\ \Phi_2(k) &= Y_{PQP}(1)U_{PQP}(k+1) + Y_{PQP}(2)U_{PQP}(k)PQP. \end{aligned}$$

In [3, Theorem 3.9] it is shown that

$$\Phi_1(k) = U_{PQP}(k)(PQP)^\pi + (-1)^{k+1} Y_{PQP}(k+1)(PQP)^{k+1}.$$

Analogous to the proof of [3, Theorem 3.9], we can show that $\Phi_2(k) = \Phi_1(k-1)$.

Therefore,

$$\begin{aligned} TM^{k+1} &= \begin{pmatrix} U_{PQP}(k-1)(PQP)^{\pi} + U_{PQP}(k-2)(PQP)^{\pi}PQ & U_{PQP}(k-1)(PQP)^{\pi}Q \\ U_{PQP}(k-2)(PQP)^{\pi} + U_{PQP}(k-3)(PQP)^{\pi}PQ & U_{PQP}(k-2)(PQP)^{\pi}Q \end{pmatrix} \\ &\quad +(-1)^k \begin{pmatrix} \Gamma_{PQP}(k) - \Gamma_{PQP}(k-1)PQ & \Gamma_{PQP}(k)PQ \\ -\Gamma_{PQP}(k-1) + \Gamma_{PQP}(k-2)PQ & -\Gamma_{PQP}(k-1)PQ \end{pmatrix}. \end{aligned} \tag{2.13}$$

Combining (2.11), (2.12) and (2.13) gives (2.9). $\square$

Similarly, we state the symmetrical formulations of Lemma 2.5 and Lemma 2.6.

Lemma 2.7. Let $M = \begin{pmatrix} P & I \\ QP & 0 \end{pmatrix}$, where $P, Q \in \mathbb{C}^{m\times m}$, with $P^2 = P$ and $\mathrm{ind}(PQP) = r$. Then, for all $k \geqslant 1$,

$$\begin{aligned} (M^D)^k &= \begin{pmatrix} Z_{PQP}(k-2) & Z_{PQP}(k-1) \\ PQPZ_{PQP}(k-1) & PQPZ_{PQP}(k) \end{pmatrix} \\ &\quad + \begin{pmatrix} P^{\pi}QP & 0 \\ 0 & P^{\pi}QP \end{pmatrix} \begin{pmatrix} Z_{PQP}(k) & Z_{PQP}(k+1) \\ Z_{PQP}(k-1) & Z_{PQP}(k) \end{pmatrix}. \end{aligned}$$

Moreover,

$$M^{\pi} = \begin{pmatrix} \Gamma_{PQP}(1) & -P\Gamma_{PQP}(0) \\ -QP\Gamma_{PQP}(0) & Y_{PQP}(-1) \end{pmatrix} - \begin{pmatrix} P^{\pi}QPZ_{PQP}(0) - P^{\pi} & P^{\pi}QPZ_{PQP}(1) \\ 0 & P^{\pi}QPZ_{PQP}(0) \end{pmatrix}.$$

where $Y_{PQP}(k)$ and $Z_{PQP}(k)$ are given by (1.1) and (1.3) with $E = PQP$ respectively.

Lemma 2.8. Let $M = \begin{pmatrix} P & I \\ QP & 0 \end{pmatrix}$, where $P, Q \in \mathbb{C}^{m\times m}$, with $P^2 = P$ and $\mathrm{ind}(PQP) = r$. Then, for all $k \geqslant 2$,

$$M^k = \begin{pmatrix} PU_{PQP}(k-1) + QPU_{PQP}(k-2) & PU_{PQP}(k-2) + QPU_{PQP}(k-3) \\ QPU_{PQP}(k-1) & QPU_{PQP}(k-2) \end{pmatrix},$$

and consequently,

$$M^kM^{\pi} = (-1)^k \begin{pmatrix} -\Gamma_{PQP}(k) + QP\Gamma_{PQP}(k-1) & \Gamma_{PQP}(k-1) - QP\Gamma_{PQP}(k-2) \\ -QP\Gamma_{PQP}(k) & QP\Gamma_{PQP}(k-1) \end{pmatrix},$$

where $U_{PQP}(k)$ and $\Gamma_{PQP}(k)$ are given by (1.4) and (1.5) with $E = PQP$ respectively.

3 Main results

Based on the auxiliary conclusions presented in previous section, next, we derive the formula for the Drazin inverse of $P + Q$ under the assumptions $P^2 = P$ and $PQ^2 = 0$.

Theorem 3.1. Let P, $Q \in C^{m\times m}$, such that $P^2 = P$ and $\mathrm{ind}(PQP) = r$, $\mathrm{ind}(Q) = s$. If $PQ^2 = 0$, then

$$\begin{aligned}(P+Q)^D &= Q^\pi \sum_{k=0}^{s-1} Q^k [Z_{PQP}(k-1)P + Z_{PQP}(k+1)PQP^\pi] \\ &\quad + \sum_{k=0}^{r-1} (-1)^k (Q^D)^{k+2} [\Gamma_{PQP}(k+1) - \Gamma_{PQP}(k)PQ] \\ &\quad + Q^D[\Gamma_{PQP}(1) + P^\pi - Z_{PQP}(0)PQP^\pi], \end{aligned} \tag{3.1}$$

where $Y_{PQP}(k)$, $Z_{PQP}(k)$ and $\Gamma_{PQP}(k)$ are given by (1.1), (1.3) and (1.5) with $E = PQP$ respectively.

Proof. Using the fact $(AB)^D = A((BA)^D)^2B$, we have

$$(P+Q)^D = \left(\begin{pmatrix} I & Q \end{pmatrix} \begin{pmatrix} P \\ I \end{pmatrix} \right)^D = \begin{pmatrix} I & Q \end{pmatrix} \left(\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^D \right)^2 \begin{pmatrix} P \\ I \end{pmatrix}. \tag{3.2}$$

Denote $\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix} = M$, and rewrite M as $M = G + F$, where $G = \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$. Since $PQ^2 = 0$, thus $GF = 0$. So from Lemma 2.4, it follows that

$$\begin{aligned}(M^D)^2 &= F^\pi \sum_{k=0}^{s-1} F^k (G^D)^{k+2} + \sum_{k=0}^{t-1} (F^D)^{k+2} G^k G^\pi - F^D G^D \\ &= F^\pi \sum_{k=0}^{s-1} F^k (G^D)^{k+2} + \sum_{k=2}^{t-1} (F^D)^{k+2} G^k G^\pi \\ &\quad + (F^D)^2 G^\pi + (F^D)^3 G G^\pi - F^D G^D, \end{aligned} \tag{3.3}$$

where $t = \mathrm{ind}(G)$.

Furthermore, we have

$$F^D = \begin{pmatrix} 0 & 0 \\ 0 & Q^D \end{pmatrix}, \quad F^\pi = \begin{pmatrix} I & 0 \\ 0 & Q^\pi \end{pmatrix}. \tag{3.4}$$

Applying Proposition 2.1, Lemma 2.5 and Lemma 2.6, combining (3.2), (3.3) and (3.4), we compute

$$\begin{aligned}
\begin{pmatrix} I & Q \end{pmatrix} F^\pi \sum_{k=0}^{s-1} F^k (G^D)^{k+2} \begin{pmatrix} P \\ I \end{pmatrix} &= Z_{PQP}(-1)P + Z_{PQP}(1)PQP^\pi \\
&\quad + Q^\pi \sum_{k=0}^{s-2} Q^{k+1}[Z_{PQP}(k)P + Z_{PQP}(k+2)PQP^\pi], \\
\begin{pmatrix} I & Q \end{pmatrix} \sum_{k=2}^{t-1} (F^D)^{k+2} G^k G^\pi \begin{pmatrix} P \\ I \end{pmatrix} &= \sum_{k=2}^{r} (-1)^k (Q^D)^{k+1}[-\Gamma_{PQP}(k) + \Gamma_{PQP}(k-1)PQ], \\
\begin{pmatrix} I & Q \end{pmatrix} (F^D)^3 G G^\pi \begin{pmatrix} P \\ I \end{pmatrix} &= (Q^D)^2[-\Gamma_{PQP}(0)P + Y_{PQP}(-1)P - \Gamma_{PQP}(0)PQ] \\
&= (Q^D)^2[-Y_{PQP}(0)P + Y_{PQP}(-1)P - \Gamma_{PQP}(0)PQ] \\
&= (Q^D)^2[\Gamma_{PQP}(1) - \Gamma_{PQP}(0)PQ], \\
\begin{pmatrix} I & Q \end{pmatrix} (F^D)^2 G^\pi \begin{pmatrix} P \\ I \end{pmatrix} &= Q^D[\Gamma_{PQP}(1) + P^\pi - Z_{PQP}(0)PQP^\pi], \\
\begin{pmatrix} I & Q \end{pmatrix} F^D G^D \begin{pmatrix} P \\ I \end{pmatrix} &= QQ^D[Z_{PQP}(0)P + Z_{PQP}(1)PQP + Z_{PQP}(1)PQP^\pi] \\
&= QQ^D[Z_{PQP}(-1)P + Z_{PQP}(1)PQP^\pi].
\end{aligned}$$

By substituting the above computations in (3.2) and rearranging terms we obtain

$$\begin{aligned}
(P+Q)^D &= Q^\pi \sum_{k=-1}^{s-2} Q^{k+1}[Z_{PQP}(k)P + Z_{PQP}(k+2)PQP^\pi] \\
&\quad + \sum_{k=1}^{r} (-1)^k (Q^D)^{k+1}[-\Gamma_{PQP}(k) + \Gamma_{PQP}(k-1)PQ] \\
&\quad + Q^D[Y_{PQP}(-1) - \Gamma_{PQP}(0)P - Z_{PQP}(0)PQP^\pi].
\end{aligned}$$

Therefore, (3.1) is evident. □

Specially, if $PQP = 0$, then we have the following result which can also be deduced by [21, Theorem 2.1].

Corollary 3.1. Let P, $Q \in C^{m\times m}$, such that $P^2 = P$ and $\mathrm{ind}(Q) = s$. If $PQ^2 = 0$ and $PQP = 0$, then

$$(P+Q)^D = Q^\pi \sum_{k=0}^{s-1} Q^k (P + PQ) + Q^D P^\pi - Q^D PQ - (Q^D)^2 PQ.$$

Similarly, we state the symmetrical formulation of Theorem 3.1.

Theorem 3.2. Let P, $Q \in C^{m\times m}$, such that $P^2 = P$ and $\mathrm{ind}(PQP) = r$, $\mathrm{ind}(Q) = s$. If $Q^2 P = 0$, then

$$\begin{aligned}
(P+Q)^D &= \sum_{k=0}^{s-1} [PZ_{PQP}(k-1) + P^\pi QPZ_{PQP}(k+1)]Q^k Q^\pi \\
&\quad + \sum_{k=0}^{r-1} (-1)^k [\Gamma_{PQP}(k+1) - QP\Gamma_{PQP}(k)](Q^D)^{k+2} \\
&\quad + [\Gamma_{PQP}(1) + P^\pi - P^\pi QPZ_{PQP}(0)]Q^D,
\end{aligned}$$

where $Y_{PQP}(k)$, $Z_{PQP}(k)$ and $\Gamma_{PQP}(k)$ are given by (1.1), (1.3) and (1.5) with $E = PQP$ respectively.

Proof. Since $P + Q$ can also be expressed as

$$P + Q = \begin{pmatrix} P & I \end{pmatrix} \begin{pmatrix} I \\ Q \end{pmatrix}.$$

Hence,

$$(P + Q)^D = \begin{pmatrix} P & I \end{pmatrix} \left(\begin{pmatrix} P & I \\ QP & Q \end{pmatrix}^D \right)^2 \begin{pmatrix} I \\ Q \end{pmatrix}.$$

Let $M = \begin{pmatrix} P & I \\ QP & Q \end{pmatrix}$. Consider the splitting $M = G + F$, where $G = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ and $F = \begin{pmatrix} P & I \\ QP & 0 \end{pmatrix}$. Since $Q^2P = 0$, thus $GF = 0$. Using Lemma 2.4, Lemma 2.7 and Lemma 2.8, this result may be proved in much the same way as Theorem 3.1. Hence, we omit the details. $\square$

As a special case of Theorem 3.2, we can deduce the following result.

Corollary 3.2. Let P, $Q \in C^{m\times m}$, such that $P^2 = P$ and $\text{ind}(Q) = s$. If $Q^2P = 0$ and $PQP = 0$, then

$$(P + Q)^D = \sum_{k=0}^{s-1} (P + QP)Q^k Q^\pi + P^\pi Q^D - QPQ^D - QP(Q^D)^2.$$

4 Applications

Consider the block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.1}$$

where A and D are square complex matrices but need not to be the same size. In the following, we illustrate some applications of our results obtained in the previous section to establish representations for M^D under some conditions, which extend some results in the literature, especially those in [3, 6, 22].

Theorem 4.1. Let M be a given matrix of form (4.1), such that $A^2 = A$ with $\text{ind}(BCA) = r$ and $\text{ind}(D) = s$. If $AB = B$, $BDC = 0$ and $BD^2 = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} Z_{BCA}(0)A + Z_{BCA}(1)BC & Z_{BCA}(0)B + Z_{BCA}(1)BD \\ -D^DC[Z_{BCA}(0)A + Z_{BCA}(1)BC] & -D^DC[Z_{BCA}(0)B + Z_{BCA}(1)BD] \end{pmatrix} \\ &+ \sum_{k=1}^{s} \begin{pmatrix} 0 & 0 \\ D^\pi D^{k-1}C & 0 \end{pmatrix} \begin{pmatrix} Z_{BCA}(k)A + Z_{BCA}(k+1)BC & \begin{matrix} Z_{BCA}(k)B+ \\ Z_{BCA}(k+1)BD \end{matrix} \\ 0 & 0 \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & 0 \\ (D^D)^{k+3}C & 0 \end{pmatrix} \begin{pmatrix} \Gamma_{BCA}(k+1) - \Gamma_{BCA}(k)BC & \begin{matrix} \Gamma_{BCA}(k+1)B \\ -\Gamma_{BCA}(k)BD \end{matrix} \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ (D^D)^2C & D^D \end{pmatrix} \begin{pmatrix} \begin{matrix} \Gamma_{BCA}(1) + A^\pi \\ -Z_{BCA}(0)BCA^\pi \end{matrix} & \begin{matrix} \Gamma_{BCA}(1)B - B- \\ Z_{BCA}(0)(BD - BCB) \end{matrix} \\ 0 & I \end{pmatrix}, \end{aligned} \tag{4.2}$$

where $Z_{BCA}(k)$ and $\Gamma_{BCA}(k)$ are given by (1.3) and (1.5) with $E = BCA$ respectively.

Proof. Consider the splitting of matrix $M = P + Q$, where $P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$. According to the assumptions, we have $P^2 = P$ and $PQ^2 = 0$. From Theorem 3.1 it follows that

$$\begin{aligned} M^D = (P+Q)^D &= Q^\pi \sum_{k=0}^{s} Q^k [Z_{PQP}(k-1)P + Z_{PQP}(k+1)PQP^\pi] \\ &+ \sum_{k=0}^{r} (-1)^k (Q^D)^{k+2} [\Gamma_{PQP}(k+1) - \Gamma_{PQP}(k)PQ] \\ &+ Q^D[\Gamma_{PQP}(1) + P^\pi - Z_{PQP}(0)PQP^\pi]. \end{aligned} \tag{4.3}$$

Further, we can compute

$$\begin{aligned} Q^D &= \begin{pmatrix} 0 & 0 \\ (D^D)^2 C & D^D \end{pmatrix}; \\ Q^\pi &= \begin{pmatrix} I & 0 \\ -D^D C & D^\pi \end{pmatrix}; \\ Y_{PQP}(n) &= \begin{pmatrix} Y_{BCA}(n) & Y_{BCA}(n)B - B \\ 0 & I \end{pmatrix}; \\ \Gamma_{PQP}(n) &= \begin{pmatrix} \Gamma_{BCA}(n) & \Gamma_{BCA}(n)B \\ 0 & 0 \end{pmatrix}, \quad n \geqslant 1; \\ X_{PQP}(n)(PQP)^D &= \begin{pmatrix} X_{BCA}(n)(BCA)^D & X_{BCA}(n)(BCA)^D B \\ 0 & 0 \end{pmatrix}; \\ Z_{PQP}(n) &= \begin{pmatrix} Z_{BCA}(n) & Z_{BCA}(n)B - B \\ 0 & I \end{pmatrix}. \end{aligned}$$

By substituting the above comutations in (4.3) the formula (4.2) readily follows. □

Applying Theorem 3.2, similar as in Theorem 4.1, we get the following result.

Theorem 4.2. Let M be a given matrix of form (4.1), such that $A^2 = A$ with $\mathrm{ind}(BCA) = r$ and $\mathrm{ind}(D) = s$. If $AB = B$ and $DCA = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} AZ_{BCA}(0) & Z_{BCA}(0)B \\ CAZ_{BCA}(1) & CZ_{BCA}(1)B \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^D C & D^\pi \end{pmatrix} \\ &+ \sum_{k=1}^{s} \begin{pmatrix} 0 & Z_{BCA}(k)B \\ 0 & CZ_{BCA}(k+1)B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D^\pi D^{k-1} C & D^\pi D^k \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & \Gamma_{BCA}(k+1)B \\ 0 & -C\Gamma_{BCA}(k)B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (D^D)^{k+3} C & (D^D)^{k+2} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 2\Gamma_{BCA}(1)B - B(BCA)^\pi \\ 0 & I - CZ_{BCA}(0)B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (D^D)^2 C & D^D \end{pmatrix}, \end{aligned}$$

where $Z_{BCA}(k)$ and $\Gamma_{BCA}(k)$ are given by (1.3) and (1.5) with $E = BCA$ respectively.

The following special cases can be deduced by Theorem 4.1 immediately.

Corollary 4.1. Let M be a given matrix of form (4.1), such that $A = I$ with $\mathrm{ind}(BC) = r$ and $\mathrm{ind}(D) = s$.

If $DC = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} Z_{BC}(-1) & Z_{BC}(0)BD^\pi \\ CZ_{BC}(0) & CZ_{BC}(1)BD^\pi \end{pmatrix} + \sum_{k=1}^{s} \begin{pmatrix} 0 & Z_{BC}(k)BD^\pi D^k \\ 0 & CZ_{BC}(k+1)BD^\pi D^k \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & \Gamma_{BC}(k+1)B(D^D)^{k+2} \\ 0 & -C\Gamma_{BC}(k)B(D^D)^{k+2} \end{pmatrix} + \begin{pmatrix} 0 & [2\Gamma_{BC}(1)B - B(BC)^\pi]D^D \\ 0 & [I - CZ_{BC}(0)B]D^D \end{pmatrix}, \end{aligned}$$

where $Z_{BC}(k)$ and $\Gamma_{BC}(k)$ are given by (1.3) and (1.5) with $E = BC$ respectively.

Corollary 4.2. ([22]) Let M be a given matrix of form (4.1), such that $A^2 = A$ and $D = 0$ with $\mathrm{ind}(BCA) = r$. If $AB = B$, then

$$M^D = \begin{pmatrix} AZ_{BCA}(0) + Z_{BCA}(1)BC & Z_{BCA}(0)B \\ CAZ_{BCA}(1) + CZ_{BCA}(2)BC & CZ_{BCA}(1)B \end{pmatrix},$$

where $Z_{BCA}(k)$ is given by (1.3) with $E = BCA$.

Corollary 4.3. ([22]) Let M be a given matrix of form (4.1), such that $B = A = A^2$ and $D = 0$ with $\mathrm{ind}(ACA) = r$. Then

$$M^D = \begin{pmatrix} AZ_{ACA}(0) + Z_{ACA}(1)AC & Z_{ACA}(0)A \\ CAZ_{ACA}(1) + CZ_{ACA}(2)AC & CZ_{ACA}(1)A \end{pmatrix},$$

where $Z_{ACA}(k)$ is given by (1.3) with $E = ACA$.

Next, we consider another splitting of block matrix M and present some alternative results.

Theorem 4.3. Let M be a given matrix of form (4.1), such that $A^2 = A$ with $\mathrm{ind}(ABC) = r$ and $\mathrm{ind}(D) = s$. If $CA = C$ and $ABD = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} I & -BD^D \\ 0 & D^\pi \end{pmatrix} \begin{pmatrix} Z_{ABC}(0)A & Z_{ABC}(1)AB \\ CZ_{ABC}(0) & CZ_{ABC}(1)B \end{pmatrix} \\ &+ \sum_{k=1}^{s} \begin{pmatrix} 0 & BD^{k-1}D^\pi \\ 0 & D^kD^\pi \end{pmatrix} \begin{pmatrix} 0 & 0 \\ CZ_{ABC}(k) & CZ_{ABC}(k+1)B \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & B(D^D)^{k+3} \\ 0 & (D^D)^{k+2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C\Gamma_{ABC}(k+1) & -C\Gamma_{ABC}(k)B \end{pmatrix} \\ &+ \begin{pmatrix} 0 & B(D^D)^2 \\ 0 & D^D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2C\Gamma_{ABC}(1) - C(ABC)^\pi & I - CZ_{ABC}(0)B \end{pmatrix}, \end{aligned}$$

where $Z_{ABC}(k)$ and $\Gamma_{ABC}(k)$ are given by (1.3) and (1.5) with $E = ABC$ respectively.

As consequence of above theorem we point out the following result of interest.

Corollary 4.4. Let M be a given matrix of form (4.1), such that $A = I$ with $\mathrm{ind}(BC) = r$ and $\mathrm{ind}(D) = s$.

If $BD = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} Z_{BC}(-1) & Z_{BC}(0)B \\ D^\pi C Z_{BC}(0) & D^\pi C Z_{BC}(1)B \end{pmatrix} + \sum_{k=1}^{s} \begin{pmatrix} 0 & 0 \\ D^k D^\pi C Z_{BC}(k) & D^k D^\pi C Z_{BC}(k+1)B \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & 0 \\ (D^D)^{k+2} C\Gamma_{BC}(k+1) & -(D^D)^{k+2} C\Gamma_{BC}(k)B \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ D^D[2C\Gamma_{BC}(1) - C(BC)^\pi] & D^D - D^D C Z_{BC}(0)B \end{pmatrix}, \end{aligned}$$

where $Z_{BC}(k)$ and $\Gamma_{BC}(k)$ are given by (1.3) and (1.5) with $E = BC$ respectively.

Theorem 4.4. Let M be a given matrix of form (4.1), such that $A^2 = A$ with $\mathrm{ind}(ABC) = r$. If $CA = C$, $BDC = 0$ and $D^2C = 0$, then

$$\begin{aligned} M^D &= \begin{pmatrix} AZ_{ABC}(0) + BCZ_{ABC}(1) & -[AZ_{ABC}(0) + BCZ_{ABC}(1)]BD^D \\ CZ_{ABC}(0) + DCZ_{ABC}(1) & -[CZ_{ABC}(0) + DCZ_{ABC}(1)]BD^D \end{pmatrix} \\ &+ \sum_{k=1}^{s} \begin{pmatrix} 0 & [AZ_{ABC}(k) + BCZ_{ABC}(k+1)]BD^{k-1}D^\pi \\ 0 & [CZ_{ABC}(k) + DCZ_{ABC}(k+1)]BD^{k-1}D^\pi \end{pmatrix} \\ &+ \sum_{k=0}^{r} (-1)^k \begin{pmatrix} 0 & [\Gamma_{ABC}(k+1) - BC\Gamma_{ABC}(k)]B(D^D)^{k+3} \\ 0 & [C\Gamma_{ABC}(k+1) - DC\Gamma_{ABC}(k)]B(D^D)^{k+3} \end{pmatrix} \\ &+ \begin{pmatrix} \Gamma_{ABC}(1) + A^\pi - A^\pi BCZ_{ABC}(0) & 0 \\ C\Gamma_{ABC}(1) - C - (DC - CBC)Z_{ABC}(0) & I \end{pmatrix} \begin{pmatrix} 0 & B(D^D)^2 \\ 0 & D^D \end{pmatrix}, \end{aligned}$$

where $Z_{ABC}(k)$ and $\Gamma_{ABC}(k)$ are given by (1.3) and (1.5) with $E = ABC$ respectively.

The following results are straightforward applications of Theorem 4.4.

Corollary 4.5. ([3]) Let M be a given matrix of form (4.1), such that $A = I$, $D = 0$, with $\mathrm{ind}(BC) = r$. Then

$$M^D = \begin{pmatrix} Z_{BC}(-1) & Z_{BC}(0)B \\ CZ_{BC}(0) & CZ_{BC}(1)B \end{pmatrix},$$

where $Z_{BC}(k)$ is given by (1.3) with $E = BC$.

Corollary 4.6. Let M be a given matrix of form (4.1), such that $A^2 = A$ and $D = 0$, with $\mathrm{ind}(ABC) = r$. If $CA = C$, then

$$M^D = \begin{pmatrix} AZ_{ABC}(0) + BCZ_{ABC}(1) & AZ_{ABC}(1)B + BCZ_{ABC}(2)B \\ CZ_{ABC}(0) & CZ_{ABC}(1)B \end{pmatrix},$$

where $Z_{ABC}(k)$ is given by (1.3) with $E = ABC$.

Remark: The condition $AB = B$ (or $CA = C$) in the above results can also be weakened by $CA^\pi B = 0$, in this case, by Lemma 2.4, we can deduce the representations of M^D by a similar approach. For example, we can split M as $M = P + Q$, where $P = \begin{pmatrix} A & AB \\ C & D \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$. Since $A^2 = A$ and $CA^\pi B = 0$, therefore $PQ = 0$ and $Q^2 = 0$. Hence

$$M^D = P^D + Q(P^D)^2,$$

where P^D can be computed by Theorem 4.1 or Theorem 4.2.

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Five-Order Algorithms for Solving Laplace's Steklov Eigenvalue on Polygon by Mechanical Quadrature Methods*

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Abstract

By the potential theory, Steklov eigenvalue problems of Laplace equation on polygon are converted into boundary integral equations(BIEs). In this paper, the singularities at corners and in the integral kernels are studied to obtain five order approximate solution. Firstly,a sin transformation is used to deal with the boundary condition. Secondly, a Sidi's quadrature formula is introduced to approximate the logarithmic singularity integral operator with $O(h^3)$ approximate accuracy order. Then a similar approximate equation is also constructed for the logarithmic singular operator, which is based on coarse grid with mesh width $2h$. So an extrapolation algorithm is applied to approximate the logarithmic operator and the accuracy order is improved to $O(h^5)$. Moreover, the accuracy order is based on fine grid h. Furthermore, an asymptotic expansion with odd powers of the errors is presented with convergence rate $O(h^5)$. The efficiency of the algorithms is illustrated by the example.

Keywords:Laplace's equation, mechanical quadrature method, singularity, eigenvalue

2000 MSC: 65N25, 65N38

1 Introduction

The Steklov eigenvalue problems[1,2] of Laplace equation on polygon are defined as follows:

$$\begin{cases} \triangle u = 0, \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u, \text{ on } \Gamma, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a piecewise smooth boundary Γ and $\Gamma(= \cup_{m=1}^{d} \Gamma_m)$ is a closed curve, $\partial/(\partial n)$ is an outward normal derivative on Γ, and λ is an eigenvalue.

The problem arises from many applications, e.g. the free membranes and heat flow problems, the analysis of stability of mechanical oscillators, and the study of vibration modes of a structure interaction. Andreev and Todorov[3], Armentano and Padra[4] and Hadjesfandiari and Dargush[5] studied finite element methods and carried out the error estimation. Liu and Ortiz[6] provided finite difference methods and Tao-methods. Tang, Guan and Han[7] derived boundary element methods for smooth boundary Γ and the accuracy orders of their approximation are $O(h^2)$. Huang and Lü[8] constructed the mechanical quadrature methods(MQMs) with the accuracy orders $O(h^3)$ for smooth boundary Γ.

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By means of the potential theory, Eq.(1) will be transformed into general eigenvalue problems of boundary integral equations(BIEs) as follow[9,10,11]:

$$\alpha_i(y)u(y) - \int_\Gamma k^*(y,x)u(x)ds_x = \int_\Gamma h^*(y,x)\frac{\partial u(x)}{\partial n_x}ds_x, \quad y \in \Gamma_i, \tag{2}$$

where $\alpha_i(y) = \theta(y)/(2\pi)$ is related to the interior angle $\theta(y)$ of Ω at $y \in \Gamma_i$, especially when y is on a smooth part of the boundary Γ, $\alpha_i(y) = 1/2$, $h^*(y,x) = -1/(2\pi)\ln r$ is the fundamental solution and $k^*(y,x) = \partial h^*(y,x)/\partial n_x$ with $r = |x-y|$.

The $k^*(y,x)$ is a smooth integral kernel and $h^*(y,x)$ is a logarithmic singular kernel in Eq.(2), moreover, $\partial u/\partial n$ is discontinuous with singularity at the corners. Quadrature method[12,13] is presented for solving the boundary integral equation, where the generation of the discrete matrixes does not require any calculations of singular integrals. The logarithmic integral kernel is approximated by extrapolation algorithms derived from Sidi's quadrature rule. An asymptotic expansion[14,15] about the error is obtained with convergence rate $O(h^5)$.

Note that the five order approximate solution is obtained directly and is based on fine grid h. Although there are some papers[15−18] also obtain the same accuracy order, there are three main priority for our paper: firstly, the accuracy orders are based on fine grid; secondly, because the accuracy order is not derived from the extrapolation algorithms but from the linear equations directly, so there are not any errors generated from the extrapolation algorithms of approximate solution; finally, when an linear equation with n order is solved, there are n approximate eigenvector solutions u_h can be obtained on boundary Γ with accuracy order $O(h^5)$, while not $n/2$ values from extrapolation algorithms.

The left terms in Eq.(2) are smooth integrals and the right hand side term is characterized as a logarithmic singularity. Various numerical methods have been proposed for dealing with the singularity, such as Galerkin methods in Stephan and Wendland[19], Chandler[20], Sloan and Spence[14], and Amini and Nixon[5], collocation methods in Elschner and Graham[21] and Yan[22], quadrature methods in Sidi and Israeli[13], Saranen[23,24], Huang and Lü[17,18] and combined Trefftz methods in Li[25].

This paper is organized as follows: In Section 2 the singularities of the solutions at corners and in integral kernels are removed by $\sin^p$-transformation. In Section 3 the mechanical quadrature methods(MQMs) combined with extrapolation algorithms are described to obtain an asymptotic expansion of the eigenvalues. In Section 4 numerical example is given to show the significance of the algorithms.

2 Singularity of integral kernels and solutions

Suppose $\Gamma = \cup_{m=1}^d \Gamma_m (d > 1)$ be a closed polygonal curve, and $\Gamma_m (m = 1, \ldots, d)$ are the piecewise smooth curves. Define the boundary integral operators on Γ_m as following:

$$(K_{qm}u_m)(y) = -\frac{1}{2\pi}\int_{\Gamma_m}\frac{\partial u_m(x)}{\partial n}\ln|y-x|ds_x, \quad y \in \Gamma_q, \tag{3}$$

$$(\bar{C}_{qm}u_m)(y) = -\frac{1}{2\pi}\int_{\Gamma_m}u_m(x)\frac{\partial \ln|y-x|}{\partial n}ds_x, \quad y \in \Gamma_q. \tag{4}$$

Then Eq.(2) can be converted into an operator equation:

$$(\alpha(y)I - \bar{C})u = \lambda K u \tag{5}$$

where $K = [K_{qm}]_{q,m=1}^d$, $u = (u_1(x), \ldots, u_d(x))^T$, $\bar{C} = [\bar{C}_{qm}]_{q,m=1}^d$, $\alpha(y) = diag(\alpha_1(y), \ldots, \alpha_d(y))$ and $I = diag(I_1, \ldots, I_d)$ with identity operator I_m.

Assume that Γ_m can be described by the parameter mapping: $x_m(s) = (x_{m1}(s), x_{m2}(s))$: $[0,1] \to \Gamma_m$ with $|x'_m(s)| = (|(x'_{m1}(s)|^2 + |x'_{m2}(s)|^2)^{1/2} > 0$. For simplicity we assume that the functions $x_m(s)$ are infinitely often differentiable.

The singularity of the solutions and integral kernels in Eq.(5) will be analyzed. Denote the values $0 = s_0 < s_1 < \cdots < s_{d-1}$ corresponding to the corner points $Q_0, Q_1, \ldots, Q_{d-1}, Q_d$ on Γ with $Q_0 = Q_d$, and $(1-\xi_m)\pi$ is the corner Q_m with $\xi_m \in [0,1]$. Based on the potential theory[20,24], the singularities of the $\partial u/\partial n$ at the corner point Q_m have the form $\partial u/\partial n = O(s-s_m)^{\beta_m}$, where $\beta_m = -\xi_m/(\xi_m+1) \geq -1/2$ and s is the arc parameter at Q_m. About the singularity of the integral kernels, we have the following conclusion.

Lemma 1[2]**:** *Suppose $|q-m| = 1$ or $d-1$, then the kernel $\eta_{qm} = \partial \ln |x_m(t) - x_q(\tau)|/\partial n_t$ is at most the Cauchy singularity at corner point $(\tau, t) = (1,0)$ or $(0,1)$.*

In order to remove the singularity at these corner points, a $\sin^p-$ transformation[2,16] is introduced into Eq.(5):

$$s = \varphi_{p+1}(t) : [0,1] \to [0,1],\ p \in N, \tag{6}$$

where $\varphi_{p+1}(t) = \vartheta_{p+1}(t)/\vartheta_{p+1}(1)$ with $\vartheta_{p+1}(t) = \int_0^t (\sin \pi\tau)^{p+1} d\tau$, and the derivative $\varphi'_{p+1}(t) = (\sin \pi t)^{p+1}/\vartheta_{p+1}(1)$ has the zero points with degree $p+1$ at 0 and 1. The operators in Eq.(5) are converted into integral operators on $[0,1]$, Since the Jacobi of the integrals is $|x'(\varphi_{p+1}(\tau))|\varphi'_{p+1}(\tau)$, the Jacobi is divided into two parts with the two kind of combinations: $\eta_{qm} \sin \pi\tau$ and $\partial u_m/(\partial n)(\sin \pi\tau)^p$. When $\tau \to 0$, the constant p can be chosen so that

$$\begin{aligned}
&\lim_{\tau\to 0} \frac{\partial u_m(x_m(\varphi_{p+1}(\tau)))}{\partial n}(\sin \pi\tau)^p = \lim_{\tau\to 0} c(\varphi_{p+1}(\tau) - 0)^{\beta_m}(\sin \pi\tau)^p \\
&= \lim_{\tau\to 0} c\frac{(\vartheta_{p+1}(\tau)/\vartheta_{p+1}(1) - 0)^{\beta_m}}{(\tau-0)^{(p+2)\beta_m}}(\tau-0)^{(p+2)\beta_m}(\sin \pi\tau)^p \\
&= \lim_{\tau\to 0} c(\tau-0)^{(p+2)\beta_m}(\sin \pi\tau)^p = 0,\ \texttt{as}\ (p+2)\beta_m + p > 0,
\end{aligned}$$

where c is a constant, and similar results are obtained when $\tau \to 1$. So the singularity of $\partial u/\partial n$ at corner points has been removed in the integration.

Define the following operators:

$$(C_{qm}w_m)(t) = \int_0^1 c_{qm}(t,\tau)w_m(\tau)d\tau, \quad t \in [0,1], \tag{7}$$

$$(A_{qq}w_q)(t) = \int_0^1 a_{qq}(t,\tau)w_q(\tau)d\tau, \quad t \in [0,1], \tag{8}$$

and

$$(B_{qm}w_m)(t) = \int_0^1 b_{qm}(t,\tau)w_m(\tau)d\tau, \quad t \in [0,1], \tag{9}$$

where $w_m(\tau) = u_m(x_m(\varphi_{p+1}(\tau)))(\sin \pi\tau)^p$,

$$a_{qq}(t,\tau) = -\frac{1}{2\pi}\rho_q(t,\tau)\ln|2e^{-1/2}\sin \pi(t-\tau)|,$$

$$c_{qm}(t,\tau) = -\frac{1}{2\pi}\rho_m(t,\tau)\frac{\partial \ln|x_q(t) - x_m(\tau)|}{\partial n},$$

and

$$b_{qm}(t,\tau) = \begin{cases} -\dfrac{1}{2\pi}\rho_m(t,\tau)\ln|\dfrac{x_q(t)-x_m(\tau)}{2e^{-1/2}\sin \pi(t-\tau)}|,\ q = m, \\ \qquad -\dfrac{1}{2\pi}\rho_m(t,\tau)\ln|x_q(t) - x_m(\tau)|,\ q \neq m, \end{cases}$$

with $x_m(t) = (x_{m1}(\varphi_{p+1}(t)), x_{m2}(\varphi_{p+1}(t)))$ $(m = 1,\ldots,d)$, $|x_q(t) - x_m(\tau)|^2 = (x_{q1}(t) - x_{m1}(\tau))^2 + (x_{q2}(t) - x_{m2}(\tau))^2$, and $\rho_m(t,\tau) = |x'_m(\varphi_p(\tau))|(\sin \pi\tau)\ (\sin \pi t)^p$.

Then Eq.(5) becomes

$$(\alpha(t)I - C)W = \lambda(A+B)W, \tag{10}$$

where $C=[C_{qm}]_{q,m=1}^d$, $W=(w_1,\ldots,w_d)^T$, $A=diag(A_{11},\ldots,A_{dd})$, $B=[B_{qm}]_{q,m=1}^d$. Since $\varphi_{p+1}(t)\in C^\infty[0,1]$ increases monotonously on $[0,1]$ with $\varphi_{p+1}(0)=0$ and $\varphi_{p+1}(1)=1$, the solutions of Eq.(10) are equivalent to those of Eq.(5)[16].

From Lemma 1, we conclude that operator C is a continuous operator, and Huang etc.[18] have proofed that operator B is a continuous operator and A is a logarithmic singular operator.

3 Mechanical quadrature methods

To approximate the integral operators and obtain the discrete equations, a Lemma is introduced:

Lemma 2:[13,14] *Consider the integral* $\int_0^{2\pi}G(x)dx$ *with integral kernel* $G(x)$ *and* $h=2\pi/n$. *Assume that the functions* $g(x)$, $\tilde{g}(x)$ *are* $2m$ *times differentiable on* $[0,2\pi]$. *Also assume that the integral kernel* $G(x)$ *are periodic function with period* 2π. *Then the following conclusion can be drawn:*

(a). If $G(x)=g(x)/(x-t)+\tilde{g}(x)$, *and* $Q_n[G]=h\sum_{j=1,x_j\neq t}^n G(x_j)$, *then*

$$E_n[G]=h[\tilde{g}(t)+g'(t)]+O(h^{2m})\quad \texttt{as}\ \ h\to 0,$$

where $E_n[G]=\int_0^{2\pi}G(x)dx-Q_n[G]$ *in all cases;*

(b). If $G(x)=g(x)(x-t)^s+\tilde{g}(x), s>-1$, *and*

$$Q_n[G]=h\sum_{j=1,x_j\neq t}^{n}G(x_j)+h\tilde{g}(t)-2\zeta(-s)g(t)h^{s+1},$$

then

$$E_n[G]=-2\sum_{\mu=1}^{m-1}\frac{\zeta(-s-2\mu)}{(2\mu)!}g^{(2\mu)}(t)h^{2\mu+s+1}+O(h^{2m}),\quad \texttt{as}\ \ h\to 0;$$

where $\varsigma(t)$ *is the Riemann zeta function.*

(c). If $G(x)=g(x)(x-t)^s\log|x-t|+\tilde{g}(x), s>-1$, *and*

$$Q_n[G]=h\sum_{j=1,x_j\neq t}^{n}G(x_j)+h\tilde{g}(t)+2[\zeta'(-s)-\zeta(-s)\log h]g(t)h^{s+1},$$

then

$$E_n[G]=-2\sum_{\mu=1}^{m-1}[\zeta'(-s-2\mu)-\zeta(-s-2\mu)\log h]\frac{g^{(2\mu)}(t)}{(2\mu)!}h^{2\mu+s+1}$$
$$+O(h^{2m}),\quad \texttt{as}\ \ h\to 0;$$

Especially, when $s=0$, *then* $\zeta'(0)=-(1/2)\log(2\pi)$, *and we have*

$$Q_n[G]=h\sum_{j=1,x_j=t}^{n}G(x_j)+h\tilde{g}(t)+\log\Big(\frac{h}{2\pi}\Big)g(t)h,$$

then

$$E_n[G]=2\sum_{\mu=1}^{m-1}\zeta'(-2\mu)\frac{g^{(2\mu)}(t)}{(2\mu)!}h^{2\mu+1}+O(h^{2m}),\quad \texttt{as}\ \ h\to 0.$$

Let $h_m=1/n_m$ ($n_m\in N, m=1,\ldots,d$ and n_m is supposed to be an even number and so $n_m/2\in N$) be the mesh width and $t_{mj}=\tau_{mj}=(j-1/2)h_m$, ($j=1,\ldots,n_m$) be the nodes on $\Gamma_m(m=1,\ldots,d)$, then $\alpha_m(t_{mj})=1/2$.

Since B_{qm} is a smooth integral operator with period 2π, we obtain a high accuracy approximation when set $g(x) \equiv 0$ in case (a) of Lemma 2:

$$(B^h_{qm}w_m)(t_{qi}) = h_m \sum_{j=1}^{n_m} b_{qm}(t_{qi}, t_{mj})w_m(t_{mj}),\ (q, m = 1, \ldots, d). \tag{11}$$

We have the error bounds[1,2,16]

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h_m^{2l}) \texttt{ for } q = m \texttt{ or } \Gamma_q \cap \Gamma_m = \varnothing, \tag{12}$$

and

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h_m^{\omega}), \quad \Gamma_q \cap \Gamma_m = Q \in \{Q_m\} \tag{13}$$

with $\omega = (p + 2)\beta_m + (p + 1)$. Similar approximate operator C^h_{qm} can be achieved for continuous operator C_{qm}.

We construct the approximate operator J^h_{qq} for the logarithmic singular operators A_{qq} following Lemma 2,

$$(J^h_{qq}w_q)(t) = h_q \sum_{j=0}^{n_q-1} \Im_{qq}(t, t_{qj})w(t_{qj}),$$

with

$$\Im_{qq}(t, s) = \begin{cases} a_{qq}(t, s), & |t - s| \geq h_q, \\ \dfrac{1}{2\pi}\rho_q \ln\Big(\dfrac{h_q}{2\pi}|x'(s)|\Big), & |t - s| < h_q, \end{cases} \tag{14}$$

and $\rho_q(t_{qi}, t_{qj}) = |x'_q(\varphi_p(t_{qj}))|(\sin \pi t_{qj})\ (\sin \pi t_{qi})^p$.

Then we have the error bounds

$$(A_{qq}w_q)(t) - (J^h_{qq}w_q)(t) = \frac{2}{\pi}\sum_{\mu=1}^{l-1} \frac{\zeta'(-2\mu)}{(2\mu)!}(\rho_q w_q)^{(2\mu)}(t)h_q^{2\mu+1} + O(h_q^{2l}). \tag{15}$$

where $\zeta(x)$ is the Riemann Zeta function.

We can find that there is an asymptotic expansion with accuracy order $O(h_q^3)$ for the logarithmic singular operator. In order to improve the accuracy order from $O(h_q^3)$ to $O(h_q^5)$, a coarse grid $2h_q = 2\pi/(n/2) = 4\pi/n$ is obtained. The approximate operator based on coarse grid $2h$ is shown as:

$$(J^{2h}_{qq}w_q)(t) = 2h_q \sum_{j=0}^{n_q-1} \Im_{qq}(t, t_{qj})w(t_{qj})\vartheta_j, \tag{16}$$

where

$$\vartheta_j = \begin{cases} 0, & \text{j is an odd number,} \\ 1, & \text{j is an even number,} \end{cases}$$

and when $t_{qj} = t$ and j is an even number, then $\Im_{qq}(t, t_{qj}) = \frac{1}{2\pi}\rho_q \ln\Big(\frac{2h_q}{2\pi}|x'(t)|\Big)$.

The error estimate is:

$$\begin{aligned}(A_{qq}w_q)(t) - (J^{2h}_{qq}w_q)(t) &= \frac{2}{\pi}(2h_q)^3\frac{\varsigma'(-2)}{2!}(\rho_q w_q)^{(2)}(t) \\ &+2\sum_{\mu=2}^{l-1} \frac{\varsigma'(-2\mu)}{(2\mu)!}(\rho_q w_q)^{(2\mu)}(t)(2h_q)^{2\mu+1} + O((2h_q)^{2l}).\end{aligned} \tag{17}$$

An extrapolation algorithm is used to counteract the item $O(h_q^3)$ in Eqs (15) and (17):

$$(A^h_{qq}w_q)(t) = \frac{8}{7}(J^h_{qq}w_q)(t) - \frac{1}{7}(J^{2h}_{qq}w_q)(t). \tag{18}$$

The error for the approximate operator will be improved from $O(h_q^3)$ to $O(h_q^5)$:

$$(A_{qq}w_q)(t) - (A_{qq}^h w_q)(t) = \sum_{\mu=2}^{m-1} \eta_{q\mu}(t) h_q^{2\mu+1} + O(h_q^{2m}), \tag{19}$$

where $\eta_{q\mu}$ is some coefficients combination of the item $h_q^{2\mu+1}$. So the accuracy order is not only improved to $O(h_q^5)$, but also built on the fine grid h_q.

So we obtain the discrete equations of Eq.(10)

$$(\frac{1}{2}I - C_h)W_h = \lambda_h(A_h + B_h)W_h, \tag{20}$$

where

$$\begin{aligned}
&W_h = (w_1^h(t_{11}), \ldots, w_1^h(t_{1n_1}), \ldots, w_d^h(t_{d1}), \ldots, w_d^h(t_{dn_d}))^T, \\
&A_h = diag(A_{11}^h, \ldots, A_{dd}^h), \quad A_{qq}^h = [\Im\Im_{qq}(t_{qi}, t_{qj})]_{i,j=1}^n, \\
&B_h = [B_{qm}^h]_{q,m=1}^n, \quad B_{qm}^h = [b_{qm}(t_{qi}, t_{mj})]_{i,j=1}^n, \\
&C_h = [C_{qm}^h]_{q,m=1}^n, \quad C_{qm}^h = [c_{qm}(t_{qi}, t_{mj})]_{i,j=1}^n,
\end{aligned}$$

where $\Im\Im_{qq}(t_{qi}, t_{qj})$ is the combination of $\Im_{qq}(t_{qi}, t_{qj})$ derived from Eq.(18), especially, when i is an odd number for t_{qi}, the ϑ_j will be changed to $1 - \vartheta_j$ in Eq.(16) to construct the extrapolation algorithm.

According to logarithmic capacity theory[14], the eigenvalue of both C and C_h are without 1/2, then the Eqs.(10), and (20) can be rewritten as follows: find γ and $W \in V^{(0)}$ satisfying

$$LW = ((1/2)I - C)^{-1}(A + B)W = \gamma W, \tag{21}$$

and find γ_h and W_h satisfying

$$L^h W_h = ((1/2)I - C_h)^{-1}(A_h + B_h)W_h = \gamma_h W_h, \tag{22}$$

where $\gamma = 1/\lambda$, and $\gamma_h = 1/\lambda_h$, and the space $V^{(m)} = (C^{(m)}[0, 2\pi])^d$, $m = 0, 1, 2, \ldots$.

Since γ is an isolated eigenvalue, the dimension of its eigenvectors is finite[9] and the conjugate complex $\bar{\gamma}$ is also an eigenvalue of the conjugate operator $\bar{L}$. Let the space $\bar{V}_\gamma$ =span$\{\bar{W}_{(1)}, \cdots, \bar{W}_{(\chi)}\}$ and V_γ =span$\{W_{(1)}, \cdots, W_{(\chi)}\}$ be the eigenspace of $\bar{L}$ and L respectively, which construct the biorthogonal system

$$\langle W_{(i)}, \bar{W}_{(j)} \rangle = \delta_{ij}, \quad i, j = 1, \cdots, \chi. \tag{23}$$

Let γ_h, $V_{\gamma h} = \{W_{(i)h}\}, (i = 1, 2, \ldots)$ be the eigenvalue and the eigenspace of L^h, and $\bar{\gamma}_h$, $\bar{V}_{\gamma h} = \{\bar{W}_{(i)h}\}$ be the eigenvalue and the eigenspace of $\bar{L}^h$, which satisfies the following normalized conditions

$$\begin{cases} \langle W_{(i)h}, \bar{W}_{(j)h} \rangle = \delta_{ij}, & i, j = 1, 2, \ldots \\ \langle W_{(i)h}, \bar{W}_{(i)} \rangle = 1, & i = 1, 2, \ldots \end{cases} \tag{24}$$

Lemma 3[2,9]. *The approximate operator sequence $\{L^h\}$ is the asymptotically compact sequence and convergence to L in $V^{(0)}$, i,e.*

$$L^h \stackrel{a.c}{\to} L, \tag{25}$$

where $\stackrel{a.c}{\to}$ shows the asymptotically compact convergence.

Theorem 1. *Suppose $\varphi(s) \in V^{(3)}$, and p is large enough such that $\omega \geq 5$. Then we have the following asymptotic expansion:*

$$(L^h - L)\varphi(s) = \sum_{k=1}^{d} h_k^5 \psi_k(s) + o(h_0^5), \tag{26}$$

where $\psi_k(s) = (\psi_{k1}(s), \ldots, \psi_{kd}(s))^T \in V^{(1)}, k = 1, \cdots, d$, are functions independent of h, and $h_0 = \max\{h_1, \ldots, h_d\}$.

Proof. Because A is weak logarithmic singular operator and B, C are continuous operators, when $\omega \geq 5$, we have

$$(A + B - A_h - B_h)\varphi(s) = diag(h_1^5, \ldots, h_d^5)\bar{\omega} + o(h_0^5), \tag{27}$$

and

$$(C - C_h)\varphi(s) = o(h_0^5), \tag{28}$$

where $\bar{\omega}^T = (\bar{\omega}_1, \ldots, \bar{\omega}_d)^T$ and $\bar{\omega}_m = \zeta'(-2)\varphi''(s)$.

$$\begin{aligned}(L^h - L)\varphi(s) &= ((1/2)I - C)^{-1}(A + B - A_h - B_h)\varphi(s) \\ &+ ((1/2)I - C)^{-1}(C - C^h)((1/2)I - C_h)^{-1}(A_h + B_h)\varphi(s) \\ &= ((1/2)I - C)^{-1} diag(h_1^5, \ldots, h_d^5)\bar{\omega}(s) + o(h_0^5).\end{aligned} \tag{29}$$

Suppose $((1/2)I - C)^{-1} = (M_{ij})_{i,j=1}^d$, then

$$(L^h - L)\varphi(s) = \Big(\sum_{j=1}^{d} h_j^5 M_{1j}\bar{\omega}_j(s), \ldots, \sum_{j=1}^{d} h_j^5 M_{dj}\bar{\omega}_j(s)\Big)^T + o(h_0^5). \tag{30}$$

when set $\psi_{kj}(s) = M_{kj}\bar{\omega}_j(s), (k, j = 1, \ldots, d)$, we obtain the conclusion. □

Theorem 2. *Suppose the polygon $\Gamma = \bigcup_{m=1}^d \Gamma_m$ are piecewise smooth and p is chosen such that $\omega \geq 5$, and $(\lambda_{(i)}, W_{(i)})$ and $(\lambda_{(i)h^{(0)}}, W_{(i)h^{(0)}})$ are the eigenvalue of Eq.(10) and Eq.(20) respectively. Then there exist constants $a_{(i)m}$, independent of $h^{(0)} = (h_1, \ldots, h_d)$, such that*

$$\lambda_{(i)h^{(0)}} - \lambda_{(i)} = \sum_{m=1}^{d} a_{(i)m} h_m^5 + o(h_0^5), \tag{31}$$

where $\lambda_{(i)}$ is the $i-th$ eigenvalue.

Proof. From Theorem 1, we obtain

$$\begin{aligned}&L^h(W_{(i)} + \sum_{k=1}^{d} v_k h_k^5) - (\gamma_{(i)} + \sum_{k=1}^{d} a_{(i)k} h_k^5)(W_{(i)} + \sum_{k=1}^{d} v_k h_k^5) \\ &= (L^h - L)W_{(i)} + \sum_{k=1}^{d} h_k^5(L^h v_k - a_{(i)k}W_{(i)} - \gamma_{(i)}v_k) - \sum_{k=1}^{d} a_{(i)k} h_k^5 \sum_{k=1}^{d} v_k h_k^5 \\ &= \sum_{k=1}^{d} h_k^5(L^h v_k - a_{(i)k}W_{(i)} - \gamma_{(i)}v_k + \psi_k) + o(h_0^5).\end{aligned} \tag{32}$$

Choose the constant $a_{(i)k}$ and function v_k satisfy the following operator equations:

$$\begin{cases} L^h v_k - \gamma_{(i)}v_k = a_{(i)k}W_{(i)} - \psi_k,\ k = 1, \ldots, d \\ \langle a_{(i)k}W_{(i)} - \psi_k, \phi\rangle = 0, \forall\phi \in \bar{V}_{\gamma^{(i)}}^{\perp}. \end{cases} \tag{33}$$

Obviously, under the restriction, there exists a unique solution v_k in Eq.(33). Taking $\phi = \bar{W}_{(i)}$, we obtain $a_{(i)k} = \langle \psi_k, \bar{W}_{(i)} \rangle$.

Thus, Eq.(32) is converted to be

$$L^h(W_{(i)} + \sum_{k=1}^{d} v_k h_k^5) - (\gamma_{(i)} + \sum_{k=1}^{d} a_{(i)k} h_k^5)(W_{(i)} + \sum_{k=1}^{d} v_k h_k^5) = o(h_0^5).$$

Since $\{\gamma_{(i)h^{(0)}}, W_{(i)h^{(0)}}\}$ satisfies $L^h W_{(i)h^{(0)}} = \gamma_{(i)h^{(0)}} W_{(i)h^{(0)}}$, we obtain

$$L^h(W_{(i)h^{(0)}} - W_{(i)} - \sum_{k=1}^{d} v_k h_k^5) - \gamma_{(i)h^{(0)}}(W_{(i)h^{(0)}} - W_{(i)} - \sum_{k=1}^{d} v_k h_k^5)$$

$$-(\gamma_{(i)h^{(0)}} - \gamma_{(i)} - \sum_{k=1}^{d} a_{(i)k} h_k^5)(W_{(i)} + \sum_{k=1}^{d} v_k h_k^5) = o(h_0^5). \quad (34)$$

Taking the inner product on the both sides of Eq.(34) by $\bar{W}_{(i)h^{(0)}}$ and using the results of Eqs.(23),(24), we obtain

$$\gamma_{(i)h^{(0)}} - \gamma_{(i)} - \sum_{k=1}^{d} a_{(i)k} h_k^5 = o(h_0^5). \quad (35)$$

According to the relationship of λ and γ, we obtain the conclusion. □

4 Numerical example

Suppose $e_i^{h^{(0)}} = |\lambda_{(i)h^{(0)}} - \lambda_{(i)}|$ be the errors, $\tilde{e}_i^{h^{(0)}} = |\lambda^*_{(i)h^{(0)}} - \lambda_{(i)}|$ be the errors after SEAs, $r_i^{h^{(0)}} = e_i^{h^{(0)}}/e_i^{h^{(0)}/2}$, and $\bar{r}_i^{h^{(0)}} = \tilde{e}_i^{h^{(0)}}/\tilde{e}_i^{h^{(0)}/2}$.

Example 1 : We carry out the numerical experiments for singularity problem as shown in Fig.1, where $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 = \{(\cos \pi s, \sin \pi s) : 0 \leq s \leq 1\}$, and $\Gamma_2 = \{(a \cos \pi(s + 1), b \sin \pi(s + 1)) : 0 \leq s \leq 1, a = 1, b = 1/2\}$. There is no singularity for $\partial u/\partial n$ at the corners $(\pm 1, 0)$ since $\beta_m = 0$. The eigenvalues r_1, r_2, r_3 are calculated in paper[2] without sin^p-transformation in table 1.

Table 1. The errors of $i-$th eigenvalue without sin^p-transformation.

n	$(2^3, 2^3)$	$(2^4, 2^4)$	$(2^5, 2^5)$	$(2^6, 2^6)$	$(2^7, 2^7)$	$(2^8, 2^8)$
$e_1^{h^{(0)}}$	4.755E-3	1.871E-3	6.53E-4	2.102E-4	6.334E-5	1.761E-5
$r_1^{h^{(0)}}$		$2^{1.35}$	$2^{1.52}$	$2^{1.64}$	$2^{1.73}$	$2^{1.85}$
$e_2^{h^{(0)}}$	5.780E-3	1.810E-3	5.333E-4	1.510E-4	4.131E-5	1.068E-5
$r_2^{h^{(0)}}$		$2^{1.68}$	$2^{1.76}$	$2^{1.82}$	$2^{1.87}3.655$	$2^{1.95}$
$e_3^{h^{(0)}}$	1.337E-2	7.282E-4	3.180E-4	1.859E-4	6.933E-5	2.14E-5
$r_3^{h^{(0)}}$		$2^{4.20}$	$2^{1.20}$	$2^{0.7748}$	$2^{1.42}$	$2^{1.70}$

Because of the singularities in integral kernels and low smoothness about u, from Table 1, we can numerically see that the accuracy is low. Moreover, the convergent rates are slowly and ruleless.

To overcome the shortcoming, these eigenvalues are calculated in paper[2] by taking three order methods with $s = \varphi_3(t)$. The splitting extrapolation results are also listed in Table 2.

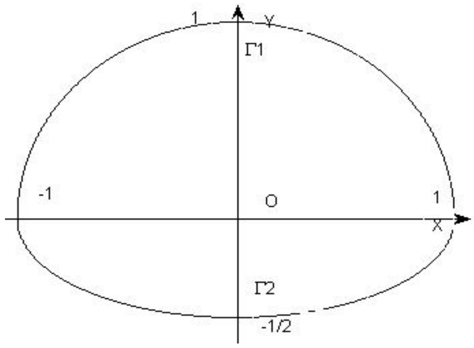

Figure 1: The singularity of the kernels at $(\pm 1, 0)$.

Table 2. The errors of i-th eigenvalue with $\sin^3$-transformation.

n	$(2^3,2^3)$	$(2^4,2^4)$	$(2^5,2^5)$	$(2^6,2^6)$	$(2^7,2^7)$
$e_1^{h^{(0)}}$	9.701E-3	1.183E-3	1.450E-4	1.805E-5	2.254E-6
$r_1^{h^{(0)}}$		$2^{3.04}$	$2^{3.03}$	$2^{3.01}$	$2^{3.00}$
$\bar{r}_1^{h^{(0)}}$		$2^{4.55}$	$2^{4.63}$	$2^{4.78}$	$2^{4.90}$
$e_2^{h^{(0)}}$	2.023E-2	2.437E-3	2.990E-4	3.723E-5	4.649E-6
$r_2^{h^{(0)}}$		$2^{3.05}$	$2^{3.03}$	$2^{3.01}$	$2^{3.00}$
$\bar{r}_2^{h^{(0)}}$		$2^{4.32}$	$2^{4.28}$	$2^{4.76}$	$2^{4.89}$
$e_3^{h^{(0)}}$	1.664E-1	2.017E-2	2.504E-3	3.099E-4	3.866E-5
$r_3^{h^{(0)}}$		$2^{3.04}$	$2^{3.03}$	$2^{3.01}$	$2^{3.00}$
$\bar{r}_3^{h^{(0)}}$		$2^{4.43}$	$2^{8.23}$	$2^{4.83}$	$2^{4.89}$

Next, we will calculated the same eigenvalues with our new method. When $s = \varphi_5(t)$ is selected, so the ω equal to 6 and the results is listed in Table 3.

Table 3. The errors of $i-$ th eigenvalue without sin^p-transformation.

n	$(2^3,2^3)$	$(2^4,2^4)$	$(2^5,2^5)$	$(2^6,2^6)$	$(2^7,2^7)$
$e_1^{h^{(0)}}$	4.185E-5	1.189E-6	3.507E-8	1.072E-9	3.329E-11
$r_1^{h^{(0)}}$		35.2	33.9	32.7	32.2
$e_2^{h^{(0)}}$	5.913E-5	1.709E-6	4.983E-8	1.501E-9	4.632E-11
$r_2^{h^{(0)}}$		34.6	34.3	33.2	32.4
$e_3^{h^{(0)}}$	1.591E-4	4.533E-6	1.357E-7	4.137E-9	1.285E-10
$r_3^{h^{(0)}}$		35.1	33.4	32.8	32.2

Evidently, from Table 3, we can numerically see $\log_2 r_i^{h^{(0)}} \approx 5$ which agree with the theorem 2 very well. Comparing with Table 2, our method is advanced for the numerical accuracy and the convergence rate.

Concluding remarks

In this paper, the mechanical quadrature methods are used to solve Steklov eigenvalue problems on polygons. The following conclusions can be drawn:

1. The eigenvectors in paper[2] are hardly to improve the numerical accuracy by splitting extrapolation algorithms. So the eigenvectors calculated with the new method will own higher numerical accuracy

2. In this paper we only discuss the MQMs for problems on polygon. It can be viewed as the first step toward the efficient solution of boundary value problems on very general singular problems such as for notches, cracks, bi-material problems, and so on.

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Infinite families of integral graphs

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In this article, we shall give the adjacency characteristic polynomial, Laplacian characteristic polynomial and the signless Laplacian characteristic polynomial of the join-based compositions of arbitrary graphs in terms of the corresponding characteristic polynomial of $G_i (i = 1, 2, 3)$. These characterizations allow us to exhibit many infinite families of integral graphs.

1 Introduction

Throughout this article, all graphs considered are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G and $D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Given an $n \times n$ matrix M, denoted by

$$P_A(M; x) = det(xI_n - M)$$

the $A-$ characteristic polynomial of M, where I_n is the identity matrix of size n. The adjacency eigenvalues of G, denoted by $\lambda_1(G) \leq \lambda_2(G) \leq ... \leq \lambda_n(G)$ are called the A-spectrum of G. Similarly, the eigenvalue of $L(G)$ and $Q(G)$, denoted by $0 = \mu_1(G) \leq \mu_2(G) \leq ... \leq \mu_n(G)$ and $\nu_1(G) \leq \nu_2(G) \leq ... \leq \nu_n(G)$ respectively, are called the $L-$ spectrum and $Q-$ spectrum of G accordingly.

A graph G is $A-$ integral $(respectively, L-integral, Q-integral)$ if the spectrum of $A(G)(respectively,\ L(G), Q(G))$ consists only of integers[2, 3]. Some work on these lines pertaining to the class of trees is in [9]. Moreover, several graph operations such as cartesian product, strong sum and product on integral graphs can be used for constructing infinite families of integral graphs[1]. For some other work see [4, 5, 6] and also the references cited therein.

Definition 1 The union of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$.

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Definition 2 The complete product or join of graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2.

In [6], the signless Laplacian characteristic polynomials for graphs $G_1 \vee G_2$ and $G_1 \vee (G_2 \cup G_3)$, where G_i is r_i regular, for $i = 1, 2$ and 3. In this paper, our main result(Section 3) is the description of the the adjacency (Laplacian, signless Laplacian) characteristic polynomial of the join of arbitrary graphs in terms of their characteristic polynomial. This description is then used to show that there exist a number of infinite families of integral graphs. In Section 4, we further describe the adjacency (Laplacian, signless Laplacian) characteristic polynomial of the join of an arbitrary graph with the union of two arbitrary graphs, and provide further infinite families of integral graphs.

2 Preliminaries

In this section, we determine the characteristic polynomials of graphs with the help of the *coronal* of a matrix. The $M-coronal$ $T_M(x)$ of an $n \times n$ matrix M is defined $[7, 8]$ to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is $T_M(x) = j_n^T(xI_n - M)^{-1}j_n$ where j_n denotes the column vector of dimension n with all the entries equal one.

It is well known $[7, Proposition 2]$ that, if M is an $n \times n$ matrix with each row sum equal to a constant t, then

$$T_M(x) = \frac{n}{x-t}. \tag{1}$$

3 Integral graphs of joins of graphs

In this section we shall give the $A-(L-, Q-)$characteristic polynomial for graphs $G_1 \vee G_2$ in terms of the corresponding characteristic polynomial of $G_i (i = 1, 2)$ and give necessary and sufficient conditions for the join of two graphs to be $A-(L-, Q-)$ integral.

3.1 A-integral graphs of joins of graphs

Theorem 3.1 Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Also let $T_{A_i}(\lambda)(i = 1, 2)$ be the A_i- coronal of G_i. Then the characteristic polynomials of the matrix $A(G_1 \vee G_2)$ is

$$P_A(G_1 \vee G_2, x) = P_A(G_1, x)P_A(G_2, x)(1 - T_{A(G_2)}(x)T_{A(G_1)}(x)).$$

Proof With a proper labeling of vertices, the adjacency characteristic poly-

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nomial of $G_1 \vee G_2$ is given by

$$\begin{aligned} P_A(G_1 \vee G_2) &= det\begin{pmatrix} xI_{n_1} - A(G_1) & -j_{n_1\times n_2} \\ -j_{n_2\times n_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= det(xI_{n_2} - A(G_2))det(B) \\ &= P_{A(G_2)}(x)det(B) \end{aligned}$$

where

$$B = xI_{n_1} - A(G_1) - j_{n_1\times n_2}(xI_{n_2} - A(G_2))^{-1}j_{n_2\times n_1}$$

is the Schur complement of $xI_{n_2} - A(G_2)$. Thus, the result follows from

$$\begin{aligned} detB &= det(xI_{n_1} - A(G_1) - j_{n_1\times n_2}(xI_{n_2} - A(G_2))^{-1}j_{n_2\times n_1}) \\ &= det(xI_{n_1} - A(G_1) - T_{A(G_2)}(x)j_{n_1\times n_1}) \\ &= det(xI_{n_1} - A(G_1) - T_{A(G_2)}(x)j_{n_1}^T adj(xI_{n_1} - A(G_1))j_{n_1}) \\ &= det(xI_{n_1} - A(G_1))(1 - T_{A(G_2)}(x)j_{n_1}^T(xI_{n_1} - A(G_1))^{-1}j_{n_1}) \\ &= det(xI_{n_1} - A(G_1))(1 - T_{A(G_2)}(x)T_{A(G_1)}(x)) \end{aligned}$$

where $adj(M)$ is the adjugate matrix of M.

Hence, the adjacency characteristic polynomial of $G_1 \vee G_2$ is

$$P_A(G_1 \vee G_2, x) = P_A(G_1, x)P_A(G_2, x)(1 - T_{A(G_2)}(x)T_{A(G_1)}(x))$$

as desired.

Theorem 3.1 implies the following result

Corollary 3.2 For $i = 1, 2$, let G_i be a r_i regular graph with n_i vertices. Then the characteristic polynomials of the matrix $A(G_1 \vee G_2)$ is

$$P_A(G_1 \vee G_2, x) = \frac{P_A(G_1, x)P_A(G_2, x)}{(x - r_1)(x - r_2)} f(x)$$

where $f(x) = x^2 - (r_1 + r_2)x + r_1r_2 - n_2n_1$.

Corollary 3.3 Let G_i be a complete bipartite graph K_{p_i,q_i} with n_i vertices. Then the characteristic polynomials of the matrix $A(G_1 \vee G_2)$ is

$$P_A(G_1 \vee G_2, x) = \frac{P_A(G_1, x)P_A(G_2, x)}{(x^2 - p_1q_1)(x^2 - p_2q_2)} g(x)$$

where $g(x) = x^4 - (p_1q_1 + p_2q_2 + n_1n_2)x^2 - 2(n_1p_2q_2 + n_2p_1q_1)x - 3p_1p_2q_1q_2$.

Corollary 3.4 Let $G_i(i = 1, 2)$ be a r_i regular graph with n_i vertices and $G_j(j = 1, 2)$ be a complete bipartite graph K_{p_j,q_j} with n_j vertices. Then the characteristic polynomials of the matrix $A(G_i \vee G_j)$ is

$$P_A(G_i \vee G_j, x) = \frac{P_A(G_i, x)P_A(G_j, x)}{(x - r_i)(x^2 - p_jq_j)} h(x)$$

where $h(x) = x^3 - r_ix^2 - (n_in_j + p_jq_j)x + r_ip_jq_j - 2n_ip_jq_j$.

Up till now, many infinite families of integral graphs are generated by using graph operations (for example, [1, 6]). Now, we can use the join of two graphs to obtain many new classes of integral graphs.

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Theorem 3.5 (a) For $i = 1, 2$, let G_i be a r_i regular graph with n_i vertices. The graph $G_1 \vee G_2$ is $A-$ integral if and only if G_1 and G_2 are $A-$ integral and $f(x)$ are integral.

(b) For $i = 1, 2$, Let G_i be a complete bipartite graph K_{p_i,q_i} with n_i vertices. The graph $G_1 \vee G_2$ is $A-$ integral if and only if G_1 and G_2 are $A-$ integral and $g(x)$ are integral.

(c) Let $G_i (i = 1, 2)$ be a r_i regular graph with n_i vertices and $G_j (j = 1, 2)$ be a complete bipartite graph K_{p_j,q_j} with n_j vertices. The graph $G_1 \vee G_2$ is $A-$ integral if and only if G_1 and G_2 are $A-$ integral and $h(x)$ are integral.

3.2 L-integral graphs of joins of graphs

Theorem 3.6 Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Also let $T_{L_i}(x)(i = 1, 2)$ be the L_i- coronal of G_i. Then the characteristic polynomials of the matrix $L(G_1 \vee G_2)$ is

$$P_L(G_1 \vee G_2, x) = P_L(G_1, x - n_2)P_L(G_2, x - n_1)(1 - T_{L(G_2)}(x - n_1)T_{L(G_1)}(x - n_2)).$$

Proof With a proper labeling of vertices, the Laplacian characteristic polynomial of $G_1 \vee G_2$ is given by

$$\begin{aligned} P_L(G_1 \vee G_2) &= det \begin{pmatrix} xI_{n_1} - L(G_1) - n_2 I_{n_1} & j_{n_1 \times n_2} \\ j_{n_2 \times n_1} & xI_{n_2} - L(G_2) - n_1 I_{n_2} \end{pmatrix} \\ &= det((x - n_1)I_{n_2} - L(G_2))det(B) \\ &= P_{L(G_2)}(x - n_1)det(B) \end{aligned}$$

where $B = xI_{n_1} - L(G_1) - n_2 I_{n_1} - j_{n_1 \times n_2}((x - n_1)I_{n_2} - L(G_2))^{-1} j_{n_2 \times n_1}$ is the Schur complement of $xI_{n_2} - L(G_2) - n_1 I_{n_2}$. The result follows from

$$\begin{aligned} detB &= det(xI_{n_1} - L(G_1) - n_2 I_{n_1} - j_{n_1 \times n_2}((x - n_1)I_{n_2} - L(G_2))^{-1} j_{n_2 \times n_1}) \\ &= det((x - n_2)I_{n_1} - L(G_1) - T_{L(G_2)}(x - n_1)j_{n_1 \times n_1}) \\ &= det((x - n_2)I_{n_1} - L(G_1) - T_{L(G_2)}(x - n_1)j_{n_1}^T adj((x - n_2)I_{n_1} - L(G_1)j_{n_1})) \\ &= det((x - n_2)I_{n_1} - L(G_1))(1 - T_{L(G_2)}(x - n_1)j_{n_1}^T((x - n_2)I_{n_1} - L(G_1))^{-1} j_{n_1}) \\ &= det((x - n_2)I_{n_1} - L(G_1))(1 - T_{L(G_2)}(x - n_1)T_{L(G_1)}(x - n_2)) \end{aligned}$$

Hence, the Laplacian characteristic polynomial of $G_1 \vee G_2$

$$P_L(G_1 \vee G_2, x) = P_L(G_1, x - n_2)P_L(G_2, x - n_1)(1 - T_{L(G_2)}(x - n_1)T_{L(G_1)}(x - n_2))$$

as desired.

Theorem 3.6 implies the following result.

Corollary 3.7 Let G_i be any graph with n_i vertices. Then the characteristic polynomials of the matrix $L(G_1 \vee G_2)$ is

$$P_L(G_1 \vee G_2, x) = \frac{P_L(G_1, x)P_L(G_2, x)}{(x - n_1)(x - n_2)} f(x)$$

where $f(x) = x^2 - (n_1 + n_2)x$.

Theorem 3.8 For $i = 1, 2$, let G_i be a graph with n_i vertices. The graph $G_1 \vee G_2$ is $L-$ integral if and only if G_1 and G_2 are $L-$ integral.

3.3 Q-integral graphs of joins of graphs

Theorem 3.9 Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Also let $T_{Q_i}(\lambda)(i=1,2)$ be the Q_i- coronal of G_i. Then the signless Laplacian characteristic polynomials of the matrix $Q(G_1 \vee G_2)$ is

$$P_Q(G_1 \vee G_2, x) = P_Q(G_1, x-n_2)P_Q(G_2, x-n_1)(1 - T_{Q(G_2)}(x-n_1)T_{Q(G_1)}(x-n_2)).$$

Proof With a proper labeling of vertices, the signless Laplacian matrix of $G_1 \vee G_2$ can be written as

$$Q = Q(G_1 \vee G_2) = \begin{pmatrix} Q(G_1) + n_2 I_{n_1} & j_{n_1 \times n_2} \\ j_{n_2 \times n_1} & Q(G_2) + n_1 I_{n_2} \end{pmatrix}$$

The result refines the arguments used to prove Theorem 3.6.

Again, by applying (1), Theorem 3.9 implies the following result.

Corollary 3.10 For $i = 1, 2$, let G_i be a r_i regular graph with n_i vertices. The signless Laplacian characteristic polynomials of the matrix $P(G_1 \vee G_2)$ is

$$P_Q(G_1 \vee G_2, x) = \frac{P_Q(G_1, x-n_2)P_Q(G_2, x-n_1)}{(x-2r_1-n_2)(x-2r_2-n_1)} f(x)$$

where $f(x) = x^2 - (2(r_1+r_2) + (n_1+n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

Similarly, we can compute the characteristic polynomials of the matrix $Q(G_1 \vee G_2)$, when G_1 and G_2 are complete bipartite graphs.

Corollary 3.11 Let G_i be a complete bipartite graph K_{p_i,q_i} with n_i vertices. Then the signless Laplacian characteristic polynomials of the matrix $P(G_1 \vee G_2)$ is

$$P_Q(G_1 \vee G_2, x) = \frac{P_Q(G_1, x-n_2)P_Q(G_2, x-n_1)}{[(x-n_1)^2 - (x-n_1)n_2)][(x-n_2)^2 - (x-n_2)n_1)]} g(x)$$

where $g(x) = x^4 - 3(n_1+n_2)x^3 + (3(n_1+n_2)^2 - n_2)x^2 - (4n_1n_2(n_1+n_2) + (n_1^3 + n_2^3) - n_2(p_1-q_1)^2 - n_1(p_2-q_2)^2)$.

Corollary 3.12 Let $G_i(i=1,2)$ be a r_i regular graph with n_i vertices and $G_j(j=1,2)$ be a complete bipartite graph K_{p_j,q_j} with n_j vertices. Then the signless Laplacian characteristic polynomials of the matrix $G_i \vee G_j$ is

$$P_Q(G_i \vee G_j, x) = \frac{P_Q(G_1, x-n_2)P_Q(G_2, x-n_1)}{(x-n_i-2r_j)[(x-n_j)^2 - (x-n_j)(p_i+q_j)]} h(x)$$

where $h(x) = x^3 - 2(n_i + r_j + n_j)x^2 + ((n_i+n_j)^2 + 2n_ir_j)x - n_2(p_i-q_i)^2 + 2r_j + n_i^2 + 2n_ir_j)$.

The Theorem below gives necessary and sufficient conditions for the join of two $Q-$ integral graphs to be $Q-$ integral. The result of Theorem 3.13(a) had been obtained the Corollary 2.1 in [6]. Here for completeness, we give the complete proof but in a different way and give the two new classes $Q-$integral graph.

Theorem 3.13 (a) For $i = 1, 2$, let G_i be a r_i regular graph with n_i vertices. The graph $G_1 \vee G_2$ is $Q-$ integral if and only if G_1 and G_2 are $Q-$ integral and $f(x)$ are integral.

(b) For $i = 1, 2$, Let G_i be a complete bipartite graph K_{p_i,q_i} with n_i vertices. The graph $G_1 \vee G_2$ is $Q-$ integral if and only if G_1 and G_2 are $Q-$ integral and $g(x)$ are integral.

(c)Let $G_i(i = 1, 2)$ be a r_i regular graph with n_i vertices and $G_j(j = 1, 2)$ be a complete bipartite graph K_{p_j,q_j} with n_j vertices. The graph $G_1 \vee G_2$ is $Q-$ integral if and only if G_1 and G_2 are $Q-$ integral and $h(x)$ are integral.

4 Integral graphs of the joins of graphs with the union of graphs

In this section, we give $A-(L-, Q-)$ characteristic polynomial of a graph obtained from three arbitrary graphs by union and join and build many infinite families of $A-(L-, Q-)$integral graphs.

4.1 A-integral graphs of the joins of graphs with the union of graphs

Theorem 4.1 Let $G_i(i = 1, 2, 3)$ be three graphs on n_i vertices. Also let $T_{A_i}(x)(i = 1, 2, 3)$ be the A_i- coronal of G_i. Then the characteristic polynomials of the matrix $A(G_1 \vee (G_2 \cup G_3))$ is

$$\begin{aligned} P_A(G_1 \vee (G_2 \cup G_3)) &= P_A(G_1, x)P_A(G_2, x)P_A(G_3, x)(1 - T_{A(G_3)}(x)T_{A(G_1)}(x) \\ &\quad -T_{A(G_3)}(x)T_{A(G_1)}(x)) \end{aligned}$$

Proof With a proper labeling of vertices, the adjacency characteristic polynomial of $A(G_1 \vee (G_2 \cup G_3))$ is given by

$$\begin{aligned} P_A(G_1 \vee (G_2 \cup G_3)) &= det\begin{pmatrix} xI_{n_1} - A(G_1) & -j_{n_1\times n_2} & -j_{n_1\times n_3} \\ -j_{n_2\times n_1} & xI_{n_2} - A(G_2) & 0_{n_2\times n_3} \\ -j_{n_3\times n_1} & 0_{n_3\times n_2} & xI_{n_3} - A(G_3) \end{pmatrix} \\ &= det(xI_{n_3} - A(G_3))det(B) \\ &= P_{A(G_3)}(x)det(B) \end{aligned}$$

where

$$B = \begin{pmatrix} xI_{n_1} - A(G_1) & -j_{n_1\times n_2} \\ -j_{n_2\times n_1} & xI_{n_2} - A(G_2) \end{pmatrix} - \begin{pmatrix} -j_{n_1\times n_3} \\ 0_{n_2\times n_3} \end{pmatrix} (xI_{n_3} - A(G_3))^{-1} \begin{pmatrix} -j_{n_3\times n_1} & 0_{n_3\times n_2} \end{pmatrix}$$

is the Schur complement of $xI_{n_3} - A(G_3)$

Thus, the result follows from

$$\begin{aligned} detB &= det\begin{pmatrix} xI_{n_1} - A(G_1) - T_{A(G_3)}(x)j_{n_1\times n_1} & -j_{n_1\times n_2} \\ -j_{n_2\times n_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= det(xI_{n_2} - A(G_2))det(M) \\ &= P_{A(G_2)}(x)det(M) \end{aligned}$$

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and

$$\begin{aligned} detM &= det(xI_{n_1} - A(G_1) - T_{A(G_3)}(x)j_{n_1\times n_1} - j_{n_1\times n_2}(xI_{n_2} - A(G_2))^{-1}j_{n_2\times n_1})\} \\ &= det(xI_{n_1} - A(G_1) - T_{A(G_3)}(x)j_{n_1}^T adj(xI_{n_1} - A(G_1))j_{n_1}) \\ &\quad -T_{A(G_2)}(x)j_{n_1}^T adj(xI_{n_1} - A(G_1))j_{n_1}) \\ &= det(xI_{n_1} - A(G_1))(1 - T_{A(G_3)}(x)T_{A(G_1)}(x) - T_{A(G_2)}(x)T_{A(G_1)}(x)) \end{aligned}$$

Hence, the characteristic polynomial of $A(G_1 \vee (G_2 \cup G_3))$ is

$$\begin{aligned} P_A(G_1 \vee (G_2 \cup G_3)) &= P_A(G_1, x)P_A(G_2, x)P_A(G_3, x)(1 - T_{A(G_3)}(x)T_{A(G_1)}(x) \\ &\quad -T_{A(G_2)}(x)T_{A(G_1)}(x)) \end{aligned}$$

as desired.

Theorem 4.1 implies the following result.

Corollary 4.2 Let $G_i(i = 1, 2, 3)$ be a r_i regular graph with n_i vertices. Then the characteristic polynomials of the matrix $A(G_1 \vee (G_2 \cup G_3)$ is

$$P_A(G_1 \vee (G_2 \cup G_3)) = \frac{P_A(G_1, x)P_A(G_2, x)P_A(G_3, x)}{(x - r_1)(x - r_2)(x - r_3)} f(x)$$

where $f(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_1r_3 - n_1n_3 - n_2n_1)x + n_1n_3r_2 + n_1n_2r_3 - r_1r_2r_3$.

Corollary 4.3 Let G_1 be a complete bipartite graph K_{p_1,q_1} with n_1 vertices, $G_i(i = 2, 3)$ be a r_i- regular graph with n_i vertices. Then the characteristic polynomials of the matrix $A(G_1 \vee (G_2 \cup G_3))$ is

$$P_A(G_1 \vee (G_2 \cup G_3)) = \frac{P_A(G_1, \lambda)P_A(G_2, \lambda)P_A(G_3, \lambda)}{(x^2 - p_1q_1)(x - r_2)(x - r_3)} g(x)$$

where $g(x) = x^4 - (r_2 + r_3)x^3 + (r_2r_3 - p_1q_1 - n_1(n_2 + n_3))x^2 + (p_1q_1(r_2 + r_3) + n_1(n_2r_3 + n_3r_2) - 2p_1q_1(n_2 + n_3))x - p_1q_1r_2r_3 + 2p_1q_1n_2r_3 + 2p_1q_1r_2n_3$.

Theorem 4.4 (a) Let $G_i(i = 1, 2, 3)$ be a r_i regular graph with n_i vertices. The graph $G_1 \vee (G_2 \cup G_3)$ is $A-$ integral if and only if G_1, G_2 and G_3 are $A-$ integral and $f(x)$ are integral.

(b)Let G_1 be a complete bipartite graph K_{p_1,q_1} with n_1 vertices, $G_i(i = 2, 3)$ be a r_i- regular graph with n_i vertices. The graph $G_1 \vee (G_2 \cup G_3)$ is $A-$ integral if and only if G_1, G_2 and G_3 are $A-$ integral and $g(x)$ are integral.

4.2 Q-integral graphs of the joins of graphs with the union of graphs

Theorem 4.5 Let $G_i(i = 1, 2, 3)$ be three graphs on n_i vertices. Also let $T_{Q_i}(\lambda)(i = 1, 2, 3)$ be the Q_i- coronal of G_i. Then the signless Laplacian characteristic polynomials of the matrix $Q(G_1 \vee (G_2 \cup G_3))$ is

$$\begin{aligned} P_Q(G_1 \vee (G_2 \cup G_3)) &= P_Q(G_1, x - n_2 - n_3)P_Q(G_2, x - n_1)P_Q(G_3, x - n_1) \\ &\quad (1 - T_{Q(G_3)}(x - n_1)T_{Q(G_1)}(x - n_2 - n_3) \\ &\quad -T_{Q(G_2)}(x - n_1)T_{Q(G_1)}(x - n_2 - n_3)). \end{aligned}$$

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Proof With a proper labeling of vertices, the signless Laplacian characteristic polynomial of $Q(G) = Q(G_1 \vee (G_2 \cup G_3))$ is given by

$$\begin{aligned} P_Q(G) &= det\begin{pmatrix} xI_{n_1} - Q(G_1) - (n_2+n_3)I_{n_1} & -j_{n_1\times n_2} & -j_{n_1\times n_3} \\ -j_{n_2\times n_1} & xI_{n_2} - Q(G_2) - n_1 I_{n_2} & 0_{n_2\times n_3} \\ -j_{n_3\times n_1} & 0_{n_3\times n_2} & xI_{n_3} - Q(G_3) - n_1 I_{n_3} \end{pmatrix} \\ &= det(xI_{n_3} - Q(G_3) - n_1 I_{n_3})det(B) \\ &= P_{Q(G_3)}(x-n_1)det(B) \end{aligned}$$

where

$$\begin{aligned} B &= \begin{pmatrix} xI_{n_1} - Q(G_1) - (n_2+n_3)I_{n_1} & -j_{n_1\times n_2} \\ -j_{n_2\times n_1} & xI_{n_2} - Q(G_2) - n_1 I_{n_2} \end{pmatrix} - \begin{pmatrix} -j_{n_1\times n_3} \\ 0_{n_2\times n_3} \end{pmatrix} \\ &\quad ((x-n_1)I_{n_3} - Q(G_3))^{-1} \begin{pmatrix} -j_{n_3\times n_1} & 0_{n_3\times n_2} \end{pmatrix} \end{aligned}$$

is the Schur complement of $\lambda I_{n_3} - Q(G_3) - n_1 I_{n_3}$. The result refines the arguments used to prove Theorem 4.1

Hence, the signless Laplacian characteristic polynomial of $G_1 \vee (G_2 \cup G_3)$

$$\begin{aligned} P_Q(G_1 \vee (G_2 \cup G_3)) &= P_Q(G_1, x-n_2-n_3)P_Q(G_2, x-n_1)P_Q(G_3, x-n_1) \\ &\quad (1 - T_{Q(G_3)}(x-n_1)T_{Q(G_1)}(x-(n_2+n_3)) - T_{Q(G_2)} \\ &\quad (x-n_1)T_{Q(G_1)}(x-(n_2+n_3)) \end{aligned}$$

as desired.

Theorem 4.5 implies the following result.

Corollary 4.6 let G_i be a r_i regular graph with n_i vertices. The characteristic polynomials $P_Q(G_1 \vee (G_2 \vee G_3))$ of the matrix $Q(G_1 \vee (G_2 \vee G_3))$ is

$$P_Q(G_1 \vee (G_2 \vee G_3)) = \frac{P_Q(G_1, x-n_2-n_3)P_Q(G_2, x-n_1)P_Q(G_3, x-n_1)}{(x-2r_1-n_2-n_3)(x-2r_2-n_1)(x-2r_3-n_1)} f(x).$$

where $f(x) = x^3 - (2(r_1+r_2+r_3) + 2n_1 + n_2 + n_3)x^2 + ((n_1+n_2+n_3)(n_1 + 2(r_2+r_3)) + 4(r_1(n_1+n_3) + r_2(r_1+r_3)))x - (2n_1(n_1r_1 + n_2r_2 + n_3r_3 + 2r_1(r_2 + r_3)) + 4r_2r_3(2r_1+n_2+n_3))$.

Now we will give the characterization to one case of $Q-$ integral graphs. The result of Theorem 4.7 had been obtained in [6].

Theorem 4.7 Let $G_i(i = 1, 2, 3)$ be a r_i regular graph with n_i vertices. The graph $G_1 \vee (G_2 \cup G_3)$ is $Q-$ integral if and only if G_1, G_2 and G_3 are $Q-$ integral and $f(x)$ are integral.

4.3 L-integral graphs of the joins of graphs with the union of graphs

Theorem 4.8 Let $G_i(i = 1, 2, 3)$ be three graphs on n_i vertices. Also let $T_{L_i}(\lambda)(i = 1, 2, 3)$ be the L_i- coronal of G_i. Then the Laplacian characteristic polynomials of the matrix $L(G_1 \vee (G_2 \cup G_3))$ is

$$\begin{aligned} P_L(G_1 \vee (G_2 \cup G_3)) &= P_L(G_1, x-n_2-n_3)P_L(G_2, x-n_1)P_L(G_3, x-n_1) \\ &\quad (1 - T_{L(G_3)}(x-n_1)T_{L(G_1)}(x-n_2-n_3) \\ &\quad -T_{L(G_2)}(x-n_1)T_{L(G_1)}(x-n_2-n_3)) \end{aligned}$$

Proof With a proper labeling of vertices, the Laplacian matrix of $G_1 \vee (G_2 \cup G_3)$ can be written as

$$L(G_1 \vee (G_2 \cup G_3)) = \begin{pmatrix} L(G_1) + (n_2+n_3)I_{n_1} & -j_{n_1 \times n_2} & -j_{n_1 \times n_3} \\ -j_{n_2 \times n_1} & L(G_2) + n_1 I_{n_2} & 0_{n_2 \times n_3} \\ -j_{n_3 \times n_1} & 0_{n_3 \times n_2} & L(G_3) + n_1 I_{n_3} \end{pmatrix}$$

The result refines the arguments used to prove Theorem 3.5.

Again, by applying (1), Theorem 4.8 implies the following result.

Corollary 4.9 Let G_i be any graph with n_i vertices. Then the Laplacian characteristic polynomials of the matrix $G_1 \vee (G_2 \cup G_3)$ is

$$P_L(G_1 \vee (G_2 \cup G_3), x) = \frac{P_L(G_1, x - n_2 - n_3) P_L(G_2, x - n_1)(G_3, x - n_1)}{(x - n_1)(x - n_2 - n_3)} f(x)$$

where $f(x) = x^2 - (n_1 + n_2 + n_3)x$.

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Gradient Superconvergence Post-processing of the Tetrahedral Quadratic Finite Element *

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Abstract

In this article, we will apply the superconvergence patch recovery (SPR) technique to the tetrahedral quadratic finite element over fully uniform meshes. First, the supercloseness property of the gradients of the finite element solution u_h and the interpolant u_I is given. We then analyze a SPR scheme to obtain the recovered gradient from the finite element solution at the internal vertices over the partitions. Finally, we show that this recovered gradient is superconvergent to that of the exact solution u.

Keywords: tetrahedral finite element; SPR; post-processing; superconvergence

Mathematics Subject Classification (2000): 65N30

I. INTRODUCTION

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic; see, for example, Babuška and Strouboulis [1], Chen [2], Chen and Huang [3], Lin and Yan [4], Wahlbin [5], Zhu and Lin [6], and Zhu [7] for overviews of this field. Nevertheless, how to obtain the superconvergent numerical solution is an issue to researchers. In general, it needs to use post-processing techniques to get recovered gradients with high order accuracy from the finite element solution. Usual post-processing techniques include interpolation technique, projection technique, average technique, extrapolation technique, SPR technique introduced by Zienkiewicz

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and Zhu [8–10] and polynomial patch recovery (PPR) technique raised by Zhang and Naga [11]. In previous works, except for [8–10], there are some researches on the SPR technique, such as [12–18] and so on. As for the tetrahedral linear element, Chen and Wang [12] obtained the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm by using SPR. Brandts and Křížek [19] obtained by using the interpolation technique the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm. Using the L^2-projection technique, in the average sense of the L^2-norm, Chen [20] got the recovered gradient with $\mathcal{O}(h^{1+min(\sigma,\frac{1}{2})})$ order accuracy. Goodsell [21] derived by using the average technique the pointwise superconvergence estimate of the recovered gradient with $\mathcal{O}(h^{2-\varepsilon})$ order accuracy. As for the tetrahedral quadratic element, using the interpolation post-processing technique, Brandts and Křížek [22] obtained the recovered gradient with $\mathcal{O}(h^3)$ order accuracy in the average sense of the L^2-norm. This article will discuss by using the SPR technique the superconvergence of the recovered gradient from the tetrahedral quadratic finite element solution at the internal vertices.

In this article, we shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

II. DISCRETIZATION OF THE FINITE ELEMENT

Suppose $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. We consider a general variable coefficient second-order elliptic problem

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{3} a_i\partial_i u + a_0 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We also assume that the given functions $a_{ij}, a_i \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. In addition, we write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$, which are usual partial derivatives. Thus, the variational formulation of (2.1) is

$$a(u\,,\,v) = (f\,,\,v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u\,,\,v) \equiv \int_\Omega (\sum_{i,j=1}^{3} a_{ij}\partial_i u\partial_j v + \sum_{i=1}^{3} a_i\partial_i uv + a_0 uv)\,dxdydz$$

and

$$(f\,,\,v) = \int_\Omega fv\,dxdydz.$$

To discretize the problem (2.2), one proceeds as follows. The domain Ω is firstly partitioned into cubes of side h, and each of these is then subdivided into six tetrahedra (see Fig. 1). We denote by $\mathcal{T}^h$ this partition.

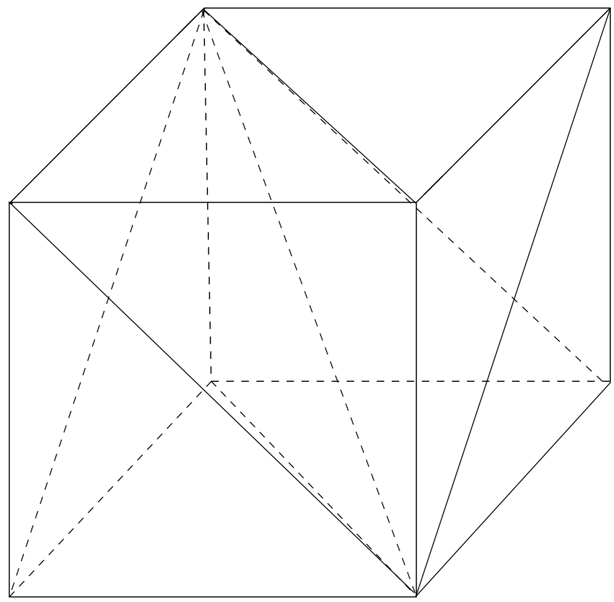

FIG. 1. Tetrahedral partition

For this fully uniform mesh of tetrahedral elements, let $S_0^h(\Omega) \subset H_0^1(\Omega)$ be the piecewise quadratic tetrahedral finite element space, and $u_I \in S_0^h(\Omega)$ the Lagrange interpolant to the solution u of (2.2).

Discretizing (2.2) using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\,v) = (f\,,\,v)$ for all $v \in S_0^h$. Here u_h is a finite element approximation to u. Thus we have the following result (see [23])

Lemma 2.1. *Let $\{\mathcal{T}^h\}$ be a uniform family of tetrahedral partitions of Ω and $u \in W^{4,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h the tetrahedral quadratic finite element approximation, and u_I the corresponding interpolant to u, the solution of (2.2). Then we have the supercloseness estimate*

$$|u_h - u_I|_{1,\infty,\Omega} \leq Ch^3 \,|\ln h|^{\frac{4}{3}}\, \|u\|_{4,\infty,\Omega}. \tag{2.3}$$

III. SPR TECHNIQUE AND SUPERCONVERGENCE

In this section, we consider a gradient recovery scheme by using the SPR technique for $v \in S_0^h(\Omega)$ at the internal vertices of the elements over the partitions and denote by R_h this recovery operator. In addition, we denote by R_x, R_y and R_z the recovery operators of x-derivative, y-derivative and z-derivative, respectively. Thus $R_h = (R_x, R_y, R_z)$.

Suppose N is an internal vertex of the element $e \in \mathcal{T}^h$, and denote by ω the element patch around N containing 24 tetrahedra (see Fig. 2).

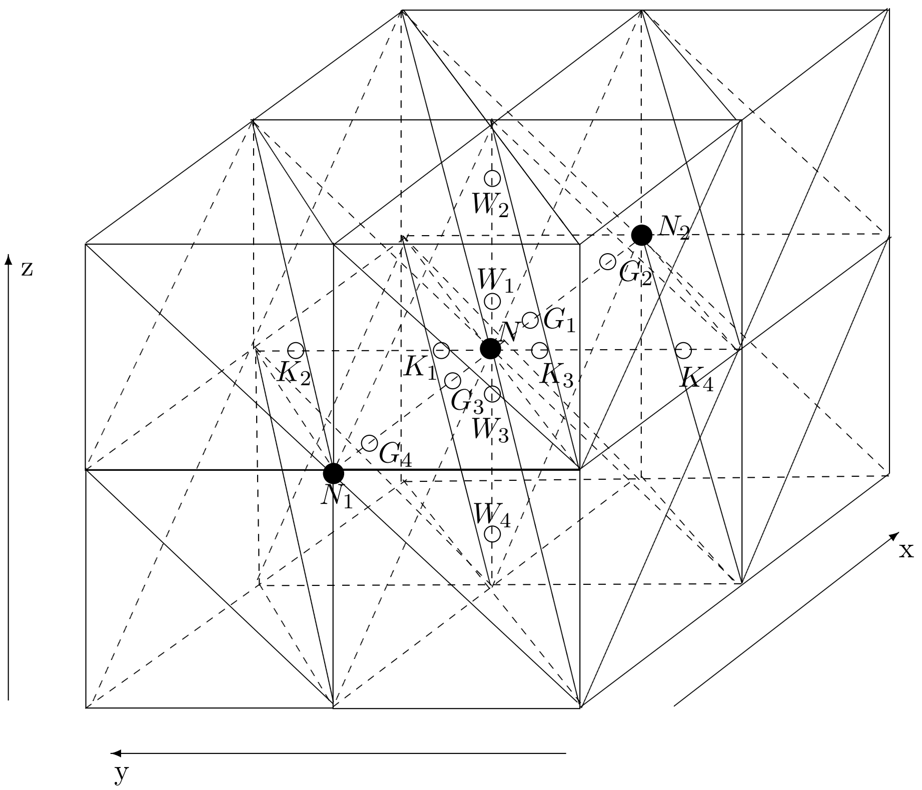

FIG. 2. The element patch around N containing 24 tetrahedra

Under the local coordinate system centered N, we choose Gauss points G_i, $i = 1, \cdots, 4$ in the direction x as sample points to recover the x-derivative of $v \in S_0^h(\Omega)$ at the point N. Clearly, the y-derivative and the z-derivative of v at the point N can be recovered similarly by choosing Gauss points K_i, $i = 1, \cdots, 4$ in the direction y and Gauss points W_i, $i = 1, \cdots, 4$ in the direction z as sample points, respectively. We denote by ω_x the line segment N_1N_2 going through the Gauss points G_i, $i = 1, \cdots, 4$ (see Fig. 2). SPR uses the discrete least-squares fitting to seek quadratic function $p \in P_2(\omega_x)$, such that

$$|||p - \partial_1 v||| = \min_{q \in P_2(\omega_x)} |||q - \partial_1 v|||, \tag{3.1}$$

where $v \in S_0^h(\Omega)$ and $|||w|||^2 = \sum_{i=1}^4 |w(G_i)|^2$. The problem (3.1) is equivalent to

$$\sum_{i=1}^{4} [p(G_i) - \partial_1 v(G_i)] q(G_i) = 0 \quad \forall\, q \in P_2(\omega_x). \tag{3.2}$$

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We define $R_x v(N) = p(0,0,0)$ and call it a x-derivative recovered value of v at the point N. It is easy to prove $|||R_x v||| = |||p||| \leq |||\partial_1 v|||$ (see [7]). Then the following Lemma 3.1 holds.

Lemma 3.1. *Let ω be the element patch around an internal vertex N, and $u \in W^{4,\infty}(\omega)$. For $u_I \in S_0^h(\Omega)$ the interpolant to u, we have*

$$|\partial_1 u(N) - R_x u_I(N)| \leq Ch^3 \|u\|_{4,\infty,\omega}, \tag{3.3}$$

if $u \in W^{5,\infty}(\omega)$, we have

$$|\partial_1 u(N) - R_x u_I(N)| \leq Ch^4 \|u\|_{5,\infty,\omega}. \tag{3.4}$$

Proof. For $q \in P_3(\omega)$, we have $\partial_1 q \in P_2(\omega_x)$ and $\partial_1 q(G_i) = \partial_1 q_I(G_i)$, $i = 1, \cdots, 4$. Thus $R_x q = R_x q_I$. Moreover, $R_x q = \partial_1 q$. Therefore we have $\partial_1 q = R_x q_I$, that is

$$\partial_1 q - R_x q_I = 0 \ \text{ in } \omega_x, \ \forall\, q \in P_3(\omega). \tag{3.5}$$

Thus

$$\begin{aligned} |\partial_1 u(N) - R_x u_I(N)| &= |\partial_1 (u-q)(N) - R_x (u-q)_I(N)| \\ &\leq |\partial_1 (u-q)(N)| + |R_x (u-q)_I(N)| \\ &\leq \|\partial_1 (u-q)\|_{0,\infty,\omega} + \|R_x (u-q)_I\|_{0,\infty,\omega_x}. \end{aligned} \tag{3.6}$$

Using the norm equivalence of the finite-dimensional space and the inverse property, we have

$$\begin{aligned} \|R_x (u-q)_I\|_{0,\infty,\omega_x} &\leq C|||R_x (u-q)_I||| \\ &\leq C|||\partial_1 (u-q)_I||| \\ &\leq C\, \|\partial_1 (u-q)_I\|_{0,\infty,\omega} \\ &\leq Ch^{-1}\, \|u-q\|_{0,\infty,\omega}. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$|\partial_1 u(N) - R_x u_I(N)| \leq \|\partial_1 (u-q)\|_{0,\infty,\omega} + Ch^{-1}\, \|u-q\|_{0,\infty,\omega}. \tag{3.8}$$

Let $\Pi_3 u$ be an interpolant of degree three to u. Choosing $q = \Pi_3 u$ in (3.8), we have by the interpolation error estimate

$$|\partial_1 u(N) - R_x u_I(N)| \leq Ch^3 \|u\|_{4,\infty,\omega},$$

which is the result (3.3). In addition, we need to consider that q is a four-degree monomial. Set $q = x^i y^j z^k$, $i + j + k = 4$, where i, j, k are non-negative integers. When $0 \leq i \leq 3$, by the arguments similar to the proof of the result (3.3) we can verify

$$\partial_1 q - R_x q_I = 0 \ \text{ in } \omega_x. \tag{3.9}$$

When $q = x^4$, we easily obtain

$$\partial_1 q(N) = R_x q(N) = R_x q_I(N) = 0. \tag{3.10}$$

From (3.5), (3.9) and (3.10), we have

$$\partial_1 q(N) - R_x q_I(N) = 0 \;\forall\, q \in P_4(\omega). \tag{3.11}$$

If $u \in W^{5,\infty}(\omega)$, let q be an interpolant of degree four to u in (3.8), we obtain the result (3.4) by using the interpolation error estimate. Thus we complete the proof of the Lemma 3.1. □

Lemma 3.2. *For $u_h \in S_0^h(\Omega)$ the tetrahedral quadratic finite element approximation to $u \in W^{4,\infty}(\Omega)$, the solution of (2.2), R_x the x-derivative recovered operator defined by (3.1), and N an internal vertex of the element e over the uniform partition. we have the superconvergent estimate*

$$|\partial_1 u(N) - R_x u_h(N)| \le Ch^3 |\ln h|^{\frac{4}{3}} \|u\|_{4,\infty,\Omega}. \tag{3.12}$$

Proof. Using the triangle inequality and the norm equivalence of the finite-dimensional space, we have

$$\begin{aligned} |\partial_1 u(N) - R_x u_h(N)| &\le |R_x(u_h - u_I)(N)| + |\partial_1 u(N) - R_x u_I(N)| \\ &\le \|R_x(u_h - u_I)\|_{0,\infty,\omega_x} + |\partial_1 u(N) - R_x u_I(N)| \\ &\le C|||R_x(u_h - u_I)||| + |\partial_1 u(N) - R_x u_I(N)| \\ &\le C|||\partial_1(u_h - u_I)||| + |\partial_1 u(N) - R_x u_I(N)| \\ &\le C\,|\partial_1(u_h - u_I)|_{0,\infty,\omega} + |\partial_1 u(N) - R_x u_I(N)|\,. \end{aligned} \tag{3.13}$$

Combining (2.3), (3.3) and (3.13) yields the desired result (3.12). □

Similar to the x-derivative recovery operator R_x, we denote by R_y and R_z the y-derivative recovery operator and the z-derivative recovery operator, respectively. With the arguments similar to the proof procedure of the result (3.12), we have the results

$$|\partial_2 u(N) - R_y u_h(N)| \le Ch^3 |\ln h|^{\frac{4}{3}} \|u\|_{4,\infty,\Omega}, \tag{3.14}$$

and

$$|\partial_3 u(N) - R_z u_h(N)| \le Ch^3 |\ln h|^{\frac{4}{3}} \|u\|_{4,\infty,\Omega}. \tag{3.15}$$

From (3.12), (3.14) and (3.15), we immediately obtain the following theorem.

Theorem 3.1. *For $u_h \in S_0^h(\Omega)$ the tetrahedral quadratic finite element approximation to $u \in W^{4,\infty}(\Omega)$, the solution of (2.2), $R_h = (R_x, R_y, R_z)$ the gradient recovered operator, and N an internal vertex of the element e over the uniform partition. Then we have the superconvergent estimate*

$$|\nabla u(N) - R_h u_h(N)| \le Ch^3 |\ln h|^{\frac{4}{3}} \|u\|_{4,\infty,\Omega}. \tag{3.16}$$

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On the Behaviour of Solutions for Some Systems of Difference Equations

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In this paper, we investigate the forms of the solutions of difference equation systems

$$x_{n+1}=\frac{y_{n-2}x_{n-3}y_{n-4}}{y_nx_{n-1}\left(\pm 1\pm y_{n-2}x_{n-3}y_{n-4}\right)},\ y_{n+1}=\frac{x_{n-2}y_{n-3}x_{n-4}}{x_ny_{n-1}\left(\pm 1\pm x_{n-2}y_{n-3}x_{n-4}\right)},$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero. Also we deal with the behavior of the solutions of these systems.

Keywords: system of difference equations; explicit solutions; equilibrium point; periodic solution.
AMS Classification: 39A10, 39A12.

1 Introduction

Since the end of the twentieth century, the theory of discrete dynamical systems and difference equations have gained a great importance. Most of the recent applications of these theories appeared in many scientific areas such as biology, economics, physics, resource management. Especially, nonlinear difference equations and their systems second order and higher order have great importance in applications. Also, there are studies which these equations and their systems appear as discrete analogues and numerical solutions of differential equations modeling some problems in some branches of science. It is very worthy to examine the behavior of solutions of a system of higher-order rational difference equations and to discuss the stability character of their equilibrium

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points. Recently, many researchers have investigated periodic solutions of difference equations or systems and also have suggested some diverse methods for the qualitative behavior of the their solutions. For example see Refs. [1-26]. Tollu et al. [3] investigated the solutions of two special types of Riccati difference equations

$$x_{n+1} = \frac{1}{1+x_n} \text{ and } y_{n+1} = \frac{1}{-1+y_n}$$

such that their solutions are associated with Fibonacci numbers. El-Metwally and Elsayed [4-5] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}\right)}, \quad x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm x_n x_{n-3}\right)}.$$

Ibrahim [6] got the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + b x_n x_{n-2}\right)}.$$

The periodicity of the positive solutions of the rational difference system

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$$

has been studied by Cinar in [14]. In [13], Kurbanli et al. have studied the positive solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In [19], authors studied the dynamical behavior of positive solution for a system of a rational third-order difference equation

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n},\ y_{n+1} = \frac{y_{n-2}}{B + x_{n-2}x_{n-1}x_n}.$$

Touafek et al., in [8], investigated the form of the solutions of the following some difference systems

$$x_{n+1} = \frac{y_n}{x_{n-1}\left(\pm 1 \pm y_n\right)},\ y_{n+1} = \frac{x_n}{y_{n-1}\left(\pm 1 \pm x_n\right)}.$$

Similar nonlinear rational difference systems were investigated; (see[9,12,15-18,20-23]). Our aim in this paper is to get the form of the solutions of the following rational difference systems

$$x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}\left(\pm 1 \pm y_{n-2}x_{n-3}y_{n-4}\right)},\ y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}\left(\pm 1 \pm x_{n-2}y_{n-3}x_{n-4}\right)},$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero.

2 Preliminaries

Let $I_x,\ I_y$ some intervals of real numbers and $f:\ I_x^5 \times I_y^5 \rightarrow I_x,\ g:\ I_x^5 \times I_y^5 \rightarrow I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -4, -3, -2, -1, 0)$, the system of difference equations

$$\left\{ \begin{array}{l} x_{n+1} = f\left(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, y_n, y_{n-1}, y_{n-2}, y_{n-3}, y_{n-4}\right) \\ y_{n+1} = g\left(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, y_n, y_{n-1}, y_{n-2}, y_{n-3}, y_{n-4}\right) \end{array} \right., \ n = 0, 1, 2, \ldots, \tag{1}$$

has a unique solution of $\{(x_n, y_n)\}_{n=-4}^{\infty}$. Also, an equilibrium point of system (1) is a point $(\overline{x}, \overline{y})$ that satisfies

$$\begin{aligned} \overline{x} &= f\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}, \overline{y}, \overline{y}\right), \\ \overline{y} &= f\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}, \overline{y}, \overline{y}\right). \end{aligned}$$

Definition 1 *Assume that $(\overline{x}, \overline{y})$ is equilibrium point of system (1). Then*

i) *An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if for every $\epsilon > 0$, there exist $\delta > 0$ such that for every initial condition $(x_i, y_i) \in I_x \times I_y$ $(i = -4, -3, -2, -1, 0)$ if $\left\| \sum_{i=-4}^{0} \left((x_i, y_i) - (\overline{x}, \overline{y})\right) \right\| < \delta$ implies $\|(x_i, y_i) - (\overline{x}, \overline{y})\| < \epsilon$, for all $n > 0$, where $\|.\|$ is the usual Euclidian norm in $\mathbb{R}^2$.*

ii) *An equilibrium point $(\overline{x}, \overline{y})$ is said to be unstable if it is not stable.*

iii) *An equilibrium point $(\overline{x}, \overline{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\left\| \sum_{i=-4}^{0} \left((x_i, y_i) - (\overline{x}, \overline{y})\right) \right\| < \eta$ and $(x_n, y_n) \rightarrow (\overline{x}, \overline{y})$ as $n \rightarrow \infty$.*

iv) *An equilibrium point $(\overline{x}, \overline{y})$ is called a global attractor if $(x_n, y_n) \rightarrow (\overline{x}, \overline{y})$ as $n \rightarrow \infty$.*

Theorem 1 *[25]Assume that $X(n+1) = F(X(n))$, $n = 0, 1, 2...$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, i.e., $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix J_F, evaluated at $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

Theorem 2 *[26] Assume that $X(n+1) = F(X(n))$, $n = 0, 1, 2...$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\overline{X}$ is $P(\lambda) = a_0\lambda^n + a_1^{n-1}\lambda + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $p(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if $\triangle_k > 0$, for $k = 1, 2, \ldots, n$, where $\triangle_k$ is the principal minor of order k of the $n \times n$ matrix*

$$\triangle_n = \begin{bmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{bmatrix}.$$

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3 The System $x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}(1+y_{n-2}x_{n-3}y_{n-4})}$, $y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}(1+x_{n-2}y_{n-3}x_{n-4})}$

In this section, we study the solutions of the system of difference equations, for $n = 0, 1, 2, \ldots,$

$$x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}\left(1 + y_{n-2}x_{n-3}y_{n-4}\right)},\ y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}\left(1 + x_{n-2}y_{n-3}x_{n-4}\right)}, \tag{2}$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero.

Theorem 3 *Let $\{x_n, y_n\}_{n=-4}^{\infty}$ be solutions of system (2). Then, for $n = 1, 2, \ldots,$*

$$\begin{aligned}
x_{6n-5} &= \frac{Cb^{2n-1}A^{2n-1}}{E^{2n-1}d^{2n-1}} \prod_{i=0}^{2n-2} \frac{1 + iEdC}{1 + (i+1)\,CbA}, \\
y_{6n-5} &= \frac{ca^{2n-1}B^{2n-1}}{D^{2n-1}e^{2n-1}} \prod_{i=0}^{2n-2} \frac{1 + ieDc}{1 + (i+1)\,cBa}, \\
x_{6n-6} &= \frac{e^{2n-1}}{B^{2n-2}} \prod_{i=0}^{2n-2} \frac{1 + iDcB}{1 + ieDc}, \quad y_{6n-6} = \frac{E^{2n-1}}{b^{2n-2}} \prod_{i=0}^{2n-2} \frac{1 + idCb}{1 + iEdC}, \\
x_{6n-7} &= \frac{d^{2n-1}}{A^{2n-2}} \prod_{i=0}^{2n-2} \frac{1 + iCbA}{1 + idCb}, \quad y_{6n-7} = \frac{D^{2n-1}}{a^{2n-2}} \prod_{i=0}^{2n-2} \frac{1 + icBa}{1 + iDcB}, \\
x_{6n-8} &= \frac{ca^{2n-2}B^{2n-2}}{D^{2n-2}e^{2n-2}} \prod_{i=0}^{2n-3} \frac{1 + ieDc}{1 + (i+1)\,cBa}, \\
y_{6n-8} &= \frac{Cb^{2n-2}A^{2n-2}}{E^{2n-2}d^{2n-2}} \prod_{i=0}^{2n-3} \frac{1 + iEdC}{1 + (i+1)\,CbA}, \\
x_{6n-9} &= \frac{E^{2n-2}}{b^{2n-3}} \prod_{i=0}^{2n-3} \frac{1 + idCb}{1 + iEdC}, \quad y_{6n-9} = \frac{e^{2n-2}}{B^{2n-3}} \prod_{i=0}^{2n-3} \frac{1 + iDcB}{1 + ieDc}, \\
x_{6n-10} &= \frac{D^{2n-2}}{a^{2n-3}} \prod_{i=0}^{2n-3} \frac{1 + icBa}{1 + iDcB}, \quad y_{6n-10} = \frac{d^{2n-2}}{A^{2n-3}} \prod_{i=0}^{2n-3} \frac{1 + iCbA}{1 + idCb},
\end{aligned}$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c,\ x_{-1} = d,\ x_0 = e,\ y_{-4} = A,\ y_{-3} = B,\ y_{-2} = C,\ y_{-1} = D,\ y_0 = E.$

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Proof. For $n=1$, the result holds. Now, suppose that $n>1$ and our assumption holds for $n-1$, that is,

$$\begin{aligned}
x_{6n-11} &= \frac{Cb^{2n-3}A^{2n-3}}{E^{2n-3}d^{2n-3}}\prod_{i=0}^{2n-4}\frac{1+iEdC}{1+(i+1)\,CbA},\\
y_{6n-11} &= \frac{ca^{2n-3}B^{2n-3}}{D^{2n-3}e^{2n-3}}\prod_{i=0}^{2n-4}\frac{1+ieDc}{1+(i+1)\,cBa},\\
x_{6n-12} &= \frac{e^{2n-3}}{B^{2n-4}}\prod_{i=0}^{2n-4}\frac{1+iDcB}{1+ieDc},\quad y_{6n-12}=\frac{E^{2n-2}}{b^{2n-3}}\prod_{i=0}^{2n-4}\frac{1+idCb}{1+iEdC},\\
x_{6n-13} &= \frac{d^{2n-3}}{A^{2n-4}}\prod_{i=0}^{2n-4}\frac{1+iCbA}{1+idCb},\quad y_{6n-13}=\frac{D^{2n-3}}{a^{2n-4}}\prod_{i=0}^{2n-4}\frac{1+icBa}{1+iDcB},\\
x_{6n-14} &= \frac{ca^{2n-4}B^{2n-4}}{D^{2n-4}e^{2n-4}}\prod_{i=0}^{2n-5}\frac{1+ieDc}{1+(i+1)\,cBa},\\
y_{6n-14} &= \frac{Cb^{2n-4}A^{2n-4}}{E^{2n-4}d^{2n-4}}\prod_{i=0}^{2n-5}\frac{1+iEdC}{1+(i+1)\,CbA},\\
x_{6n-15} &= \frac{E^{2n-4}}{b^{2n-5}}\prod_{i=0}^{2n-5}\frac{1+idCb}{1+iEdC},\quad y_{6n-15}=\frac{e^{2n-4}}{B^{2n-5}}\prod_{i=0}^{2n-5}\frac{1+iDcB}{1+ieDc},\\
x_{6n-16} &= \frac{D^{2n-4}}{a^{2n-5}}\prod_{i=0}^{2n-5}\frac{1+icBa}{1+iDcB},\quad y_{6n-16}=\frac{d^{2n-4}}{A^{2n-5}}\prod_{i=0}^{2n-5}\frac{1+iCbA}{1+idCb}.
\end{aligned}$$

Firstly, we consider $x_{6n-10}=\frac{y_{6n-13}x_{6n-14}y_{6n-15}}{y_{6n-11}x_{6n-12}(1+y_{6n-13}x_{6n-14}y_{6n-15})}$. Therefore, we can write

$$\begin{aligned}
x_{6n-10} &= \frac{DcB\left(\prod_{i=0}^{2n-4}\frac{1+icBa}{1+iDcB}\right)\left(\prod_{i=0}^{2n-5}\frac{1+ieDc}{1+(i+1)cBa}\right)\left(\prod_{i=0}^{2n-5}\frac{1+iDcB}{1+ieDc}\right)}{\begin{array}{c}\frac{cBa^{2n-3}}{D^{2n-3}}\left(\prod_{i=0}^{2n-4}\frac{1+ieDc}{1+(i+1)cBa}\right)\left(\prod_{i=0}^{2n-4}\frac{1+iDcB}{1+ieDc}\right)\\ \left[1+DcB\left(\prod_{i=0}^{2n-4}\frac{1+icBa}{1+iDcB}\right)\left(\prod_{i=0}^{2n-5}\frac{1+ieDc}{1+(i+1)cBa}\right)\left(\prod_{i=0}^{2n-5}\frac{1+iDcB}{1+ieDc}\right)\right]\end{array}}\\
&= \frac{\frac{DcB}{1+(2n-4)DcB}}{\frac{cBa^{2n-3}}{D^{2n-3}}\left(\prod_{i=0}^{2n-4}\frac{1+iDcB}{1+(i+1)cBa}\right)\left(1+\frac{DcB}{1+(2n-4)DcB}\right)}\\
&= \frac{D^{2n-2}}{a^{2n-3}}\prod_{i=0}^{2n-4}\frac{1+(i+1)\,cBa}{1+iDcB}\left(\frac{1}{1+(2n-3)\,DcB}\right)\\
&= \frac{D^{2n-2}}{a^{2n-3}}\prod_{i=0}^{2n-3}\frac{1+icBa}{1+iDcB}.
\end{aligned}$$

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Secondly, we consider $y_{6n-10} = \frac{x_{6n-13}y_{6n-14}x_{6n-15}}{x_{6n-11}y_{6n-12}(1+x_{6n-13}y_{6n-14}x_{6n-15})}$. Then we can write

$$\begin{aligned}
y_{6n-10} &= \frac{dCb\left(\prod_{i=0}^{2n-4}\frac{1+iCbA}{1+idCb}\right)\left(\prod_{i=0}^{2n-5}\frac{1+iEdC}{1+(i+1)CbA}\right)\left(\prod_{i=0}^{2n-5}\frac{1+idCb}{1+iEdC}\right)}{\begin{array}{c}\frac{CbA^{2n-3}}{d^{2n-3}}\left(\prod_{i=0}^{2n-4}\frac{1+iEdC}{1+(i+1)CbA}\right)\left(\prod_{i=0}^{2n-4}\frac{1+idCb}{1+iEdC}\right)\\ \left[1+dCb\left(\prod_{i=0}^{2n-4}\frac{1+iCbA}{1+idCb}\right)\left(\prod_{i=0}^{2n-5}\frac{1+iEdC}{1+(i+1)CbA}\right)\left(\prod_{i=0}^{2n-5}\frac{1+idCb}{1+iEdC}\right)\right]\end{array}}\\
&= \frac{\frac{dCb}{1+(2n-4)dCb}}{\frac{CbA^{2n-3}}{d^{2n-3}}\left(\prod_{i=0}^{2n-4}\frac{1+idCb}{1+(i+1)CbA}\right)\left(1+\frac{dCb}{1+(2n-4)dCb}\right)}\\
&= \frac{d^{2n-2}}{A^{2n-3}}\prod_{i=0}^{2n-4}\frac{1+(i+1)\,CbA}{1+idCb}\left(\frac{1}{1+(2n-3)\,dCb}\right)\\
&= \frac{d^{2n-2}}{A^{2n-3}}\prod_{i=0}^{2n-3}\frac{1+iCbA}{1+idCb}.
\end{aligned}$$

Similarly one can prove the other relations. The proof is complete. ■

Theorem 4 *System (2) has a unique equilibrium point which is* $(0,0)$ *and this equilibrium point is not locally asymptotically stable.*

Proof. The linearized system of (2) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ x_{n-4} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = x^{10} + x^8 + x^6 - x^4 - x^2 - 1. \tag{3}$$

The roots of $P(\lambda)$ are $\lambda_{1,2} = \pm 1$, $\lambda_3 = \lambda_4 = \left(1 + i\sqrt{3}\right)/2$, $\lambda_5 = \lambda_6 = \left(-1 + i\sqrt{3}\right)/2$, $\lambda_7 = \lambda_8 = \left(1 - i\sqrt{3}\right)/2$, $\lambda_9 = \lambda_{10} = \left(-1 - i\sqrt{3}\right)/2$. There exist some roots of Eq.(3) such that $|\lambda| = 1$ or $|\lambda| > 1$. Hence, the equilibrium point $(0,0)$ of system (2) is unstable. ■

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4 The System $x_{n+1}=\frac{y_{n-2}x_{n-3}y_{n-4}}{y_nx_{n-1}(-1+y_{n-2}x_{n-3}y_{n-4})}$, $y_{n+1}=\frac{x_{n-2}y_{n-3}x_{n-4}}{x_ny_{n-1}(-1+x_{n-2}y_{n-3}x_{n-4})}$

In this section, we study the solutions of the system of difference equations, for $n=0,1,2,\ldots$,

$$x_{n+1}=\frac{y_{n-2}x_{n-3}y_{n-4}}{y_nx_{n-1}\left(-1+y_{n-2}x_{n-3}y_{n-4}\right)},\ y_{n+1}=\frac{x_{n-2}y_{n-3}x_{n-4}}{x_ny_{n-1}\left(-1+x_{n-2}y_{n-3}x_{n-4}\right)}, \tag{4}$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero.

Theorem 5 *Let $\{x_n,y_n\}_{n=-4}^{\infty}$ be solutions of system (4). Then, for $n=1,2,\ldots$,*

$$\begin{aligned}
x_{6n-5} &= \frac{Cb^{2n-1}A^{2n-1}\left(-1+EdC\right)^{n-1}}{d^{2n-1}E^{2n-1}\left(-1+CbA\right)^{n}},\\
y_{6n-5} &= \frac{ca^{2n-1}B^{2n-1}\left(-1+eDc\right)^{n-1}}{D^{2n-1}e^{2n-1}\left(-1+cBa\right)^{n}},\\
x_{6n-6} &= \frac{e^{2n-1}\left(-1+DcB\right)^{n-1}}{B^{2n-2}\left(-1+eDc\right)^{n-1}},\ y_{6n-6}=\frac{E^{2n-1}\left(-1+dCb\right)^{n-1}}{b^{2n-2}\left(-1+EdC\right)^{n-1}},\\
x_{6n-7} &= \frac{d^{2n-1}\left(-1+CbA\right)^{n-1}}{A^{2n-2}\left(-1+dCb\right)^{n-1}},\ y_{6n-7}=\frac{D^{2n-1}\left(-1+cBa\right)^{n-1}}{a^{2n-2}\left(-1+DcB\right)^{n-1}},\\
x_{6n-8} &= \frac{ca^{2n-2}B^{2n-2}\left(-1+eDc\right)^{n-1}}{D^{2n-2}e^{2n-2}\left(-1+cBa\right)^{n-1}},\\
y_{6n-8} &= \frac{Cb^{2n-2}A^{2n-2}\left(-1+EdC\right)^{n-1}}{d^{2n-2}E^{2n-2}\left(-1+CbA\right)^{n-1}},\\
x_{6n-9} &= \frac{E^{2n-2}\left(-1+dCb\right)^{n-1}}{b^{2n-3}\left(-1+EdC\right)^{n-1}},\ y_{6n-9}=\frac{e^{2n-2}\left(-1+DcB\right)^{n-1}}{B^{2n-3}\left(-1+eDc\right)^{n-1}},\\
x_{6n-10} &= \frac{D^{2n-2}\left(-1+cBa\right)^{n-1}}{a^{2n-3}\left(-1+DcB\right)^{n-1}},\ y_{6n-10}=\frac{d^{2n-2}\left(-1+CbA\right)^{n-1}}{A^{2n-3}\left(-1+dCb\right)^{n-1}},
\end{aligned}$$

where $x_{-4}=a,\ x_{-3}=b,\ x_{-2}=c,\ x_{-1}=d,\ x_0=e,\ y_{-4}=A,\ y_{-3}=B,\ y_{-2}=C,\ y_{-1}=D,\ y_0=E$.

Proof. For $n = 1$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$\begin{aligned}
x_{6n-11} &= \frac{Cb^{2n-3}A^{2n-3}\left(-1+EdC\right)^{n-2}}{d^{2n-3}E^{2n-3}\left(-1+CbA\right)^{n-1}}, \\
y_{6n-11} &= \frac{ca^{2n-3}B^{2n-3}\left(-1+eDc\right)^{n-2}}{D^{2n-3}e^{2n-3}\left(-1+cBa\right)^{n-1}}, \\
x_{6n-12} &= \frac{e^{2n-3}\left(-1+DcB\right)^{n-2}}{B^{2n-4}\left(-1+eDc\right)^{n-2}},\ y_{6n-12} = \frac{E^{2n-3}\left(-1+dCb\right)^{n-2}}{b^{2n-4}\left(-1+EdC\right)^{n-2}}, \\
x_{6n-13} &= \frac{d^{2n-3}\left(-1+CbA\right)^{n-2}}{A^{2n-4}\left(-1+dCb\right)^{n-2}},\ y_{6n-13} = \frac{D^{2n-3}\left(-1+cBa\right)^{n-2}}{a^{2n-4}\left(-1+DcB\right)^{n-2}}, \\
x_{6n-14} &= \frac{ca^{2n-4}B^{2n-4}\left(-1+eDc\right)^{n-2}}{D^{2n-4}e^{2n-4}\left(-1+cBa\right)^{n-2}}, \\
y_{6n-14} &= \frac{Cb^{2n-4}A^{2n-4}\left(-1+EdC\right)^{n-2}}{d^{2n-4}E^{2n-4}\left(-1+CbA\right)^{n-2}}, \\
x_{6n-15} &= \frac{E^{2n-4}\left(-1+dCb\right)^{n-2}}{b^{2n-5}\left(-1+EdC\right)^{n-2}},\ y_{6n-15} = \frac{e^{2n-4}\left(-1+DcB\right)^{n-2}}{B^{2n-5}\left(-1+eDc\right)^{n-2}}, \\
x_{6n-16} &= \frac{D^{2n-4}\left(-1+cBa\right)^{n-2}}{a^{2n-5}\left(-1+DcB\right)^{n-2}},\ y_{6n-16} = \frac{d^{2n-4}\left(-1+CbA\right)^{n-2}}{A^{2n-5}\left(-1+dCb\right)^{n-2}}.
\end{aligned}$$

From system (4), we obtain

$$\begin{aligned}
x_{6n-10} &= \frac{y_{6n-13}x_{6n-14}y_{6n-15}}{y_{6n-11}x_{6n-12}\left(-1+y_{6n-13}x_{6n-14}y_{6n-15}\right)} \\
&= \frac{DcB\frac{(-1+cBa)^{n-2}}{(-1+DcB)^{n-2}}\frac{(-1+eDc)^{n-2}}{(-1+cBa)^{n-2}}\frac{(-1+DcB)^{n-2}}{(-1+eDc)^{n-2}}}{\begin{array}{c}\frac{ca^{2n-3}B(-1+eDc)^{n-2}}{D^{2n-3}(-1+cBa)^{n-1}}\frac{(-1+DcB)^{n-2}}{(-1+eDc)^{n-2}} \\ \left(-1+DcB\frac{(-1+cBa)^{n-2}}{(-1+DcB)^{n-2}}\frac{(-1+eDc)^{n-2}}{(-1+cBa)^{n-2}}\frac{(-1+DcB)^{n-2}}{(-1+eDc)^{n-2}}\right)\end{array}} \\
&= \frac{DcB}{\frac{cBa^{2n-3}}{D^{2n-3}}\frac{(-1+DcB)^{n-2}}{(-1+cBa)^{n-1}}\left(-1+DcB\right)} \\
&= \frac{D^{2n-2}}{a^{2n-3}}\frac{\left(-1+cBa\right)^{n-1}}{\left(-1+DcB\right)^{n-1}}
\end{aligned}$$

8

and

$$\begin{aligned}
y_{6n-10} &= \frac{x_{6n-13}y_{6n-14}x_{6n-15}}{x_{6n-11}y_{6n-12}\left(-1+x_{6n-13}y_{6n-14}x_{6n-15}\right)} \\
&= \frac{dCb\frac{(-1+CbA)^{n-2}}{(-1+dCb)^{n-2}}\frac{(-1+EdC)^{n-2}}{(-1+CbA)^{n-2}}\frac{(-1+dCb)^{n-2}}{(-1+EdC)^{n-2}}}{\frac{AA^{2n-3}b(-1+EdC)^{n-2}}{d^{2n-3}(-1+CbA)^{n-1}}\frac{(-1+dCb)^{n-2}}{(-1+EdC)^{n-2}}dCb\frac{(-1+CbA)^{n-2}}{(-1+dCb)^{n-2}}\frac{(-1+EdC)^{n-2}}{(-1+CbA)^{n-2}}\frac{(-1+dCb)^{n-2}}{(-1+EdC)^{n-2}}} \\
&= \frac{dCb}{\frac{CbA^{2n-3}}{d^{2n-3}}\frac{(-1+dCb)^{n-2}}{(-1+CbA)^{n-1}}\left(-1+dCb\right)} \\
&= \frac{d^{2n-2}}{A^{2n-3}}\frac{\left(-1+CbA\right)^{n-1}}{\left(-1+dCb\right)^{n-1}}.
\end{aligned}$$

Similarly, one can prove the other relations. The proof is complete. ■

Corollary 6 *System (4) has a periodic solution of period six iff* $e = B,\ E = b,\ d = A,\ D = a,\ CEd = 2,\ ceD = 2$ *and will be taken the form*

$$\left\{ \begin{array}{c} (D,d),(E,e),(c,C),(d,D),(e,E),(C,c),(D,d),(E,e),(c,C), \\ (d,D),(e,E),(C,c),\ldots \end{array} \right\}.$$

Theorem 7 *System (4) has a unique equilibrium point which is* $(0,0)$ *and this equilibrium point is not locally asymptotically stable.*

Proof. The linearized system of (4) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ x_{n-4} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = x^{10} - 3x^8 - 3x^6 - x^4 - x^2 - 1. \tag{5}$$

Note that since $P(-2) = -1 < 0$ and $P(-1) = 4 > 0$, there exists a root of $P(\lambda)$ in the interval $(-2,-1)$. Namely, at least one of the roots of Eq.(5) has absolute value greater than one. Thus, the equilibrium point $(0,0)$ of system (4) is not locally asymptotically stable. ■

9

5 The System $x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}(1-y_{n-2}x_{n-3}y_{n-4})}$, $y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}(1-x_{n-2}y_{n-3}x_{n-4})}$

In this section, we study the solutions of the system of difference equations, for $n = 0, 1, 2, \dots,$

$$x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}\left(1 - y_{n-2}x_{n-3}y_{n-4}\right)},\ y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}\left(1 - x_{n-2}y_{n-3}x_{n-4}\right)}, \tag{6}$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero.

Theorem 8 *Let $\{x_n, y_n\}_{n=-4}^{\infty}$ be solutions of system (6). Then, for $n = 1, 2, \dots,$*

$$\begin{aligned}
x_{6n-5} &= -\frac{Cb^{2n-1}A^{2n-1}}{E^{2n-1}d^{2n-1}}\prod_{i=0}^{2n-2}\frac{-1+iEdC}{-1+(i+1)\,CbA},\\
y_{6n-5} &= -\frac{ca^{2n-1}B^{2n-1}}{D^{2n-1}e^{2n-1}}\prod_{i=0}^{2n-2}\frac{-1+ieDc}{-1+(i+1)\,cBa},\\
x_{6n-6} &= \frac{e^{2n-1}}{B^{2n-2}}\prod_{i=0}^{2n-2}\frac{-1+iDcB}{-1+ieDc},\quad y_{6n-6} = \frac{E^{2n-1}}{b^{2n-2}}\prod_{i=0}^{2n-2}\frac{-1+idCb}{-1+iEdC},\\
x_{6n-7} &= \frac{d^{2n-1}}{A^{2n-2}}\prod_{i=0}^{2n-2}\frac{-1+iCbA}{-1+idCb},\quad y_{6n-7} = \frac{D^{2n-1}}{a^{2n-2}}\prod_{i=0}^{2n-2}\frac{-1+icBa}{-1+iDcB},\\
x_{6n-8} &= -\frac{ca^{2n-2}B^{2n-2}}{D^{2n-2}e^{2n-2}}\prod_{i=0}^{2n-3}\frac{-1+ieDc}{-1+(i+1)\,cBa},\\
y_{6n-8} &= -\frac{Cb^{2n-2}A^{2n-2}}{E^{2n-2}d^{2n-2}}\prod_{i=0}^{2n-3}\frac{-1+iEdC}{-1+(i+1)\,CbA},\\
x_{6n-9} &= \frac{E^{2n-2}}{b^{2n-3}}\prod_{i=0}^{2n-3}\frac{-1+idCb}{-1+iEdC},\quad y_{6n-9} = \frac{e^{2n-2}}{B^{2n-3}}\prod_{i=0}^{2n-3}\frac{-1+iDcB}{-1+ieDc},\\
x_{6n-10} &= \frac{D^{2n-2}}{a^{2n-3}}\prod_{i=0}^{2n-3}\frac{-1+icBa}{-1+iDcB},\quad y_{6n-10} = \frac{d^{2n-2}}{A^{2n-3}}\prod_{i=0}^{2n-3}\frac{-1+iCbA}{-1+idCb},
\end{aligned}$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c,\ x_{-1} = d,\ x_0 = e,\ y_{-4} = A,\ y_{-3} = B,\ y_{-2} = C,\ y_{-1} = D,\ y_0 = E.$

Theorem 9 *System (6) has a unique equilibrium point which is $(0, 0)$ and this equilibrium point is not locally asymptotically stable.*

10

6 The System $x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}(-1-y_{n-2}x_{n-3}y_{n-4})}$, $y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}(-1-x_{n-2}y_{n-3}x_{n-4})}$

In this section, we study the solutions of the system of difference equations, for $n = 0, 1, 2, \ldots,$

$$x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_n x_{n-1}\left(-1 - y_{n-2}x_{n-3}y_{n-4}\right)}, \; y_{n+1} = \frac{x_{n-2}y_{n-3}x_{n-4}}{x_n y_{n-1}\left(-1 - x_{n-2}y_{n-3}x_{n-4}\right)}, \tag{7}$$

where the initial values are arbitrary nonzero real numbers such that the denominator is always nonzero.

Theorem 10 *Let $\{x_n, y_n\}_{n=-4}^{\infty}$ be solutions of system (7). Then, for $n = 1, 2, \ldots,$*

$$\begin{aligned}
x_{6n-5} &= -\frac{Cb^{2n-1}A^{2n-1}\left(1+EdC\right)^{n-1}}{d^{2n-1}E^{2n-1}\left(1+CbA\right)^{n}}, \; y_{6n-5} = -\frac{ca^{2n-1}B^{2n-1}\left(1+eDc\right)^{n-1}}{D^{2n-1}e^{2n-1}\left(1+cBa\right)^{n}}, \\
x_{6n-6} &= \frac{e^{2n-1}\left(1+DcB\right)^{n-1}}{B^{2n-2}\left(1+eDc\right)^{n-1}}, \; y_{6n-6} = \frac{E^{2n-1}\left(1+dCb\right)^{n-1}}{b^{2n-2}\left(1+EdC\right)^{n-1}}, \\
x_{6n-7} &= \frac{d^{2n-1}\left(1+CbA\right)^{n-1}}{A^{2n-2}\left(1+dCb\right)^{n-1}}, \; y_{6n-7} = \frac{D^{2n-1}\left(1+cBa\right)^{n-1}}{a^{2n-2}\left(1+DcB\right)^{n-1}}, \\
x_{6n-8} &= \frac{ca^{2n-2}B^{2n-2}\left(1+eDc\right)^{n-1}}{D^{2n-2}e^{2n-2}\left(1+cBa\right)^{n-1}}, \; y_{6n-8} = \frac{Cb^{2n-2}A^{2n-2}\left(1+EdC\right)^{n-1}}{d^{2n-2}E^{2n-2}\left(1+CbA\right)^{n-1}}, \\
x_{6n-9} &= \frac{E^{2n-2}\left(1+dCb\right)^{n-1}}{b^{2n-3}\left(1+EdC\right)^{n-1}}, \; y_{6n-9} = \frac{e^{2n-2}\left(1+DcB\right)^{n-1}}{B^{2n-3}\left(1+eDc\right)^{n-1}}, \\
x_{6n-10} &= \frac{D^{2n-2}\left(1+cBa\right)^{n-1}}{a^{2n-3}\left(1+DcB\right)^{n-1}}, \; y_{6n-10} = \frac{d^{2n-2}\left(1+CbA\right)^{n-1}}{A^{2n-3}\left(1+dCb\right)^{n-1}},
\end{aligned}$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c,\ x_{-1} = d,\ x_0 = e,\ y_{-4} = A,\ y_{-3} = B,\ y_{-2} = C,\ y_{-1} = D,\ y_0 = E.$

Theorem 11 *The following statements are valid:*

i) *System (7) has a unique equilibrium point which is $(0, 0)$ and this equilibrium point is not locally asymptotically stable.*

ii) *System (7) has a periodic solution of period six iff $e = B,\ E = b,\ d = A,\ D = a$ and will be taken the form*

$$\left\{ \begin{array}{c} (a, A), (E, e), (c, C), (A, a), (e, E), \left(-\frac{C}{1+CEA}, -\frac{c}{1+cea}\right), \\ (a, A), (E, e), (c, C), (A, a), (e, E), \left(-\frac{C}{1+CEA}, -\frac{c}{1+cea}\right), \ldots \end{array} \right\}$$

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Solutions and periodicity for some systems of fourth order rational difference equations

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Abstract

In this paper we deal with the existence of solutions and the periodicity character of the following systems of rational difference equations with order four

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}\left(\pm 1 \pm x_n y_{n-3}\right)},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm y_n x_{n-3}\right)},$$

with initial conditions are nonzero real numbers.

Keywords: difference equations, periodic solution, system of difference equations.
Mathematics Subject Classification: 39A10.

1 Introduction

In recent years, rational difference equations have attracted the attention of many researchers for varied reasons. On the one hand, they provide examples of nonlinear equations which are, in some cases, treatable but whose dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models, and, hence, their study is of interest also due to their applications. A good example of both facts is Ricatti difference equations; the richness of the dynamics of Ricatti equations is very well-known (see, e.g., [5]), and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of exploited fish populations [2]. Obviously, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. The reader can find in the following books [5, 12], and the works cited therein.

The periodicity of the positive solutions of the rational difference equations systems

$$x_{n+1} = \frac{m}{y_n}, \quad y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}},$$

has been obtained by Cinar in [4].
The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1+x_{n-1}y_n}, \; y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n}.$$

has been studied by Kurbanli et al. [13].

Touafek et al. [17] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \; y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

In [18] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following systems of difference equations

$$z_{n+1} = \frac{t_n z_{n-1}+a}{t_n+z_{n-1}}, \; t_{n+1} = \frac{z_n t_{n-1}+a}{z_n+t_{n-1}}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [1]-[28].
In this paper, we investigated the periodic nature and the form of the solutions of nonlinear difference equations systems of order four

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(\pm 1 \pm x_n y_{n-3})}, \; y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(\pm 1 \pm y_n x_{n-3})},$$

with initial conditions are nonzero real numbers.

2 System $x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(1+x_n y_{n-3})}$, $y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(1+y_n x_{n-3})}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(1+x_n y_{n-3})}, \; y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(1+y_n x_{n-3})}, \tag{1}$$

where the initial conditions are arbitrary nonzero real numbers.

Theorem 2.1. Assume that $\{x_n, y_n\}$ are solutions of system (1). Then for $n = 0, 1, 2, ...$

$$\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(6i)de)}{(1+(6i+3)ah)}, \quad x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1+(6i+1)de)}{(1+(6i+4)ah)}, \\
x_{6n-1} &= \frac{ba^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1+(6i+2)de)}{(1+(6i+5)ah)}, \quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1+(6i+3)de)}{(1+(6i+6)ah)}, \\
x_{6n+1} &= \frac{a^{n+1} h^{n+1}}{ge^n d^n(1+ah)} \prod_{i=0}^{n-1} \frac{(1+(6i+4)de)}{(1+(6i+7)ah)}, \; x_{6n+2} = \frac{a^{n+1} h^{n+1}}{fe^n d^n(1+2ah)} \prod_{i=0}^{n-1} \frac{(1+(6i+5)de)}{(1+(6i+8)ah)}, \\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(6i)ah)}{(1+(6i+3)de)}, \quad y_{6n-2} = \frac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+1)ah)}{(1+(6i+4)de)}, \\
y_{6n-1} &= \frac{fd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+2)ah)}{(1+(6i+5)de)}, \quad y_{6n} = \frac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+3)ah)}{(1+(6i+6)de)},
\end{aligned}$$

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$$y_{6n+1}=\frac{d^{n+1}e^{n+1}}{ca^nh^n(1+de)}\prod_{i=0}^{n-1}\frac{(1+(6i+4)ah)}{(1+(6i+7)de)},\ y_{6n+2}=\frac{d^{n+1}e^{n+1}}{ba^nh^n(1+2de)}\prod_{i=0}^{n-1}\frac{(1+(6i+5)ah)}{(1+(6i+8)de)},$$

where $x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_0=a,\ y_{-3}=h,\ y_{-2}=g,\ y_{-1}=f$ *and* $y_0=e$.

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-7} &= \frac{ba^{n-1}h^{n-1}}{e^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+2)de)}{(1+(6i+5)ah)},\ x_{6n-6}=\frac{a^nh^{n-1}}{e^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+3)de)}{(1+(6i+6)ah)},\\
x_{6n-5} &= \frac{a^nh^n}{g(ed)^{n-1}(1+ah)}\prod_{i=0}^{n-2}\frac{1+(6i+4)de}{1+(6i+7)ah},\ x_{6n-4}=\frac{a^nh^n}{f(ed)^{n-1}(1+2ah)}\prod_{i=0}^{n-2}\frac{1+(6i+5)de}{1+(6i+8)ah},\\
y_{6n-7} &= \frac{fd^{n-1}e^{n-1}}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+2)ah)}{(1+(6i+5)de)},\ y_{6n-6}=\frac{d^{n-1}e^n}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+3)ah)}{(1+(6i+6)de)},\\
y_{6n-5} &= \frac{d^ne^n}{c(ah)^{n-1}(1+de)}\prod_{i=0}^{n-2}\frac{1+(6i+4)ah}{1+(6i+7)de},\ y_{6n-4}=\frac{d^ne^n}{b(ah)^{n-1}(1+2de)}\prod_{i=0}^{n-2}\frac{1+(6i+5)ah}{1+(6i+8)de}.
\end{aligned}$$

Now it follows from Eq.(1) that

$$\begin{aligned}
x_{6n-3} &= \frac{x_{6n-4}y_{6n-7}}{y_{6n-6}(1+x_{6n-4}y_{6n-7})}\\
&= \frac{\left(\frac{a^nh^n}{fe^{n-1}d^{n-1}(1+2ah)}\prod_{i=0}^{n-2}\frac{(1+(6i+5)de)}{(1+(6i+8)ah)}\right)\left(\frac{fd^{n-1}e^{n-1}}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+2)ah)}{(1+(6i+5)de)}\right)}{\left(\frac{d^{n-1}e^n}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+3)ah)}{(1+(6i+6)de)}\right)\left(1+\frac{a^nh^n}{fe^{n-1}d^{n-1}(1+2ah)}\frac{fd^{n-1}e^{n-1}}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+5)de)}{(1+(6i+8)ah)}\frac{(1+(6i+2)ah)}{(1+(6i+5)de)}\right)}\\
&= \frac{\frac{a^nh^n}{(1+(6n-4)ah)}}{d^{n-1}e^n\prod_{i=0}^{n-2}\frac{1+(6i+3)ah}{1+(6i+6)de}\left(1+\frac{ah}{1+(6n-4)ah}\right)}=\frac{a^nh^n}{d^{n-1}e^n}\prod_{i=0}^{n-1}\frac{(1+(6i)de)}{(1+(6i+3)ah)},\\
y_{6n-3} &= \frac{y_{6n-4}x_{6n-7}}{x_{6n-6}(1+y_{6n-4}x_{6n-7})}\\
&= \frac{\left(\frac{d^ne^n}{ba^{n-1}h^{n-1}(1+2de)}\prod_{i=0}^{n-2}\frac{(1+(6i+5)ah)}{(1+(6i+8)de)}\right)\left(\frac{ba^{n-1}h^{n-1}}{e^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+2)de)}{(1+(6i+5)ah)}\right)}{\left(\frac{a^nh^{n-1}}{e^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+3)de)}{(1+(6i+6)ah)}\right)\left(1+\frac{d^ne^n}{ba^{n-1}h^{n-1}(1+2de)}\frac{ba^{n-1}h^{n-1}}{e^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(6i+5)ah)}{(1+(6i+8)de)}\frac{(1+(6i+2)de)}{(1+(6i+5)ah)}\right)}\\
&= \frac{\frac{e^nd^n}{(1+(6n-4)de)}}{a^nh^{n-1}\prod_{i=0}^{n-2}\frac{1+(6i+3)de}{1+(6i+6)ah}\left(1+\frac{de}{1+(6n-4)de}\right)}=\frac{e^nd^n}{a^nh^{n-1}}\prod_{i=0}^{n-1}\frac{1+(6i)ah}{1+(6i+3)de}.
\end{aligned}$$

Also, we see from Eq.(1) that

$$x_{6n-2}=\frac{x_{6n-3}y_{6n-6}}{y_{6n-5}(1+x_{6n-3}y_{6n-6})}$$

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$$= \frac{\frac{a^n h^n}{e^n d^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(6i)de)}{(1+(6i+3)ah)} \frac{d^{n-1}e^n}{a^{n-1}h^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(6i+3)ah)}{(1+(6i+6)de)}}{\frac{d^n e^n}{c(ah)^{n-1}(1+de)} \prod_{i=0}^{n-2} \frac{(1+(6i+4)ah)}{(1+(6i+7)de)} \left(1+\frac{a^n h^n}{e^n d^{n-1}} \frac{d^{n-1}e^n}{(ah)^{n-1}} \prod_{i=0}^{n-1} \frac{1+(6i)de}{1+(6i+3)ah} \prod_{i=0}^{n-2} \frac{1+(6i+3)ah}{1+(6i+6)de}\right)}$$

$$= \frac{\frac{ah}{(1+(6n-3)ah)}}{\frac{d^n e^n}{ca^{n-1}h^{n-1}(1+de)} \prod_{i=0}^{n-2} \frac{1+(6i+4)ah}{1+(6i+7)de} \left(1+\frac{ah}{1+(6n-3)ah}\right)} = \frac{ca^n h^n}{d^n e^n} \prod_{i=0}^{n-1} \frac{(1+(6i+1)de)}{(1+(6i+4)ah)},$$

$$y_{6n-2} = \frac{y_{6n-3}x_{6n-6}}{x_{6n-5}(1+y_{6n-3}x_{6n-6})}$$

$$= \frac{\frac{d^n e^n}{a^n h^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(6i)ah)}{(1+(6i+3)de)} \frac{a^n h^{n-1}}{e^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(6i+3)de)}{(1+(6i+6)ah)}}{\frac{a^n h^n}{g(ed)^{n-1}(1+ah)} \prod_{i=0}^{n-2} \frac{1+(6i+4)de}{1+(6i+7)ah} \left(1+\frac{d^n e^n}{a^n h^{n-1}} \frac{a^n h^{n-1}}{(ed)^{n-1}} \prod_{i=0}^{n-1} \frac{1+(6i)ah}{1+(6i+3)de} \prod_{i=0}^{n-2} \frac{1+(6i+3)de}{1+(6i+6)ah}\right)}$$

$$= \frac{\frac{de}{(1+(6n-3)de)}}{\frac{a^n h^n}{ge^{n-1}d^{n-1}(1+ah)} \prod_{i=0}^{n-2} \frac{1+(6i+4)de}{1+(6i+7)ah} \left(1+\frac{de}{1+(6n-3)de}\right)} = \frac{ge^n d^n}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+1)ah)}{(1+(6i+4)de)}.$$

Also, we can prove the other relations. The proof is complete.

The following Theorems can be proved similarly:

Theorem 2.2. Assume that $\{x_n, y_n\}$ are solutions of the system

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(1+x_n y_{n-3})},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(1-y_n x_{n-3})}.$$

Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}} \prod_{i=0}^{n-1} \frac{(1-(6i)de)}{(1+(6i+3)ah)},\ x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1-(6i+1)de)}{(1+(6i+4)ah)},\\
x_{6n-1} &= \frac{ba^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1-(6i+2)de)}{(1+(6i+5)ah)},\quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \prod_{i=0}^{n-1} \frac{(1-(6i+3)de)}{(1+(6i+6)ah)},\\
x_{6n+1} &= \frac{a^{n+1} h^{n+1}}{ge^n d^n(1+ah)} \prod_{i=0}^{n-1} \frac{(1-(6i+4)de)}{(1+(6i+7)ah)},\ x_{6n+2} = \frac{a^{n+1} h^{n+1}}{fe^n d^n(1+2ah)} \prod_{i=0}^{n-1} \frac{(1-(6i+5)de)}{(1+(6i+8)ah)},\\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(6i)ah)}{(1-(6i+3)de)},\quad y_{6n-2} = \frac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+1)ah)}{(1-(6i+4)de)},\\
y_{6n-1} &= \frac{fd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+2)ah)}{(1-(6i+5)de)},\quad y_{6n} = \frac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1} \frac{(1+(6i+3)ah)}{(1-(6i+6)de)},\\
y_{6n+1} &= \frac{d^{n+1} e^{n+1}}{ca^n h^n(1-de)} \prod_{i=0}^{n-1} \frac{(1+(6i+4)ah)}{(1-(6i+7)de)},\ y_{6n+2} = \frac{d^{n+1} e^{n+1}}{ba^n h^n(1-2de)} \prod_{i=0}^{n-1} \frac{(1+(6i+5)ah)}{(1-(6i+8)de)}.
\end{aligned}$$

Theorem 2.3. Let $\{x_n, y_n\}$ are solutions of the following system

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(1-x_n y_{n-3})},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(1+y_n x_{n-3})}.$$

Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{6n-3} &= \tfrac{a^n h^n}{e^n d^{n-1}} \prod_{i=0}^{n-1} \tfrac{(1+(6i)de)}{(1-(6i+3)ah)}, \quad x_{6n-2} = \tfrac{ca^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1+(6i+1)de)}{(1-(6i+4)ah)}, \\
x_{6n-1} &= \tfrac{ba^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1+(6i+2)de)}{(1-(6i+5)ah)}, \quad x_{6n} = \tfrac{a^{n+1} h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1+(6i+3)de)}{(1-(6i+6)ah)}, \\
x_{6n+1} &= \tfrac{a^{n+1} h^{n+1}}{ge^n d^n (1-ah)} \prod_{i=0}^{n-1} \tfrac{(1+(6i+4)de)}{(1-(6i+7)ah)}, \quad x_{6n+2} = \tfrac{a^{n+1} h^{n+1}}{fe^n d^n (1-2ah)} \prod_{i=0}^{n-1} \tfrac{(1+(6i+5)de)}{(1-(6i+8)ah)}, \\
y_{6n-3} &= \tfrac{d^n e^n}{a^n h^{n-1}} \prod_{i=0}^{n-1} \tfrac{(1-(6i)ah)}{(1+(6i+3)de)}, \quad y_{6n-2} = \tfrac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+1)ah)}{(1+(6i+4)de)}, \\
y_{6n-1} &= \tfrac{fd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+2)ah)}{(1+(6i+5)de)}, \quad y_{6n} = \tfrac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+3)ah)}{(1+(6i+6)de)}, \\
y_{6n+1} &= \tfrac{d^{n+1} e^{n+1}}{ca^n h^n (1+de)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+4)ah)}{(1+(6i+7)de)}, \quad y_{6n+2} = \tfrac{d^{n+1} e^{n+1}}{ba^n h^n (1+2de)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+5)ah)}{(1+(6i+8)de)}.
\end{aligned}$$

Theorem 2.4. The solutions of the following system

$$x_{n+1} = \tfrac{x_n y_{n-3}}{y_{n-2}(1-x_n y_{n-3})}, \; y_{n+1} = \tfrac{y_n x_{n-3}}{x_{n-2}(1-y_n x_{n-3})}.$$

are given by the following formulae

$$\begin{aligned}
x_{6n-3} &= \tfrac{a^n h^n}{e^n d^{n-1}} \prod_{i=0}^{n-1} \tfrac{(1-(6i)de)}{(1-(6i+3)ah)}, \quad x_{6n-2} = \tfrac{ca^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+1)de)}{(1-(6i+4)ah)}, \\
x_{6n-1} &= \tfrac{ba^n h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+2)de)}{(1-(6i+5)ah)}, \quad x_{6n} = \tfrac{a^{n+1} h^n}{e^n d^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+3)de)}{(1-(6i+6)ah)}, \\
x_{6n+1} &= \tfrac{a^{n+1} h^{n+1}}{ge^n d^n (1-ah)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+4)de)}{(1-(6i+7)ah)}, \quad x_{6n+2} = \tfrac{a^{n+1} h^{n+1}}{fe^n d^n (1-2ah)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+5)de)}{(1-(6i+8)ah)}, \\
y_{6n-3} &= \tfrac{d^n e^n}{a^n h^{n-1}} \prod_{i=0}^{n-1} \tfrac{(1-(6i)ah)}{(1-(6i+3)de)}, \quad y_{6n-2} = \tfrac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+1)ah)}{(1-(6i+4)de)}, \\
y_{6n-1} &= \tfrac{fd^n e^n}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+2)ah)}{(1-(6i+5)de)}, \quad y_{6n} = \tfrac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1} \tfrac{(1-(6i+3)ah)}{(1-(6i+6)de)}, \\
y_{6n+1} &= \tfrac{d^{n+1} e^{n+1}}{ca^n h^n (1-de)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+4)ah)}{(1-(6i+7)de)}, \quad y_{6n+2} = \tfrac{d^{n+1} e^{n+1}}{ba^n h^n (1-2de)} \prod_{i=0}^{n-1} \tfrac{(1-(6i+5)ah)}{(1-(6i+8)de)}.
\end{aligned}$$

Example 1. We consider interesting numerical example for the difference system (1) with the initial conditions $x_{-3} = -0.3,\; x_{-2} = 0.2,\; x_{-1} = -0.13,\; x_0 = 0.52,\; y_{-3} = 0.21,\; y_{-2} = 0.12,\; y_{-1} = -0.6$ *and* $y_0 = -0.32$. (See Fig. 1).

3 System $x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(1+x_n y_{n-3})},\; y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(-1+y_n x_{n-3})}$

In this section, we obtain the form of the solutions of the system of two difference equations

$$x_{n+1} = \tfrac{x_n y_{n-3}}{y_{n-2}(1+x_n y_{n-3})}, \; y_{n+1} = \tfrac{y_n x_{n-3}}{x_{n-2}(-1+y_n x_{n-3})}, \tag{2}$$

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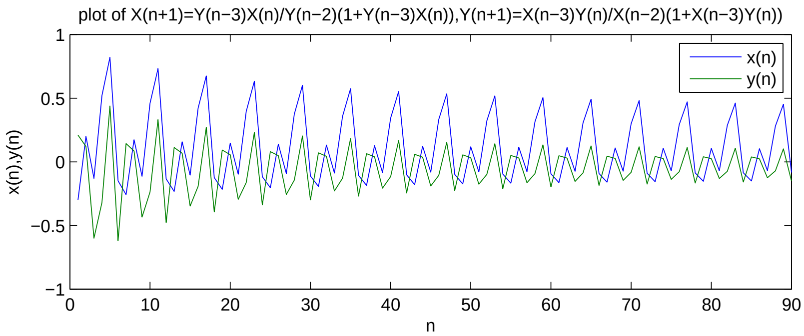

Figure 1:

where the initial conditions are arbitrary non zero real numbers with $x_{-3}y_0 \neq 1$.
Theorem 3.1. Let $\{x_n, y_n\}_{n=-3}^{+\infty}$ be solutions of system (2). Then for $n = 0, 1, 2, ...,$

$$x_{6n-3} = \frac{a^n h^n}{e^n d^{n-1}} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+3)ah)}, \quad x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(1+(6i+4)ah)},$$

$$x_{6n-1} = \frac{ba^n h^n}{e^n d^n} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+5)ah)}, \quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(1+(6i+6)ah)},$$

$$x_{6n+1} = \frac{a^{n+1} h^{n+1}}{ge^n d^n(1+ah)} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+7)ah)}, \quad x_{6n+2} = \frac{a^{n+1} h^{n+1}}{fe^n d^n(1+2ah)} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(1+(6i+8)ah)},$$

$$y_{6n-3} = \frac{d^n e^n}{a^n h^{n-1}} \frac{\prod_{i=0}^{n-1}(1+(6i)ah)}{(-1+de)^n}, \quad y_{6n-2} = \frac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1}(1+(6i+1)ah),$$

$$y_{6n-1} = \frac{fd^n e^n}{a^n h^n} \frac{\prod_{i=0}^{n-1}(1+(6i+2)ah)}{(-1+de)^n}, \quad y_{6n} = \frac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1}(1+(6i+3)ah),$$

$$y_{6n+1} = \frac{d^{n+1} e^{n+1}}{ca^n h^n} \frac{\prod_{i=0}^{n-1}(1+(6i+4)ah)}{(-1+de)^{n+1}}, \quad y_{6n+2} = \frac{d^{n+1} e^{n+1}}{ba^n h^n} \prod_{i=0}^{n-1}(1+(6i+5)ah),$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$x_{6n-7} = \frac{ba^{n-1} h^{n-1}}{e^{n-1} d^{n-1}} \frac{1}{\prod_{i=0}^{n-2}(1+(6i+5)ah)}, \quad x_{6n-6} = \frac{a^n h^{n-1}}{e^{n-1} d^{n-1}} \frac{(-1+de)^{n-1}}{\prod_{i=0}^{n-2}(1+(6i+6)ah)},$$

$$x_{6n-5} = \frac{a^n h^n}{g(ed)^{n-1}(1+ah)} \frac{1}{\prod_{i=0}^{n-2} 1+(6i+7)ah}, x_{6n-4} = \frac{a^n h^n}{f(ed)^{n-1}(1+2ah)} \frac{(-1+de)^{n-1}}{\prod_{i=0}^{n-2} 1+(6i+8)ah},$$

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$$\begin{aligned} y_{6n-7} &= \frac{fd^{n-1}e^{n-1}}{a^{n-1}h^{n-1}}\frac{\prod\limits_{i=0}^{n-2}(1+(6i+2)ah)}{(-1+de)^{n-1}},\ y_{6n-6}=\frac{d^{n-1}e^{n}}{a^{n-1}h^{n-1}}\prod_{i=0}^{n-2}(1+(6i+3)ah),\\ y_{6n-5} &= \frac{d^{n}e^{n}}{ca^{n-1}h^{n-1}}\frac{\prod\limits_{i=0}^{n-2}(1+(6i+4)ah)}{(-1+de)^{n}},\ y_{6n-4}=\frac{d^{n}e^{n}}{ba^{n-1}h^{n-1}}\prod_{i=0}^{n-2}(1+(6i+5)ah), \end{aligned}$$

Now it follows from Eq.(2) that

$$x_{6n-3}=\frac{x_{6n-4}y_{6n-7}}{y_{6n-6}(1+x_{6n-4}y_{6n-7})}$$

$$=\frac{\frac{a^{n}h^{n}}{fe^{n-1}d^{n-1}(1+2ah)}\frac{(-1+de)^{n-1}}{\prod\limits_{i=0}^{n-2}(1+(6i+8)ah)}\frac{fd^{n-1}e^{n-1}}{a^{n-1}h^{n-1}}\frac{\prod\limits_{i=0}^{n-2}(1+(6i+2)ah)}{(-1+de)^{n-1}}}{\frac{d^{n-1}e^{n}}{a^{n-1}h^{n-1}}\prod\limits_{i=0}^{n-2}1+(6i+3)ah\left(1+\frac{a^{n}h^{n}}{f(ed)^{n-1}(1+2ah)}\frac{(-1+de)^{n-1}}{\prod\limits_{i=0}^{n-2}(1+(6i+8)ah)}\frac{f(ed)^{n-1}}{(ah)^{n-1}}\frac{\prod\limits_{i=0}^{n-2}1+(6i+2)ah}{(-1+de)^{n-1}}\right)}$$

$$=\frac{\frac{ah}{(1+(6n+2)ah)}}{\frac{d^{n-1}e^{n}}{a^{n-1}h^{n-1}}\prod\limits_{i=0}^{n-2}(1+(6i+3)ah)\left(1+\frac{ah}{(1+(6n+2)ah)}\right)}=\frac{a^{n}h^{n}}{d^{n-1}e^{n}\prod\limits_{i=0}^{n-1}1+(6i+3)ah},$$

$$y_{6n-3}=\frac{y_{6n-4}x_{6n-7}}{x_{6n-6}\left(-1+y_{6n-4}x_{6n-7}\right)}$$

$$=\frac{\frac{d^{n}e^{n}}{ba^{n-1}h^{n-1}}\prod\limits_{i=0}^{n-2}(1+(6i+5)ah)\frac{ba^{n-1}h^{n-1}}{e^{n-1}d^{n-1}}\frac{1}{\prod\limits_{i=0}^{n-2}(1+(6i+5)ah)}}{\frac{a^{n}h^{n-1}}{e^{n-1}d^{n-1}}\frac{(-1+de)^{n-1}}{\prod\limits_{i=0}^{n-2}1+(6i+6)ah}\left(-1+\frac{d^{n}e^{n}}{b(ah)^{n-1}}\prod\limits_{i=0}^{n-2}1+(6i+5)ah\frac{b(ah)^{n-1}}{(ed)^{n-1}}\frac{1}{\prod\limits_{i=0}^{n-2}1+(6i+5)ah}\right)}$$

$$=\frac{de}{\frac{a^{n}h^{n-1}}{e^{n-1}d^{n-1}}\frac{(-1+de)^{n-1}}{\prod\limits_{i=0}^{n-2}(1+(6i+6)ah)}(-1+de)}=\frac{e^{n}d^{n}}{a^{n}h^{n-1}(-1+de)^{n}}\prod_{i=0}^{n-1}(1+(6i)ah).$$

Similarly, we can prove the other relations. This completes the proof.

We consider the following systems and the proof of the theorems are similarly to above theorem and so, left to the reader.

$$x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(1+x_ny_{n-3})},\ y_{n+1}=\frac{y_nx_{n-3}}{x_{n-2}(-1-y_nx_{n-3})}. \tag{3}$$

$$x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(1-x_ny_{n-3})},\ y_{n+1}=\frac{y_nx_{n-3}}{x_{n-2}(-1+y_nx_{n-3})}. \tag{4}$$

$$x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(1-x_ny_{n-3})},\ y_{n+1}=\frac{y_nx_{n-3}}{x_{n-2}(-1-y_nx_{n-3})}. \tag{5}$$

Theorem 3.2. Let $\{x_n, y_n\}_{n=-3}^{+\infty}$ be solutions of system (3) and $x_{-3}y_0 \neq -1$.

Then

$$
\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+3)ah)}, \quad x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \frac{(-1-de)^n}{\prod_{i=0}^{n-1}(1+(6i+4)ah)}, \\
x_{6n-1} &= \frac{ba^n h^n}{e^n d^n} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+5)ah)}, \quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \frac{(-1-de)^n}{\prod_{i=0}^{n-1}(1+(6i+6)ah)}, \\
x_{6n+1} &= \frac{a^{n+1} h^{n+1}}{ge^n d^n (1+ah)} \frac{1}{\prod_{i=0}^{n-1}(1+(6i+7)ah)}, \quad x_{6n+2} = \frac{a^{n+1} h^{n+1}}{fe^n d^n (1+2ah)} \frac{(-1-de)^n}{\prod_{i=0}^{n-1}(1+(6i+8)ah)}, \\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}} \frac{\prod_{i=0}^{n-1}(1+(6i)ah)}{(-1-de)^n}, \quad y_{6n-2} = \frac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1}(1+(6i+1)ah), \\
y_{6n-1} &= \frac{fd^n e^n}{a^n h^n} \frac{\prod_{i=0}^{n-1}(1+(6i+2)ah)}{(-1-de)^n}, \quad y_{6n} = \frac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1}(1+(6i+3)ah), \\
y_{6n+1} &= \frac{d^{n+1} e^{n+1}}{ca^n h^n} \frac{\prod_{i=0}^{n-1}(1+(6i+4)ah)}{(-1-de)^{n+1}}, \quad y_{6n+2} = \frac{d^{n+1} e^{n+1}}{ba^n h^n} \prod_{i=0}^{n-1}(1+(6i+5)ah).
\end{aligned}
$$

Theorem 3.3. Assume that $\{x_n, y_n\}$ are solutions of system (4) with $x_{-3}y_0 \neq 1$. Then

$$
\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}} \frac{1}{\prod_{i=0}^{n-1}(1-(6i+3)ah)}, \quad x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(-1+(6i+4)ah)}, \\
x_{6n-1} &= \frac{ba^n h^n}{e^n d^n} \frac{1}{\prod_{i=0}^{n-1}(1-(6i+5)ah)}, \quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(-1+(6i+6)ah)}, \\
x_{6n+1} &= \frac{a^{n+1} h^{n+1}}{ge^n d^n (1-ah)} \frac{1}{\prod_{i=0}^{n-1}(1-(6i+7)ah)}, \quad x_{6n+2} = \frac{a^{n+1} h^{n+1}}{fe^n d^n (-1+2ah)} \frac{(-1+de)^n}{\prod_{i=0}^{n-1}(-1+(6i+8)ah)}, \\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}} \frac{\prod_{i=0}^{n-1}(1-(6i)ah)}{(-1+de)^n}, \quad y_{6n-2} = \frac{gd^n e^n}{a^n h^n} \prod_{i=0}^{n-1}(1-(6i+1)ah), \\
y_{6n-1} &= \frac{fd^n e^n}{a^n h^n} \frac{\prod_{i=0}^{n-1}(1-(6i+2)ah)}{(-1+de)^n}, \quad y_{6n} = \frac{d^n e^{n+1}}{a^n h^n} \prod_{i=0}^{n-1}(1-(6i+3)ah), \\
y_{6n+1} &= \frac{d^{n+1} e^{n+1}}{ca^n h^n} \frac{\prod_{i=0}^{n-1}(1-(6i+4)ah)}{(-1+de)^{n+1}}, \quad y_{6n+2} = \frac{d^{n+1} e^{n+1}}{ba^n h^n} \prod_{i=0}^{n-1}(1-(6i+5)ah).
\end{aligned}
$$

Theorem 3.4. Suppose that $\{x_n, y_n\}$ are solutions of system (5) such that $x_{-3}y_0 \neq -1$. Then

$$
\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}} \frac{1}{\prod_{i=0}^{n-1}(1-(6i+3)ah)}, \quad x_{6n-2} = \frac{ca^n h^n}{e^n d^n} \frac{(-1-de)^n}{\prod_{i=0}^{n-1}(1-(6i+4)ah)}, \\
x_{6n-1} &= \frac{ba^n h^n}{e^n d^n} \frac{1}{\prod_{i=0}^{n-1}(1-(6i+5)ah)}, \quad x_{6n} = \frac{a^{n+1} h^n}{e^n d^n} \frac{(-1-de)^n}{\prod_{i=0}^{n-1}(1-(6i+6)ah)},
\end{aligned}
$$

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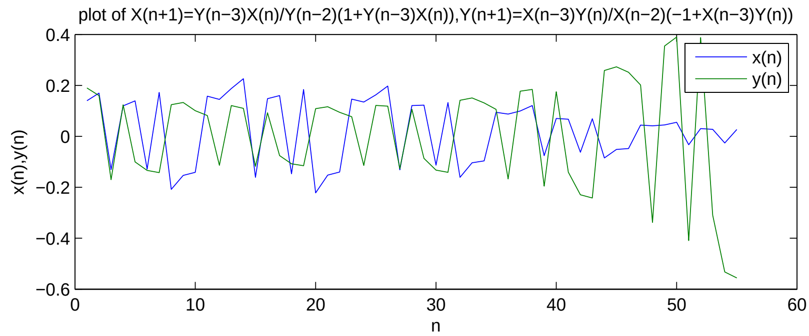

Figure 2:

$$x_{6n+1} = \frac{a^{n+1}h^{n+1}}{ge^nd^n(1-ah)}\frac{1}{\prod\limits_{i=0}^{n-1}(1-(6i+7)ah)},\ x_{6n+2} = \frac{a^{n+1}h^{n+1}}{fe^nd^n(1-2ah)}\frac{(-1-de)^n}{\prod\limits_{i=0}^{n-1}(1-(6i+8)ah)},$$

$$y_{6n-3} = \frac{d^ne^n}{a^nh^{n-1}}\frac{\prod\limits_{i=0}^{n-1}(1-(6i)ah)}{(-1-de)^n},\ y_{6n-2} = \frac{gd^ne^n}{a^nh^n}\prod_{i=0}^{n-1}(1-(6i+1)ah),$$

$$y_{6n-1} = \frac{fd^ne^n}{a^nh^n}\frac{\prod\limits_{i=0}^{n-1}(1-(6i+2)ah)}{(-1-de)^n},\ y_{6n} = \frac{d^ne^{n+1}}{a^nh^n}\prod_{i=0}^{n-1}(1-(6i+3)ah),$$

$$y_{6n+1} = \frac{d^{n+1}e^{n+1}}{ca^nh^n}\frac{\prod\limits_{i=0}^{n-1}(1-(6i+4)ah)}{(-1-de)^{n+1}},\ y_{6n+2} = \frac{d^{n+1}e^{n+1}}{ba^nh^n}\prod_{i=0}^{n-1}(1-(6i+5)ah).$$

Example 2. See Figure (2) when we take system (3) with the initial conditions $x_{-3} = 0.14,\ x_{-2} = 0.17,\ x_{-1} = -0.13,\ x_0 = 0.12,\ y_{-3} = 0.19,\ y_{-2} = 0.16,\ y_{-1} = -0.17$ *and* $y_0 = 0.124$.

The following cases can be treated similarly.

4 System $x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1+x_ny_{n-3})},\ y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(1+y_nx_{n-3})}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}\left(-1+x_ny_{n-3}\right)},\ y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}\left(1+y_nx_{n-3}\right)}, \tag{6}$$

where the initial conditions are arbitrary nonzero real numbers with $x_0y_{-3} \neq 1$.
Theorem 4.1. If $\{x_n, y_n\}$ are solutions of difference equation system (6). Then

$$x_{6n-3} = \frac{a^nh^n}{e^nd^{n-1}}\frac{\prod\limits_{i=0}^{n-1}(1+(6i)de)}{(-1+ah)^n},\ x_{6n-2} = \frac{ca^nh^n}{e^nd^n}\prod_{i=0}^{n-1}(1+(6i+1)de),$$

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$$\begin{aligned}
x_{6n-1} &= \frac{ba^nh^n}{e^nd^n}\frac{\prod_{i=0}^{n-1}(1+(6i+2)de)}{(-1+ah)^n}, \quad x_{6n} = \frac{a^{n+1}h^n}{e^nd^n}\prod_{i=0}^{n-1}(1+(6i+3)de), \\
x_{6n+1} &= \frac{a^{n+1}h^{n+1}}{ge^nd^n}\frac{\prod_{i=0}^{n-1}(1+(6i+4)de)}{(-1+ah)^{n+1}}, \quad x_{6n+2} = \frac{a^{n+1}h^{n+1}}{fe^nd^n}\prod_{i=0}^{n-1}(1+(6i+5)de), \\
y_{6n-3} &= \frac{d^ne^n}{a^nh^{n-1}}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+3)de)}, \quad y_{6n-2} = \frac{gd^ne^n}{a^nh^n}\frac{(-1+ah)^n}{\prod_{i=0}^{n-1}(1+(6i+4)de)}, \\
y_{6n-1} &= \frac{fd^ne^n}{a^nh^n}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+5)de)}, \quad y_{6n} = \frac{d^ne^{n+1}}{a^nh^n}\prod_{i=0}^{n-1}\frac{(-1+ah)^n}{(1+(6i+6)de)}, \\
y_{6n+1} &= \frac{d^{n+1}e^{n+1}}{ca^nh^n(1+de)}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+7)de)}, \quad y_{6n+2} = \frac{d^{n+1}e^{n+1}}{ba^nh^n(1+2de)}\frac{(-1+ah)^n}{\prod_{i=0}^{n-1}(1+(6i+8)de)},
\end{aligned}$$

Theorem 4.2. If $\{x_n, y_n\}$ are solutions of the following difference equation system $x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1+x_ny_{n-3})}$, $y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(1+y_nx_{n-3})}$, with $x_{-3}y_0 \neq -1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{6n-3} &= \frac{a^nh^n}{e^nd^{n-1}}\frac{\prod_{i=0}^{n-1}(1-(6i)de)}{(-1+ah)^n}, \quad x_{6n-2} = \frac{ca^nh^n}{e^nd^n}\prod_{i=0}^{n-1}(1-(6i+1)de), \\
x_{6n-1} &= \frac{ba^nh^n}{e^nd^n}\frac{\prod_{i=0}^{n-1}(1-(6i+2)de)}{(-1+ah)^n}, \quad x_{6n} = \frac{a^{n+1}h^n}{e^nd^n}\prod_{i=0}^{n-1}(1-(6i+3)de), \\
x_{6n+1} &= \frac{a^{n+1}h^{n+1}}{ge^nd^n}\frac{\prod_{i=0}^{n-1}(1-(6i+4)de)}{(-1+ah)^{n+1}}, \quad x_{6n+2} = \frac{a^{n+1}h^{n+1}}{fe^nd^n}\prod_{i=0}^{n-1}(1-(6i+5)de), \\
y_{6n-3} &= \frac{d^ne^n}{a^nh^{n-1}}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+3)de)}, \quad y_{6n-2} = \frac{gd^ne^n}{a^nh^n}\frac{(-1+ah)^n}{\prod_{i=0}^{n-1}(1-(6i+4)de)}, \\
y_{6n-1} &= \frac{fd^ne^n}{a^nh^n}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+5)de)}, \quad y_{6n} = \frac{d^ne^{n+1}}{a^nh^n}\prod_{i=0}^{n-1}\frac{(-1+ah)^n}{(1-(6i+6)de)}, \\
y_{6n+1} &= \frac{d^{n+1}e^{n+1}}{ca^nh^n(1-de)}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+7)de)}, \quad y_{6n+2} = \frac{d^{n+1}e^{n+1}}{ba^nh^n(1-2de)}\frac{(-1+ah)^n}{\prod_{i=0}^{n-1}(1-(6i+8)de)}.
\end{aligned}$$

Theorem 4.3. If $\{x_n, y_n\}$ are solutions of the difference equations system $x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1-x_ny_{n-3})}$, $y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(1+y_nx_{n-3})}$, where $x_{-3}y_0 \neq 1$. Then

$$\begin{aligned}
x_{6n-3} &= \frac{a^nh^n}{e^nd^{n-1}}\frac{\prod_{i=0}^{n-1}(1+(6i)de)}{(-1-ah)^n}, \quad x_{6n-2} = \frac{ca^nh^n}{e^nd^n}\prod_{i=0}^{n-1}(1+(6i+1)de), \\
x_{6n-1} &= \frac{ba^nh^n}{e^nd^n}\frac{\prod_{i=0}^{n-1}(1+(6i+2)de)}{(-1-ah)^n}, \quad x_{6n} = \frac{a^{n+1}h^n}{e^nd^n}\prod_{i=0}^{n-1}(1+(6i+3)de),
\end{aligned}$$

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$$
\begin{aligned}
x_{6n+1} &= \frac{a^{n+1}h^{n+1}}{ge^nd^n}\frac{\prod_{i=0}^{n-1}(1+(6i+4)de)}{(-1-ah)^{n+1}},\ x_{6n+2}=\frac{a^{n+1}h^{n+1}}{fe^nd^n}\prod_{i=0}^{n-1}(1+(6i+5)de),\\
y_{6n-3} &= \frac{d^ne^n}{a^nh^{n-1}}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+3)de)},\ y_{6n-2}=\frac{gd^ne^n}{a^nh^n}\frac{(-1-ah)^n}{\prod_{i=0}^{n-1}(1+(6i+4)de)},\\
y_{6n-1} &= \frac{fd^ne^n}{a^nh^n}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+5)de)},\ y_{6n}=\frac{d^ne^{n+1}}{a^nh^n}\prod_{i=0}^{n-1}\frac{(-1-ah)^n}{(1+(6i+6)de)},\\
y_{6n+1} &= \frac{d^{n+1}e^{n+1}}{ca^nh^n(1+de)}\frac{1}{\prod_{i=0}^{n-1}(1+(6i+7)de)},\ y_{6n+2}=\frac{d^{n+1}e^{n+1}}{ba^nh^n(1+2de)}\frac{(-1-ah)^n}{\prod_{i=0}^{n-1}(1+(6i+8)de)}.
\end{aligned}
$$

Theorem 4.4. Assume that $\{x_n, y_n\}$ are solutions of the system $x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1-x_ny_{n-3})}$, $y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(1-y_nx_{n-3})}$, with $x_{-3}y_0 \neq -1$. Then

$$
\begin{aligned}
x_{6n-3} &= \frac{a^nh^n}{e^nd^{n-1}}\frac{\prod_{i=0}^{n-1}(1-(6i)de)}{(-1-ah)^n},\ x_{6n-2}=\frac{ca^nh^n}{e^nd^n}\prod_{i=0}^{n-1}(1-(6i+1)de),\\
x_{6n-1} &= \frac{ba^nh^n}{e^nd^n}\frac{\prod_{i=0}^{n-1}(1-(6i+2)de)}{(-1-ah)^n},\ x_{6n}=\frac{a^{n+1}h^n}{e^nd^n}\prod_{i=0}^{n-1}(1-(6i+3)de),\\
x_{6n+1} &= \frac{a^{n+1}h^{n+1}}{ge^nd^n}\frac{\prod_{i=0}^{n-1}(1-(6i+4)de)}{(-1+ah)^{n+1}},\ x_{6n+2}=\frac{a^{n+1}h^{n+1}}{fe^nd^n}\prod_{i=0}^{n-1}(1-(6i+5)de),\\
y_{6n-3} &= \frac{d^ne^n}{a^nh^{n-1}}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+3)de)},\ y_{6n-2}=\frac{gd^ne^n}{a^nh^n}\frac{(-1-ah)^n}{\prod_{i=0}^{n-1}(1-(6i+4)de)},\\
y_{6n-1} &= \frac{fd^ne^n}{a^nh^n}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+5)de)},\ y_{6n}=\frac{d^ne^{n+1}}{a^nh^n}\prod_{i=0}^{n-1}\frac{(-1-ah)^n}{(1-(6i+6)de)},\\
y_{6n+1} &= \frac{d^{n+1}e^{n+1}}{ca^nh^n(1-de)}\frac{1}{\prod_{i=0}^{n-1}(1-(6i+7)de)},\ y_{6n+2}=\frac{d^{n+1}e^{n+1}}{ba^nh^n(1-2de)}\frac{(-1-ah)^n}{\prod_{i=0}^{n-1}(1-(6i+8)de)},
\end{aligned}
$$

Example 3. Figure (3) shows the behavior of the solution of the system (6) with $x_{-3} = -0.18,\ x_{-2} = 0.12,\ x_{-1} = 0.13,\ x_0 = 0.19,\ y_{-3} = -0.17,\ y_{-2} = -0.16,\ y_{-1} = -0.107$ *and* $y_0 = 0.14$.

5 System $x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1+x_ny_{n-3})}$, $y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(-1+y_nx_{n-3})}$

In this section, we get the form of the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_ny_{n-3}}{y_{n-2}(-1+x_ny_{n-3})},\ y_{n+1} = \frac{y_nx_{n-3}}{x_{n-2}(-1+y_nx_{n-3})}, \tag{7}$$

where the initial conditions are nonzero real numbers with $x_0y_{-3},\ x_{-3}y_0 \neq 1$.

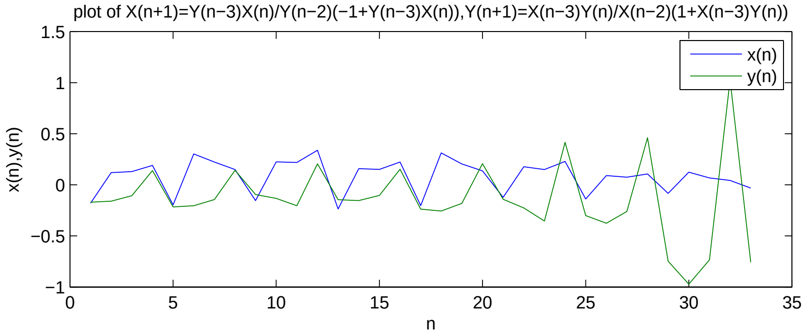

Figure 3:

Theorem 5.1. If $\{x_n, y_n\}$ are solutions of difference equation system (7). Then

$$\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}(-1+ah)^n},\ x_{6n-2} = \frac{ca^n h^n(-1+de)^n}{e^n d^n},\ x_{6n-1} = \frac{ba^n h^n}{e^n d^n(-1+ah)^n},\\
x_{6n} &= \frac{a^{n+1}h^n(-1+de)^n}{e^n d^n},\ x_{6n+1} = \frac{a^{n+1}h^{n+1}}{ge^n d^n(-1+ah)^{n+1}},\ x_{6n+2} = \frac{a^{n+1}h^{n+1}(-1+de)^n}{fe^n d^n},\\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}(-1+de)^n},\ y_{6n-2} = \frac{gd^n e^n(-1+ah)^n}{a^n h^n},\ y_{6n-1} = \frac{fd^n e^n}{a^n h^n(-1+de)^n},\\
y_{6n} &= \frac{d^n e^{n+1}(-1+ah)^n}{a^n h^n},\ y_{6n+1} = \frac{d^{n+1}e^{n+1}}{ca^n h^n(-1+de)^{n+1}},\ y_{6n+2} = \frac{d^{n+1}e^{n+1}(-1+ah)^n}{ba^n h^n}.
\end{aligned}$$

Now, we consider the following systems

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(-1+x_n y_{n-3})},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(-1-y_n x_{n-3})}. \tag{8}$$

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(-1-x_n y_{n-3})},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(-1+y_n x_{n-3})}. \tag{9}$$

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}(-1-x_n y_{n-3})},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}(-1-y_n x_{n-3})}. \tag{10}$$

Theorem 5.2. Let $\{x_n, y_n\}$ be solutions of system (8) and $x_0 y_{-3} \neq 1$, $x_{-3}y_0 \neq -1$. Then

$$\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}(-1+ah)^n},\ x_{6n-2} = \frac{ca^n h^n(-1-de)^n}{e^n d^n},\ x_{6n-1} = \frac{ba^n h^n}{e^n d^n(-1+ah)^n},\\
x_{6n} &= \frac{a^{n+1}h^n(-1-de)^n}{e^n d^n},\ x_{6n+1} = \frac{a^{n+1}h^{n+1}}{ge^n d^n(-1+ah)^{n+1}},\ x_{6n+2} = \frac{a^{n+1}h^{n+1}(-1-de)^n}{fe^n d^n},\\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}(-1-de)^n},\ y_{6n-2} = \frac{gd^n e^n(-1+ah)^n}{a^n h^n},\ y_{6n-1} = \frac{fd^n e^n}{a^n h^n(-1-de)^n},\\
y_{6n} &= \frac{d^n e^{n+1}(-1+ah)^n}{a^n h^n},\ y_{6n+1} = \frac{d^{n+1}e^{n+1}}{ca^n h^n(-1-de)^{n+1}},\ y_{6n+2} = \frac{d^{n+1}e^{n+1}(-1+ah)^n}{ba^n h^n}.
\end{aligned}$$

Theorem 5.3. Assume that $\{x_n, y_n\}$ are solutions of system (9) with $x_0 y_{-3} \neq -1$, $x_{-3}y_0 \neq 1$. Then

$$\begin{aligned}
x_{6n-3} &= \frac{a^n h^n}{e^n d^{n-1}(-1-ah)^n},\ x_{6n-2} = \frac{ca^n h^n(-1+de)^n}{e^n d^n},\ x_{6n-1} = \frac{ba^n h^n}{e^n d^n(-1-ah)^n},\\
x_{6n} &= \frac{a^{n+1}h^n(-1+de)^n}{e^n d^n},\ x_{6n+1} = \frac{a^{n+1}h^{n+1}}{ge^n d^n(-1-ah)^{n+1}},\ x_{6n+2} = \frac{a^{n+1}h^{n+1}(-1+de)^n}{fe^n d^n},\\
y_{6n-3} &= \frac{d^n e^n}{a^n h^{n-1}(-1+de)^n},\ y_{6n-2} = \frac{gd^n e^n(-1-ah)^n}{a^n h^n},\ y_{6n-1} = \frac{fd^n e^n}{a^n h^n(-1+de)^n},\\
y_{6n} &= \frac{d^n e^{n+1}(-1-ah)^n}{a^n h^n},\ y_{6n+1} = \frac{d^{n+1}e^{n+1}}{ca^n h^n(-1+de)^{n+1}},\ y_{6n+2} = \frac{d^{n+1}e^{n+1}(-1-ah)^n}{ba^n h^n}.
\end{aligned}$$

Theorem 5.4. Suppose that $\{x_n, y_n\}$ are solutions of system (10) such that $x_0y_{-3},\ x_{-3}y_0 \neq -1$. Then

$$\begin{aligned}
x_{6n-3} &= \tfrac{a^n h^n}{e^n d^{n-1}(-1-ah)^n},\ x_{6n-2} = \tfrac{ca^n h^n(-1-de)^n}{e^n d^n},\ x_{6n-1} = \tfrac{ba^n h^n}{e^n d^n(-1-ah)^n},\\
x_{6n} &= \tfrac{a^{n+1} h^n(-1-de)^n}{e^n d^n},\ x_{6n+1} = \tfrac{a^{n+1} h^{n+1}}{ge^n d^n(-1-ah)^{n+1}},\ x_{6n+2} = \tfrac{a^{n+1} h^{n+1}(-1-de)^n}{fe^n d^n},\\
y_{6n-3} &= \tfrac{d^n e^n}{a^n h^{n-1}(-1-de)^n},\ y_{6n-2} = \tfrac{gd^n e^n(-1-ah)^n}{a^n h^n},\ y_{6n-1} = \tfrac{fd^n e^n}{a^n h^n(-1-de)^n},\\
y_{6n} &= \tfrac{d^n e^{n+1}(-1-ah)^n}{a^n h^n},\ y_{6n+1} = \tfrac{d^{n+1} e^{n+1}}{ca^n h^n(-1-de)^{n+1}},\ y_{6n+2} = \tfrac{d^{n+1} e^{n+1}(-1-ah)^n}{ba^n h^n}.
\end{aligned}$$

Lemma 1. The solutions of systems (7)-(10) are unbounded except in the following cases.

Theorem 5.5. System (7) has a periodic solution of period six iff $ah = de = 2$ and it will be taken the following form $\{x_n\} = \left\{d, c, b, a, \frac{ah}{g}, \frac{ah}{f}, d, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{de}{c}, \frac{de}{b}, h, ...\right\}$.

Proof: First suppose that there exists a prime period six solution $\{x_n\} = \left\{d, c, b, a, \frac{ah}{g}, \frac{ah}{f}, d, c, b, a, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{de}{c}, \frac{de}{b}, h, g, f, e, ...\right\}$ of system (7), we see that

$$\begin{aligned}
d &= \tfrac{a^n h^n}{e^n d^{n-1}(-1+ah)^n},\quad c = \tfrac{ca^n h^n(-1+de)^n}{e^n d^n},\quad b = \tfrac{ba^n h^n}{e^n d^n(-1+ah)^n},\\
a &= \tfrac{a^{n+1} h^n(-1+de)^n}{e^n d^n},\quad \tfrac{ah}{g} = \tfrac{a^{n+1} h^{n+1}}{ge^n d^n(-1+ah)^{n+1}},\quad \tfrac{ah}{f} = \tfrac{a^{n+1} h^{n+1}(-1+de)^n}{fe^n d^n},\\
h &= \tfrac{d^n e^n}{a^n h^{n-1}(-1+de)^n},\quad g = \tfrac{gd^n e^n(-1+ah)^n}{a^n h^n},\quad f = \tfrac{fd^n e^n}{a^n h^n(-1+de)^n},\\
e &= \tfrac{d^n e^{n+1}(-1+ah)^n}{a^n h^n},\quad \tfrac{de}{c} = \tfrac{d^{n+1} e^{n+1}}{ca^n h^n(-1+de)^{n+1}},\quad \tfrac{de}{b} = \tfrac{d^{n+1} e^{n+1}(-1+ah)^n}{ba^n h^n}.
\end{aligned}$$

Then we get $de = ah$, $\quad -1 + de = -1 + ah = 1$.Thus $de = ah = 2$. Second assume that $de = ah = 2$. Then we see from the form of the solution of system (7) that

$$\begin{aligned}
x_{6n-3} &= d,\ x_{6n-2} = c,\ x_{6n-1} = b,\ x_{6n} = a,\ x_{6n+1} = \tfrac{ah}{g},\ x_{6n+2} = \tfrac{ah}{f},\\
y_{6n-3} &= h,\ y_{6n-2} = g,\ y_{6n-1} = f,\ y_{6n} = e,\ y_{6n+1} = \tfrac{de}{c},\ y_{6n+2} = \tfrac{de}{b}.
\end{aligned}$$

Thus we have a periodic solution of period six and the proof is complete.

Theorem 5.6. System (8) has a periodic solution of period six iff $ah = 2,\ de = -2$ and it will be taken the following form $\{x_n\} = \left\{d, c, b, a, \frac{ah}{g}, \frac{ah}{f}, d, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{de}{c}, \frac{de}{b}, h, ...\right\}$.

Theorem 5.7. System (9) has a periodic solution of period six iff $ah = -2,\ de = 2$ and it will be taken the following form $\{x_n\} = \left\{d, c, b, a, \frac{ah}{g}, \frac{ah}{f}, d, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{de}{c}, \frac{de}{b}, h, ...\right\}$.

Theorem 5.8. System (10) has a periodic solution of period six iff $ah = de = -2$ and it will be taken the following form $\{x_n\} = \left\{d, c, b, a, \frac{ah}{g}, \frac{ah}{f}, d, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{de}{c}, \frac{de}{b}, h, ...\right\}$.

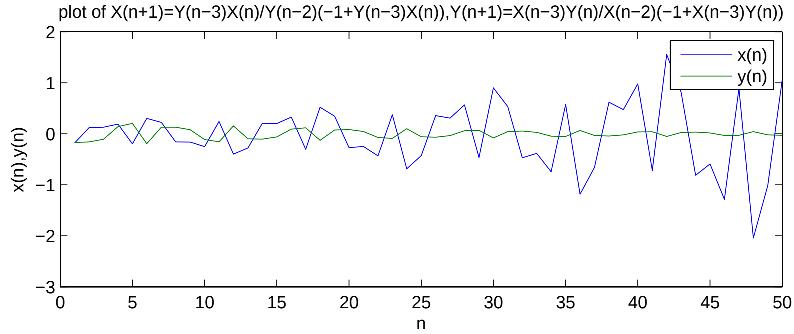

Figure 4:

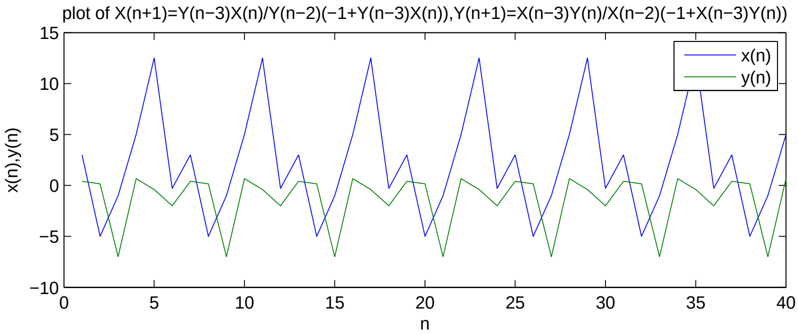

Figure 5:

Example 4. We consider numerical example for the system (7) when we put the initial conditions $x_{-3} = -0.18,\ x_{-2} = 0.12,\ x_{-1} = 0.13,\ x_0 = 0.19,\ y_{-3} = -0.17,\ y_{-2} = -0.16,\ y_{-1} = -0.107\ and\ y_0 = 0.14$. See figure 4.

Example 5. Figure (5) shows the solution of the system (7) with the initial conditions $x_{-3} = 3,\ x_{-2} = -5,\ x_{-1} = -1,\ x_0 = 5,\ y_{-3} = 0.4,\ y_{-2} = 0.16,\ y_{-1} = -7\ and\ y_0 = 2/3$.

Example 6. Figure (6) shows the behavior of the solution of the system (10) with the initial conditions $x_{-3} = 3,\ x_{-2} = -5,\ x_{-1} = -1,\ x_0 = 5,\ y_{-3} = -0.4,\ y_{-2} = 0.16,\ y_{-1} = -7\ and\ y_0 = -2/3$.

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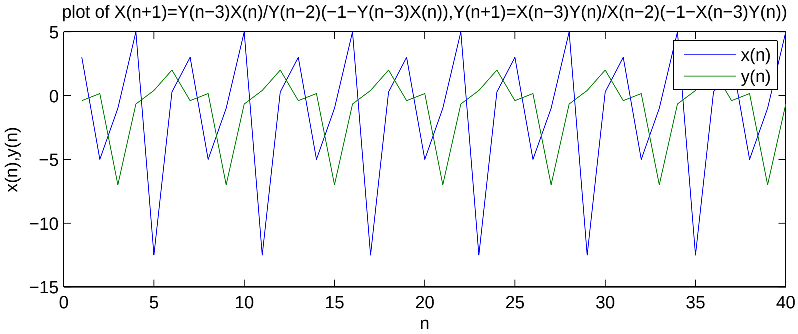

Figure 6:

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Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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On the solutions and periodic nature of some systems of rational difference equations

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Abstract

In this paper, we deal with the existence of solutions and the periodicity character of the following systems of rational difference equations with order two

$$x_{n+1}=\frac{x_n y_{n-1}}{y_{n-1}\pm y_n},\ y_{n+1}=\frac{y_n x_{n-1}}{x_{n-1}\pm x_n},$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1}\ and\ y_0$ are nonzero real numbers.

Keywords: difference equations, recursive sequences, stability, periodic solution, solution of difference equation, system of difference equations.

Mathematics Subject Classification: 39A10.

1 Introduction

Recently, rational difference equations have attracted the attention of many researchers for various reasons. On one hand, they provide examples of nonlinear equations which are, in some cases, treatable but their dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models. Hence, their study is of interest also due to their applications. A good example of both facts is Ricatti difference equations because the richness of the dynamics of Ricatti equations is very well-known (see, e.g., [8]), and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of exploited fish populations [6]. Obviously, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. The reader can find in the following books [3, 16, 17], and works cited therein, many results, applications, and open problems on higher-order equations and rational systems.

There are many papers that are related to the difference equations systems. For example, the periodicity of the positive solutions of the rational difference equations systems

$$x_{n+1}=\frac{m}{y_n},\ \ y_{n+1}=\frac{py_n}{x_{n-1}y_{n-1}},$$

has been obtained by Cinar in [7].

Din et al. [9] studied the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of a fourth-order system of rational difference equations of the form

$$x_{n+1}=\frac{\alpha x_{n-3}}{\beta+\gamma y_n y_{n-1} y_{n-2} y_{n-3}},\ y_{n+1}=\frac{\alpha_1 y_{n-3}}{\beta_1+\gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

The behavior of the positive solutions of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{1+y_{n-1}x_n}.$$

has been studied by Kurbanli et al. [18].
Mansour et al.[20] investigated the behavior of the solutions of the difference equations systems

$$x_{n+1}=\frac{x_{n-5}}{-1+x_{n-5}y_{n-2}},\ y_{n+1}=\frac{y_{n-5}}{\pm 1\pm y_{n-5}x_{n-2}},$$

In [21] Ozban studied the positive solutions of the rational difference equations system

$$x_{n+1}=\frac{1}{y_{n-k}},\ y_{n+1}=\frac{y_n}{x_{n-m}y_{n-m-k}}.$$

Touafek et al. [23] investigated the periodic nature and got the form of the solutions of the following systems of rational difference equations

$$x_{n+1}=\frac{x_{n-3}}{\pm 1\pm x_{n-3}y_{n-1}},\ y_{n+1}=\frac{y_{n-3}}{\pm 1\pm y_{n-3}x_{n-1}}.$$

In [25]-[26] Yalçınkaya investigated the sufficient conditions for the global asymptotic stability of the following systems of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}},$$

and

$$x_{n+1}=\frac{x_n+y_{n-1}}{x_n y_{n-1}-1},\ y_{n+1}=\frac{y_n+x_{n-1}}{y_n x_{n-1}-1}.$$

Zhang et al. [29] studied the dynamics of a system of the rational third-order difference equation

$$x_{n+1}=\frac{x_{n-2}}{B+y_n y_{n-1} y_{n-2}},\ y_{n+1}=\frac{y_{n-2}}{A+x_n x_{n-1} x_{n-2}}.$$

Similarly, difference equations and nonlinear systems of the rational difference equations were investigated see [1]-[33].
In this paper, we investigate the periodic nature and the form of the solutions of some nonlinear difference equations systems of order two

$$x_{n+1}=\frac{x_n y_{n-1}}{y_{n-1}\pm y_n},\ y_{n+1}=\frac{y_n x_{n-1}}{x_{n-1}\pm x_n},$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1}$ *and* y_0 are nonzero real numbers.

2 On The Solution of the System: $x_{n+1}=\frac{x_n y_{n-1}}{y_{n-1}+y_n}$, $y_{n+1}=\frac{y_n x_{n-1}}{x_{n-1}+x_n}$

In this section, we investigate the solutions of the two difference equations system

$$x_{n+1}=\frac{x_n y_{n-1}}{y_{n-1}+y_n},\ y_{n+1}=\frac{y_n x_{n-1}}{x_{n-1}+x_n},\tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions $x_{-1},\ x_0,\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers

The following theorem is devoted to the form of system (1) solutions.

Theorem 1 *Assume that $\{x_n,y_n\}$ are solutions of system (1). Then for $n=0,1,2,...,$ we see that all solutions of system (1) are given by the following formulas*

$$x_{2n-1}=b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)},\ x_{2n}=a\prod_{i=0}^{n-1}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)},$$

and

$$y_{2n-1}=e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)},\ y_{2n}=d\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)},$$

where $x_{-1}=b,\ x_0=a,\ y_{-1}=e,\ y_0=d$ *and* $.\{f_m\}_{m=-2}^{\infty}=\{1,0,1,1,2,3,5,8,13,...\}$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$x_{2n-3}=b\prod_{i=0}^{n-2}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)},\ x_{2n-2}=a\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)},$$

and

$$y_{2n-3}=e\prod_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)},\ y_{2n-2}=d\prod_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}.$$

3

Now we find from Eq.(1) that

$$
\begin{aligned}
x_{2n-1} &= \frac{x_{2n-2}y_{2n-3}}{y_{2n-3}+y_{2n-2}} \\
&= \frac{\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)\left(e\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\right)}{\left(e\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\right)+\left(d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}\right)} \\
&= \frac{e\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{e+\left(d\prod\limits_{i=0}^{n-2}\frac{(f_{2n}d+f_{2n+1}e)(f_{2n-1}d+f_{2n}e)}{(f_{2n-2}d+f_{2n-1}e)(f_{2n+1}d+f_{2n+2}e)}\right)} \\
&= \frac{e\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{e+d\left(\frac{(f_{2n-4}d+f_{2n-3}e)(f_{-1}d+f_0e)}{(f_{-2}d+f_{-1}e)(f_{2n-3}d+f_{2n-2}e)}\right)} = \frac{e\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{e+d\left(\frac{e(f_{2n-4}d+f_{2n-3}e)}{d(f_{2n-3}d+f_{2n-2}e)}\right)} \\
&= \frac{\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{1+\frac{(f_{2n-4}d+f_{2n-3}e)}{(f_{2n-3}d+f_{2n-2}e)}} = \frac{\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{\frac{(f_{2n-2}d+f_{2n-1}e)}{(f_{2n-3}d+f_{2n-2}e)}} \\
&= a\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\left(\frac{(f_{2n-3}d+f_{2n-2}e)}{(f_{2n-2}d+f_{2n-1}e)}\right) \\
&= b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)},
\end{aligned}
$$

$$
\begin{aligned}
y_{2n-1} &= \frac{y_{2n-2}x_{2n-3}}{x_{2n-3}+x_{2n-2}} \\
&= \frac{\left(d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}\right)\left(b\prod\limits_{i=0}^{n-2}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\right)}{\left(b\prod\limits_{i=0}^{n-2}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\right)+\left(a\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)} \\
&= \frac{d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}}{1+\left(\frac{a}{b}\prod\limits_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}a+f_{2n}b)}{(f_{2n-2}a+f_{2n-1}b)(f_{2n+1}a+f_{2n+2}b)}\right)} = \frac{d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}}{1+\left(\frac{a}{b}\frac{(f_{2n-4}a+f_{2n-3}b)(f_{-1}a+f_0b)}{(f_{-2}a+f_{-1}b)(f_{2n-3}a+f_{2n-2}b)}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}}{1+\frac{(f_{2n-4}a+f_{2n-3}b)}{(f_{2n-3}a+f_{2n-2}b)}} = \frac{d\prod\limits_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}}{\frac{(f_{2n-2}a+f_{2n-1}b)}{(f_{2n-3}a+f_{2n-2}b)}} \\
&= d\prod_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}\frac{(f_{2n-3}a+f_{2n-2}b)}{(f_{2n-2}a+f_{2n-1}b)} \\
&= e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}.
\end{aligned}
$$

4

Also, we infer from Eq.(1) that

$$
\begin{aligned}
x_{2n} &= \frac{x_{2n-1}y_{2n-2}}{y_{2n-2}+y_{2n-1}} \\
&= \frac{\left(b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\right)\left(d\prod_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}\right)}{\left(d\prod_{i=0}^{n-2}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}\right)+\left(e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\right)} \\
&= \frac{b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}}{1+\left(\frac{e}{d}\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\right)} \\
&= \frac{b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}}{1+\left(\frac{e(f_{-2}d+f_{-1}e)}{d(f_{-1}d+f_{0}e)}\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)}{(f_{2n}a+f_{2n+1}b)}\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)}{(f_{2n-1}a+f_{2n}b)}\right)} \\
&= \frac{b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}}{1+\left(\frac{(f_{2n-3}a+f_{2n-2}b)}{(f_{2n-2}a+f_{2n-1}b)}\right)} = \frac{b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}}{\left(\frac{(f_{2n-1}a+f_{2n}b)}{(f_{2n-2}a+f_{2n-1}b)}\right)} \\
&= b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\left(\frac{(f_{2n-2}a+f_{2n-1}b)}{(f_{2n-1}a+f_{2n}b)}\right) \\
&= a\prod_{i=0}^{n-1}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)},
\end{aligned}
$$

and so,

$$
\begin{aligned}
y_{2n} &= \frac{y_{2n-1}x_{2n-2}}{x_{2n-2}+x_{2n-1}} \\
&= \frac{\left(e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\right)\left(a\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)}{\left(a\prod_{i=0}^{n-2}\frac{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}\right)+\left(b\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\right)} \\
&= \frac{e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}}{1+\left(\frac{b}{a}\prod_{i=0}^{n-1}\frac{(f_{2n-2}a+f_{2n-1}b)(f_{2n-1}d+f_{2n}e)}{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}\prod_{i=0}^{n-2}\frac{(f_{2n+1}a+f_{2n+2}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\right)} \\
&= \frac{e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}}{1+\left(\frac{b(f_{-2}a+f_{-1}b)}{a(f_{-1}a+f_{0}b)}\left(\frac{(f_{2n-3}d+f_{2n-2}e)}{(f_{2n-2}d+f_{2n-1}e)}\right)\right)} = \frac{e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}}{\left(\frac{(f_{2n-1}d+f_{2n}e)}{(f_{2n-2}d+f_{2n-1}e)}\right)} \\
&= e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\left(\frac{(f_{2n-2}d+f_{2n-1}e)}{(f_{2n-1}d+f_{2n}e)}\right) \\
&= e\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n-2}d+f_{2n-1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n-1}d+f_{2n}e)}\left(\frac{(f_{2n-2}d+f_{2n-1}e)}{(f_{2n-1}d+f_{2n}e)}\right) \\
&= d\prod_{i=0}^{n-1}\frac{(f_{2n-1}a+f_{2n}b)(f_{2n}d+f_{2n+1}e)}{(f_{2n}a+f_{2n+1}b)(f_{2n+1}d+f_{2n+2}e)}.
\end{aligned}
$$

The proof is complete.
Lemma 1. Let $\{x_n, y_n\}$ be a positive solution of system (1), then every solution of system (1) is bounded and converges to zero.
Proof: It follows from Eq.(1) that

$$x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} + y_n} \leq x_n, \qquad y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} + x_n} \leq y_n,$$

Then the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-1}, x_0\}$. Also, the subsequences $\{y_{2n-1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{y_{-1}, y_0\}$.
Example 1. For confirming the results of this section, we consider numerical example for the difference system (1) with the initial conditions $x_{-1} = .9$, $x_0 = -.2$, $y_{-1} = -.5$ and $y_0 = .18$. (See Fig. 1).

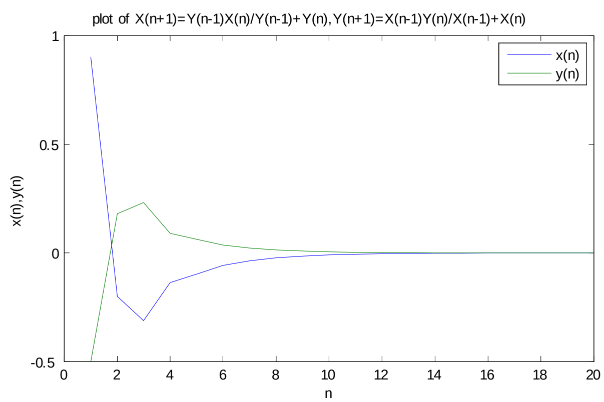

Figure 1.

3 On The Solution of the System: $x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1}+y_n}$, $y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1}-x_n}$

In this section, we obtain the form of the solutions of the two difference equations system

$$x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} + y_n}, \; y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} - x_n}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions x_{-1}, x_0, y_{-1} *and* y_0 are arbitrary non zero real numbers with $x_{-1} \neq x_0$.

Theorem 2 *Let $\{x_n, y_n\}_{n=-1}^{+\infty}$ be solutions of system (2). Then $\{x_n\}_{n=-1}^{+\infty}$ and*

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$\{y_n\}_{n=-1}^{+\infty}$ *are given by the formula for* $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n} &= \frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)},\\
x_{4n+1} &= \frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n}d+f_{2n-1}e)(f_{2n+1}d+f_{2n}e)},\\
x_{4n+2} &= \frac{-abde(a-b)}{(f_{2n-1}a-f_{2n+1}b)(f_{2n}a-f_{2n+2}b)(f_{2n}d+f_{2n-1}e)(f_{2n+1}d+f_{2n}e)},\\
x_{4n+3} &= \frac{-abde(a-b)}{(f_{2n-1}a-f_{2n+1}b)(f_{2n}a-f_{2n+2}b)(f_{2n+1}d+f_{2n}e)(f_{2n+2}d+f_{2n+1}e)},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n} &= \frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)},\ y_{4n+1} = \frac{(f_{2n-1}a-f_{2n+1}b)(f_{2n}d+f_{2n-1}e)}{(a-b)},\\
y_{4n+2} &= \frac{(f_{2n-1}a-f_{2n+1}b)(f_{2n+1}d+f_{2n}e)}{(a-b)},\ y_{4n+3} = \frac{(f_{2n}a-f_{2n+2}b)(f_{2n+1}d+f_{2n}e)}{(a-b)},
\end{aligned}$$

where $\{f_m\}_{m=-2}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$\begin{aligned}
x_{4n-4} &= \frac{-abde(a-b)}{(f_{2n-4}a-f_{2n-2}b)(f_{2n-3}a-f_{2n-1}b)(f_{2n-3}d+f_{2n-4}e)(f_{2n-2}d+f_{2n-3}e)},\\
x_{4n-3} &= \frac{-abde(a-b)}{(f_{2n-4}a-f_{2n-2}b)(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}d+f_{2n-3}e)(f_{2n-1}d+f_{2n-2}e)},\\
x_{4n-2} &= \frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-2}d+f_{2n-3}e)(f_{2n-1}d+f_{2n-2}e)},\\
x_{4n-1} &= \frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)},
\end{aligned}$$

and

$$y_{4n-4} = \frac{(f_{2n-4}a-f_{2n-2}b)(f_{2n-2}d+f_{2n-3}e)}{(a-b)},\ y_{4n-3} = \frac{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}d+f_{2n-3}e)}{(a-b)},$$

$$y_{4n-2} = \frac{(f_{2n-3}a-f_{2n-1}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)},\ y_{4n-1} = \frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}.$$

Now, we obtain from Eq.(2) that

$$\begin{aligned}
x_{4n} &= \frac{x_{4n-1}y_{4n-2}}{y_{4n-2}+y_{4n-1}}\\
&= \frac{\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)\left(\frac{(f_{2n-3}a-f_{2n-1}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}{\left(\frac{(f_{2n-3}a-f_{2n-1}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)+\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}\\
&= \frac{\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)(f_{2n-3}a-f_{2n-1}b)}{(f_{2n-3}a-f_{2n-1}b)+(f_{2n-2}a-f_{2n}b)}\\
&= \frac{\left(\frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)}{(f_{2n-1}a-f_{2n+1}b)}\\
&= \frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)},
\end{aligned}$$

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$$\begin{aligned}
y_{4n} &= \frac{y_{4n-1}x_{4n-2}}{x_{4n-2}-x_{4n-1}} \\
&= \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-2}d+f_{2n-3}e)(f_{2n-1}d+f_{2n-2}e)}\right)}{\left[\begin{array}{c}\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-2}d+f_{2n-3}e)(f_{2n-1}d+f_{2n-2}e)}\right)- \\ \left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)\end{array}\right]} \\
&= \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}{1-\left(\frac{f_{2n-2}d+f_{2n-3}e}{f_{2n}d+f_{2n-1}e}\right)} = \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}{\left(\frac{f_{2n}d+f_{2n-1}e-f_{2n-2}d-f_{2n-3}e}{f_{2n}d+f_{2n-1}e}\right)} \\
&= \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}{\left(\frac{f_{2n-1}d+f_{2n-2}e}{f_{2n}d+f_{2n-1}e}\right)} = \frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}.
\end{aligned}$$

Also, we see from Eq.(2) that

$$\begin{aligned}
x_{4n+1} &= \frac{x_{4n}y_{4n-1}}{y_{4n-1}+y_{4n}} \\
&= \frac{\left(\frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)}{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)}{(a-b)}\right)+\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}\right)} \\
&= \frac{\left(\frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n}d+f_{2n-1}e)}\right)}{(f_{2n-1}d+f_{2n-2}e)+(f_{2n}d+f_{2n-1}e)} \\
&= \frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n}d+f_{2n-1}e)(f_{2n+1}d+f_{2n}e)},
\end{aligned}$$

$$\begin{aligned}
y_{4n+1} &= \frac{y_{4n}x_{4n-1}}{x_{4n-1}-x_{4n}} \\
&= \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}\right)\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)}{\left[\begin{array}{c}\left(\frac{-abde(a-b)}{(f_{2n-3}a-f_{2n-1}b)(f_{2n-2}a-f_{2n}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right) \\ -\left(\frac{-abde(a-b)}{(f_{2n-2}a-f_{2n}b)(f_{2n-1}a-f_{2n+1}b)(f_{2n-1}d+f_{2n-2}e)(f_{2n}d+f_{2n-1}e)}\right)\end{array}\right]} \\
&= \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}\right)}{\left(1-\frac{(f_{2n-3}a-f_{2n-1}b)}{(f_{2n-1}a-f_{2n+1}b)}\right)} = \frac{\left(\frac{(f_{2n-2}a-f_{2n}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}\right)}{\frac{(f_{2n-2}a-f_{2n}b)}{(f_{2n-1}a-f_{2n+1}b)}} \\
&= \frac{(f_{2n-1}a-f_{2n+1}b)(f_{2n}d+f_{2n-1}e)}{(a-b)}.
\end{aligned}$$

Also, we can prove the other relations. This completes the proof.

Lemma 2. Let $\{x_n, y_n\}$ be a positive solution of system (2), then $\{x_n\}$ is bounded and converges to zero.

Example 2. We assume that the initial conditions for the difference system (2) are $x_{-1} = -.24$, $x_0 = -.7$, $y_{-1} = .19$ and $y_0 = -.8$. (See Fig. 2).

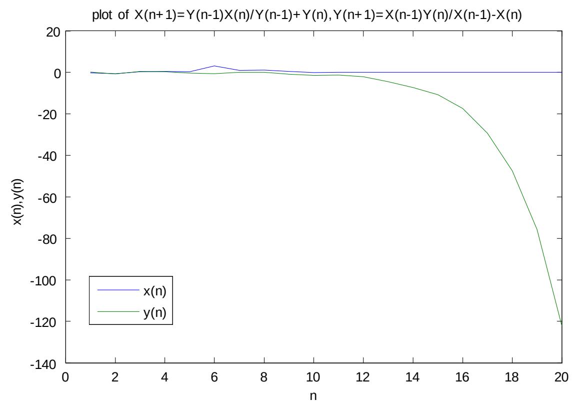

Figure 2.

4 On The Solution of the System: $x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} - y_n}$, $y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1}+x_n}$

In this section, we obtain the form of the solutions of the two difference equations system

$$x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} - y_n}, \; y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} + x_n}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions $x_{-1}, \; x_0, \; y_{-1}$ *and* y_0 are arbitrary non zero real numbers with $y_{-1} \neq y_0$.

Theorem 3 *Suppose that* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *are solutions of system (3). Then* $\{x_n\}_{n=-1}^{+\infty}$ *and* $\{y_n\}_{n=-1}^{+\infty}$ *are given by the following relations for* $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n} &= \frac{(f_{2n-2}d - f_{2n}e)(f_{2n}a + f_{2n-1}b)}{(d-e)}, \; x_{4n+1} = \frac{(f_{2n-1}d - f_{2n+1}e)(f_{2n}a + f_{2n-1}b)}{(d-e)}, \\
x_{4n+2} &= \frac{(f_{2n-1}d - f_{2n+1}e)(f_{2n+1}a + f_{2n}b)}{(d-e)}, \; x_{4n+3} = \frac{(f_{2n}d - f_{2n+2}e)(f_{2n+1}a + f_{2n}b)}{(d-e)},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n} &= \frac{-abde(d-e)}{(f_{2n-2}d - f_{2n}e)(f_{2n-1}d - f_{2n+1}e)(f_{2n-1}a + f_{2n-2}b)(f_{2n}a + f_{2n-1}b)}, \\
y_{4n+1} &= \frac{-abde(d-e)}{(f_{2n-2}d - f_{2n}e)(f_{2n-1}d - f_{2n+1}e)(f_{2n}a + f_{2n-1}b)(f_{2n+1}a + f_{2n}b)}, \\
y_{4n+2} &= \frac{-abde(d-e)}{(f_{2n-1}d - f_{2n+1}e)(f_{2n}d - f_{2n+2}e)(f_{2n}a + f_{2n-1}b)(f_{2n+1}a + f_{2n}b)}, \\
y_{4n+3} &= \frac{-abde(d-e)}{(f_{2n-1}d - f_{2n+1}e)(f_{2n}d - f_{2n+2}e)(f_{2n+1}a + f_{2n}b)(f_{2n+2}a + f_{2n+1}b)}.
\end{aligned}$$

Lemma 3. Let $\{x_n, y_n\}$ be a positive solution of system (3), then $\{y_n\}$ is bounded and converges to zero.

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Example 3. We consider numerical example for the difference system (3) with the initial conditions $x_{-1} = -4$, $x_0 = 7$, $y_{-1} = 9$ and $y_0 = 8$. See Figure (3).

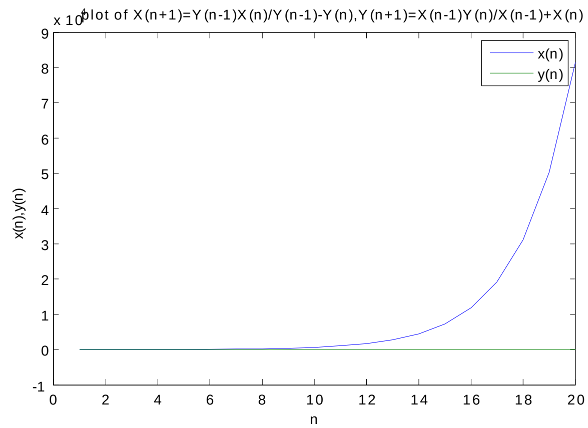

Figure 3.

5 Periodicity of the System: $x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} - y_n}$, $y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} - x_n}$

In this section, we get the form of the solutions of the difference equations system

$$x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} - y_n}, \ y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} - x_n}, \tag{4}$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-1}, \ x_0, \ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_0 \neq x_{-1}, \ y_0 \neq y_{-1}$.

Theorem 4 *If* $\{x_n, y_n\}$ *are solutions of difference equation system (4). Then all solutions of system (4) are periodic with period six and for* $n = 0, 1, 2, ...,$

$$x_{6n-1} = b, \ \ x_{6n} = a, \ \ x_{6n+1} = \frac{ae}{e-d}, \ \ x_{6n+2} = \frac{e(b-a)}{(d-e)}, \ x_{6n+3} = \frac{d(b-a)}{(d-e)}, \ \ x_{6n+4} = \frac{bd}{d-e},$$

and

$$y_{6n-1} = e, \ \ y_{6n} = d, \ \ y_{6n+1} = \frac{bd}{b-a}, \ \ y_{6n+2} = \frac{b(e-d)}{(a-b)}, \ y_{6n+3} = \frac{a(e-d)}{(a-b)}, \ \ y_{6n+4} = \frac{ae}{(a-b)}.$$

Example 4. See Figure (4) where we take system (4) with the initial conditions $x_{-1} = -2$, $x_0 = .7$, $y_{-1} = -.3$ and $y_0 = 5$.

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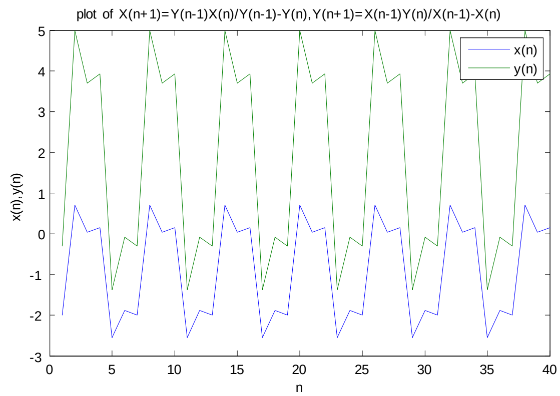

Figure 4.

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Two global iterative methods for ill-posed problems from image restoration*

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In this paper, we mainly focus on applying the global CGLS and GMRES methods for computing numerical approximate solutions of large-scale linear discrete ill-posed problems arising from image restoration. As is well known, global Krylov subspace methods are very popular iterative methods for solving linear systems of equations with multiple right-hand sides. These methods are based on global projections of the initial matrix residual onto a matrix Krylov subspace. It is shown in this paper that when equipped with a suitable stopping rule based on the discrepancy principle, the two global methods act as good regularization methods for ill-posed image restoration problems. To accelerate the convergence of the global methods, we project the computed approximate solutions onto the set of matrices with nonnegative entries before restarting. Some numerical examples from image restoration are given to illustrate the efficiency of the global methods.

Keywords: Image restoration; Global method; Krylov subspace; Project; Regularization

1 Introduction

Image restoration is one of the most important tasks in image processing. This problem is to infer, as best as possible, an original image from a blurred and noisy one. The problem is ubiquitous in science and engineering and has rightfully received a great deal of attention by applied mathematicians, statisticians and engineers [1, 2, 3, 4, 5].

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Under the assumption of linear space-invariant image formation, the noise-free image restoration problem is the convolution of a point spread function h with an original image F, i.e.,

$$G(x,y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(x-k, y-l)F(k,l), \tag{1}$$

where $G(x,y)$ is the blurred image, $F(x,y) \geq 0$ is the original image, $h(x,y)$ is the space-invariant point spread function (PSF). In this work, the PSF is assumed to be known, but it can also be satisfactorily estimated from the degraded image [13, 14, 21].

For a bandlimited image degradation, one must make full use of not only the image in the Field of View (FOV) of the given observation but part of the scenery in the area bordering it as well. Given some assumptions of the values outside FOV known as boundary conditions (BCs), (1) could be expressed in the matrix-vector equation as

$$Hf = g, \tag{2}$$

where H is an $n^2 \times n^2$ matrix associated with the known PSF and with the imposed BCs, $f = \text{vec}(F)$ is an n^2-dimensional nonnegative vector representing the original image and $g = \text{vec}(G)$ is an n^2-dimensional vector representing the blurred image. Here $x = \text{vec}(X)$ with $X \in \mathbb{R}^{n\times n}$ is an $n^2 \times 1$ vector obtained by stacking X's columns.

As we all know, the exact structure of the blurring matrix H depends on the imposed boundary conditions. In various applications, boundary conditions are often chosen for algebraic and computational convenience. Further details on boundary condition problems can be found in, e.g., [3, 17, 20, 24, 27, 28].

In many applications of image restoration, the noise-free vector g is not available. Instead, the vector $\hat{g} = g+\eta$ is known, where η represents the additive noise vector. Therefore, we would like to approximate the exact solution $f^* \geq 0$ of the noise-free linear system of equations (2) by computing an approximate solution of the available linear system of equations of the form

$$H\hat{f} = \hat{g}. \tag{3}$$

Note that since the singular values of H gradually decay to and cluster at zero, it is severely ill-conditioned. Hence, the exact solution of the system (3), if it exists, is not a meaningful approximation of f^* even when the noise η is small.

In general, a regularization method can address this problem efficiently. One may employ it to compute the approximate solutions that are less sensitive to noise than the naive solution. Some popular direct methods such as truncated SVD, Wiener filtering method and Tikhonov regularization method as well as other direct filtering methods can get an approximate solution of f^* [3, 6, 16, 19, 30, 33, 34]. However, it is generally infeasible to calculate the QR factorization or the singular value decomposition (SVD) of H explicitly when H is very large. In this way, direct methods become computationally

impractical. As an alternative to direct methods for large-scale ill-posed problems, iterative methods may be more attractive. As is well known, iterative methods such as CGLS and GMRES equipped with a suitable stopping rule, are some of the most popular and powerful iterative regularization methods [7, 8, 9, 11, 32, 5, 43]. For the purpose of presentation, define the Krylov subspaces $\mathcal{K}_m(H, \hat{g}) = \text{span}\{\hat{g}, H\hat{g}, \cdots, H^{m-1}\hat{g}\}$. In fact, the GMRES method with an initial approximate solution $\hat{f}_0$ finds approximate solutions $\hat{f}_m$ to the minimization problem $\|H\hat{f} - \hat{g}\|$ in the Krylov subspaces $\mathcal{K}_m(H, \hat{g})$ for $m = 1, 2, \cdots$, while the CGLS method with the same initial guess to compute approximate solutions that lie in the Krylov subspaces $\mathcal{K}_m(H^T H, H^T \hat{g})$ for $m = 1, 2, \cdots$. For these iterative methods, the iteration number can be thought of as a regularization parameter. For a regularization parameter k, in the first k iterations, the methods converge to the solution f^*, and then suddenly start to diverge and the noise begins to dominate the solution. Hence, at this stage the iterations should be stopped to avoid interference from the noise components. There are different methods for finding this regularization parameter, such as the discrepancy principle, the L-curve and generalized cross validation. There are advantages and disadvantages to each of these approaches especially for large-scale problems. For instance, it is necessary to have information about the noise for the use of the discrepancy principle. In the case of generalized cross validation, efficient implementation of Tikhonov regularization requires computing the singular value decomposition of the coefficient matrix, which may be computationally impractical for large-scale problems. For the L-curve, it has been advocated for many applications where no prior information about the noise is available. But it may be necessary to solve the corresponding linear systems for several regularization parameters.

The main aim of this paper is to use the global iterative regularization methods to solve the large-scale ill-posed problems arising from image restoration. For simplicity of our analysis, we first consider the case where the horizontal and vertical components of the PSF h is separable. In this case, h can be expressed as a product of a column vector with a row vector $h = cr^T$, where r represents the horizontal component of h and c represents the vertical component. In case of a separable PSF, the blurred matrix H can be written as a Kronecker product of two smaller matrices $H = A_1 \otimes B_1$. Hence, the available linear system (3) can be written as a matrix equation

$$B_1 \hat{F} A_1^T = \hat{G}, \tag{4}$$

where $\hat{F}$ is the original image to be recovered and $\hat{G}$ is the observed image. It is straightforward to see that $\hat{f} = \text{vec}(\hat{F})$ and $\hat{g} = \text{vec}(\hat{G})$. Define a linear operator $\mathcal{L}:\ X \in \mathbb{R}^{n\times n} \to B_1 X A_1^T$ and $\mathcal{L}^T:\ X \in \mathbb{R}^{n\times n} \to A_1 X B_1^T$, then the computation of the approximate solution to (3) is equal to solving the following matrix equation

$$\mathcal{L}\hat{F} = \hat{G}. \tag{5}$$

Global methods are first introduced by [23] for solving matrix equations (linear systems of equations with multiple right-hand sides). Their theoretical analy-

sis and numerical experiments have shown that global iterative algorithms are good matrix Krylov subspace methods. Some further results and applications about global methods can be found in [10, 12, 41, 42]. However, little is known about the behaviors of the global methods when they are applied to the solutions of ill-posed problems that arise from image restoration. In this work, we are concerned with the global CGLS and GMRES methods to compute a meaningful approximate solution of the large discrete ill-posed problem (5). In this research, we employ the discrepancy principle for determining a suitable value of the regularization parameter. Equipped with a stopping rule based on the discrepancy principle, the iterations with each iterative method are terminated as soon as an approximate solution $\hat{F}_m$ of (5) has been determined, such that the associated residual error

$$\|B_1\hat{F}_m A_1^T - \hat{G}\|_F \leq \alpha\delta, \qquad \|B_1\hat{F}_{m-1} A_1^T - \hat{G}\|_F > \alpha\delta,$$

where δ is the 2-norm of the noise, i.e., $\delta = \|\eta\|_2$ and $\alpha \geq 1$ is a fixed constant.

However, as the iteration proceeds, global iterative algorithms will become increasingly expensive and require more storage, because the number of the basis of the matrix Krylov subspace increases. It can be restarted, but with the dimension of the subspace limited, the convergence slows down. To accelerate the convergence of the global methods, we project the obtained approximate solution to the set of non-negative matrices before restarting the algorithms, since the desired solution F^* of (5) with $\text{vec}(F^*) = f^*$ is known to have only non-negative entries. For some discussion about other projected nonnegative methods and their applications, we refer to, e.g., [15, 25, 26, 40].

It should be noted that global methods are also suitable for unseparable cases. If the PSF h is not separable, we can still compute $h = \Sigma_{i=1}^r c_i r_i^T$, and therefore, get $H = \Sigma_{i=1}^r A_i \otimes B_i$; see [19] for details. After defining a new linear operator $\mathcal{L}:\ X \in \mathbb{R}^{n\times n} \to \Sigma_{i=1}^r B_i X A_i^T$, we know that, all results about separable image restoration problems may be easily extended in a nature way to unseparable problems. In addition, one can get an optimal Kronecker product decomposition $H = A_1 \otimes B_1$ and employ it for solving many image restoration problems; see [22, 24, 27] for more details.

The outline of the paper is as follows. In the next section, we recall some results and give some notations. In Section 3, we present the global CGLS and GMRES algorithms for large-scale discrete ill-posed problems from image restoration. For nonnegative constrained image restoration, projected restarted global iterative schemes are proposed in Section 4. Detailed experimental results reporting the performance of the proposed algorithms are given in Section 5. Finally, Section 6 contains concluding remarks.

2 Preliminaries and notations

Throughout the paper, all scalars, vectors and matrices are assumed to be real. Let $\mathscr{M} = \mathbb{R}^{n\times n}$ denotes the linear space of $n \times n$ matrices. For two matrices X and Y of $\mathscr{M}$, we define the inner product $< X, Y >_F = \text{trace}(X^T Y)$ where

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trace(Z) denotes the trace of the square matrix Z and X^T the transpose of the matrix X. It should be noted that the associated norm is just the well-known Frobenius norm denoted by $\|\cdot\|_F$. With the inner product, two matrices X and Y is said to be F-orthogonal if trace$(X^TY) = 0$.

For a matrix $V \in \mathscr{M}$, the matrix Krylov subspace

$$\mathcal{K}_m(\mathcal{L}, V) = \text{span}\{V, \mathcal{L}V, \cdots, \mathcal{L}^{m-1}V\}$$

is the subspace of $\mathscr{M}$ generated by the matrices $V, \mathcal{L}V, \cdots, \mathcal{L}^{m-1}V$. Notice that

$$Z \in \mathcal{K}_m(\mathcal{L}, V) \Leftrightarrow Z = \Sigma_{i=0}^{m-1} a_i \mathcal{L}^i V, \ a_i \in \mathbb{R},$$

In other words, $\mathcal{K}_m(\mathcal{L}, V)$ is the subspace of $\mathscr{M}$ of all $n \times n$ matrices which can be written as $Z = P(\mathcal{L})V$, where $P(\cdot)$ is a polynomial of degree not exceeding $m-1$. Let I_n be the $n \times n$ identity matrix and $a^{(m)} = (a_0, \cdots, a_{m-1})^T$. Then, we have

$$\Sigma_{i=0}^{m-1} a_i \mathcal{L}^i V = [V, \mathcal{L}V, \cdots, \mathcal{L}^{j-1}V](a^{(m)} \otimes I_n).$$

The global Arnoldi process constructs an F-orthonormal basis $V_1, V_2, \cdots, V_m$ of the matrix Krylov subspace $\mathcal{K}_m(\mathcal{L}, V)$, i.e., it holds for the matrices $V_1, V_2, \cdots, V_m$ that trace$(V_i^T V_j) = 0$ for $i \neq j$, $i, j = 1, \cdots, m$ and trace$(V_i^T V_i) = 1$ for $i = 1, \cdots, m$. The algorithm is described as follows:

Algorithm 1. Global Arnoldi process
Input: $\mathcal{L}$ and V.
Output: an F-orthonormal basis $V_1, V_2, \cdots, V_m$.
$\rho = \|V\|_F$; $V_1 = V/\rho$;
for $j = 1, 2, \cdots, m$ do
  $U = \mathcal{L}V_j$;
  for $i = 1, 2, \cdots, j$ do
    $h_{ij} = \text{trace}(U^T V_i)$;
    $U = U - h_{ij}V_i$;
  end do
  $h_{j+1,j} = \|U\|_F$; if $h_{j+1,j} = 0$ stop;
  $V_{j+1} = U/h_{j+1,j}$;
end do

Note that a breakdown occurs in the algorithm if $h_{j+1,j} = 0$ for some j. Let $\bar{H}_m$ be an $(m+1) \times m$ upper Hessenberg matrix arising from the global Arnoldi process and H_m be the $m \times m$ matrix obtained from $\bar{H}_m$ by deleting its last row. We may immediately obtain that

$$\begin{aligned} \mathcal{L}[V_1, \cdots, V_m] &= [V_1, \cdots, V_m](H_m \otimes I_n) \\ &\quad +[O_n, \cdots, O_n, V_{m+1}] \end{aligned}$$

and

$$\mathcal{L}[V_1, \cdots, V_m] = [V_1, \cdots, V_m, V_{m+1}](\bar{H}_m \otimes I_n),$$

where O_n denotes the $n \times n$ zero matrix.

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3 Global CGLS and GMRES methods

In [23], the authors first introduced a global approach for solving matrix equations and derived the global FOM and GMRES methods. They are generalizations of the global minimal residual method proposed by [39] for approximating the inverse of a matrix. These global methods are also effective, as compared to block Krylov subspace methods, when applied for solving general large-scale matrix equations [10, 41, 42]. Here we consider the performance of the global CGLS and GMRES methods when they are applied to the computation of approximate solutions of linear systems of equations with an ill-conditioned matrix. Linear systems with such matrices frequently arise in image restoration.

One of the most powerful and popular global methods is certainly the global CGLS method. The CGLS method is equal to the CG method applied to the norm equation of the linear system (5) with the initial approximate solution $\hat{F}_0$ to compute approximate solutions of the linear system (5) that lie in the Krylov subspaces $\mathcal{K}_m(\mathcal{L}^T\mathcal{L}, \mathcal{L}^T\hat{G})$ for $m = 1, 2, \cdots$. Let $\hat{F}_m$ be an approximate solution of $\mathcal{L}\hat{F} = \hat{G}$ at the mth iteration and $R_m = \hat{G} - \mathcal{L}\hat{F}_m$ be its associated residual. We have the global CGLS solution $\hat{F}_m = \operatorname{argmin}_{\hat{F}} \|\mathcal{L}\hat{F} - \hat{G}\|_F$, s.t., $\hat{F}_m \in \mathcal{K}_m(\mathcal{L}^T\mathcal{L}, \mathcal{L}^T\hat{G})$. With the discrepancy principle, the global CGLS method determining an approximate solution of a discrete ill-posed problem can be described as follows:

Algorithm 2. Global CGLS algorithm with the discrepancy principle
Input: $\mathcal{L}$, $\hat{G}$, $\hat{F}_0$, α and δ.
Output: an approximate solution $\hat{F}_m$.
Start:
$R_0 = \hat{G} - \mathcal{L}\hat{F}_0$; $D_0 = \mathcal{L}^T R_0$; $S_0 = \mathcal{L}^T R_0$; m=0;
iterate:
while $\|R_m\|_F > \alpha\delta$ do
$m = m + 1$;
$Q_m = \mathcal{L}D_{m-1}$;
$\alpha_m = \text{trace}(S_{m-1}^T S_{m-1})/\text{trace}(Q_m^T Q_m)$;
$\hat{F}_m = \hat{F}_{m-1} + \alpha_m D_{m-1}$;
$R_m = R_{m-1} - \alpha_m Q_m$;
$S_m = \mathcal{L}^T R_m$;
$\beta_k = \text{trace}(S_m^T S_m)/\text{trace}(S_{m-1}^T S_{m-1})$;
$D_m = S_k + \beta_m D_{m-1}$;
end do

In order to describe the global GMRES method, we further let $\mathfrak{V}_m$ denote the $n \times nm$ block matrix: $\mathfrak{V}_m = [V_1, \cdots, V_m]$. Assume that m steps of the global Arnoldi algorithm have been carried out, then the global GMRES method generates a new approximation $\hat{F}_m$ such that

$$\begin{aligned} \hat{F}_m &= \hat{F}_0 + \mathfrak{V}_m(a^{(m)} \otimes I_n), \\ R_m &= R_0 - \mathcal{L}\mathfrak{V}_m(a^{(m)} \otimes I_n), \end{aligned}$$

where the m-dimensional vector $a^{(m)}$ is determined by imposing a given criteria.

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The global GMRES method is characterized by selecting $a^{(m)}$ in such a way to minimize the Frobenius norm of the residual at each m-th step, i.e., $\|R_m\|_F = \min\|R_0 - \mathcal{L}\mathfrak{V}_m(a^{(m)} \otimes I_n)\|_F$. Taking into account the F-orthonormal basis $\{V_1, \cdots, V_m\}$ of the subspace $\mathcal{K}_m(\mathcal{L}, V)$, obtained from the global Arnoldi process, we can transform the above minimization problem into a minimization problem with l_2-norm. So we can determine $a^{(m)}$ by computing the solution of

$$\min\| \|R_0\|_F e_1 - \bar{H}_m a^{(m)}\|_2. \tag{6}$$

To improve the efficiency of the GMRES algorithm, it is necessary to devise a stopping criterion which does not require the explicit evaluation of the approximate solution $\hat{F}_m$ at each step. This is possible, provided that the upper Hessenberg matrix $\bar{H}_m$ is transformed into an upper triangular matrix $U_m \in R^{(m+1)\times m}$ with $u_{m+1,m} = 0$ such that $Q_m^T U_m = \bar{H}_m$, where Q_m is a matrix obtained as the product of m Givens rotations. Then, since Q_m is unitary, it is easy to see that $\min\| \|R_0\|_F e_1 - \bar{H}_k a^{(m)}\|_2 = \min\| \|R_0\|_F Q_m e_1 - U_m a^{(m)}\|_2$. It can also show that the $m+1$-th component of $\|R_0\|_F Q_m e_1$ is, in absolute value, the Frobenius norm of the residual at the m-th step. Hence, we terminate the computations as soon as the Frobenius norm of an residual is smaller than $\alpha\delta$ for the ill-posed problems. It means that the residual satisfies the discrepancy principle and an suitable approximate solution has been determined. Therefore, equipped with a stopping rule based on the discrepancy principle, the global GMRES method determining an approximate solution of a discrete ill-posed problem takes the following form:

Algorithm 3. Global GMRES method with the discrepancy principle
Input: $\mathcal{L}$, $\hat{G}$, $\hat{F}_0$, α and δ.
Output: an approximate solution $\hat{F}_k$.
Start:
$R_0 = \hat{G} - \mathcal{L}\hat{F}_0$; $\rho = \|R_0\|_F$; $V_1 = R_0/\rho$; m=0;
iterate:
while $\|R_m\|_F > \alpha\delta$ do
$m = m + 1$;
compute the mth basis matrix V_m and update $\bar{H}_m$ by Algorithm 1;
construct the QR-factorization of $\bar{H}_m$, and compute $\|R_m\|_F = \rho e_{m+1}^T Q_m e_1$;
end do
form the approximate solution:
$\hat{F}_m = \hat{F}_0 + \mathfrak{V}_m(a^{(m)} \otimes I_n)$.

4 Projected restarted global methods

The global methods entail a high computational effort and a large amount of memory, unless convergence occurs after few iterations. But, one can remedy two drawbacks by restarting the global methods periodically. Given a fixed N, the restarted global methods compute a sequence of approximate solutions $\hat{F}_i$ until $\hat{F}_i$ is acceptable or $i = N$. If the solution is not found, then a new

starting matrix is chosen on which the global methods are applied again. Often, the global methods are restarted from the last computed approximation, i.e., $\hat{F}_0 = \hat{F}_N$ to comply with the monotonicity property even when restarting. The process goes on until a good enough approximation is found.

In order to compute regularized nonnegative solutions to (5), in the present work, here we consider the projected restarted global methods. The methods are based on the following approach. A few steps of a global iterative method (global CGLS or global GMRES) are applied to the available matrix equation (5), and then the computed approximate solution is projected onto the set of matrices with non-negative entries. If the projected approximate solution satisfies the discrepancy principle, then we accept the projected approximate solution as an approximate solution of the available matrix equation (5). Otherwise we restart the associated iterative method, replacing the initial guess with the projected approximate solution.

We define P_+ by the projector onto the set of nonnegative matrices. That is,

$$P_+(X) = X_+,$$

where $X = (x_{ij})$ and $X_+ = ((x_+)_{ij})$ satisfy that $(x_+)_{ij} = x_{ij}$ for $x_{ij} \geq 0$ and $(x_+)_{ij} = 0$ for others. The projected restarted global iterative scheme with a stopping rule based on the discrepancy principle is summarized as follows:

Algorithm 4. Projected restarted global method

Input: $\mathcal{L}$, $\hat{G}$, $\hat{F}_0$, α, δ and N;

Output: an approximate solution $\hat{F}_{m+1}$.

Start:

$R_0 = \hat{G} - \mathcal{L}\hat{F}_0$; m=0;

Solve:

Let $R_m = \hat{G} - \mathcal{L}\hat{F}_m$. Compute an approximate solution $\hat{F}_m$ of the system $\mathcal{L}\hat{F} = R_m$ with a given global iterative algorithm. When either the discrepancy principle is satisfied or the maximum number of consecutive iterations, N, has been carried out, terminate the iterations.

Project:

$\hat{F}_{m+1} = P_+(\hat{F}_m)$;

Check:

if $\|R_{m+1}\|_F > \alpha\delta$, then let $m = m+1$ and go to the first step; if $\hat{F}_{m+1} \leq \alpha\delta$, then exit with an approximate solution of (5).

It should be noted that the projected restarted global methods can be looked as a inner-outer iteration scheme. In the inner iteration, we apply a given global method (Global CGLS or Global GMRES) to solve the available matrix equation (5). In this process, the iterations are terminated as soon as an acceptable solution that satisfies the discrepancy principle has been determined , or the maximum number of consecutive iterations N has been carried out. The outer iteration is just the projected process. If the projected approximate solution satisfies the discrepancy principle, then we accept it as an approximate solution of (5). The projected restarted global methods differ from the restarted global

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Table 1: Relative errors and number of iterations with different BCs.

BC	Method	Relative error	It.s	Original error
Reflexive	Global CGLS	0.0804	10	0.1726
	Global GMRES	0.0843	4	
Antireflexive	Global CGLS	0.0793	12	
	Global GMRES	0.0858	4	
Zero	Global CGLS	0.1306	10	
	Global GMRES	0.1260	4	

methods in that the projected global methods are restarted with the orthogonal projection as an initial iterate.

5 Numerical examples

In this section, we provide some experimental results of using the global CGLS and GMRES methods to solve large scale linear discrete ill-posed problems that arise from image restoration. All the experiments given in this paper were performed in Matlab 7.0. The results were obtained by running the Matlab codes on a Intel Core(TM)2 Duo CPU (2.93GHz, 2.93GHz) computer with RAM of 2048M. In all tests, the black image (zero matrix) is the initial guess and the iteration process is terminated if the stopping criterion of the discrepancy principle or the maximum number of iteration is met. In the first two numerical tests, we hope that the natural boundary elements would contribute to these blurring. So we need to perform the blurring operation on larger images, and cut out their central parts. Then using the built-in MATLAB function *randn*, we add Guassian white noise to the blurred data for generating these blurred and noisy images.

The first test data we use is shown in Figure 1. In the true image, the FOV is delimited by white lines. The 192-by-192 blurred and noisy image shown on the right side of Figure 1, has been cut out from the larger 256-by-256 image. In this test, the separable blur we consider is given by the mask $\frac{1}{100}(1,1,1,4,1,1,1)^T(1,1,1,4,1,1,1)$ arising from [28, 29]. In this test, we add 0.02 Gaussian white noise to the blurred pixel values.

In Table 1, we report relative errors and the numbers of iterations (regularization parameters) under the reflexive boundary condition , the antireflexive boundary condition and the zero boundary condition. The relative error is defined by $\frac{\|\hat{F}_k - F_{\text{true}}\|_F^2}{\|F_{\text{true}}\|_F}$, where $F_{\text{true}} = F^*$ is the original image and the $\hat{F}_k$ is the computed solution with the global methods at the kth iteration. The corresponding computed restorations by the global methods are shown in Figure 2. In this test, we choose $\alpha = 1$. We observe from Table 1 and Figure 2 that the global CGLS and GMRES methods are very effective for separable image restoration problems under different boundary conditions.

In Test 2, we are dealing with a strongly nonsymmetric PSF h arising from

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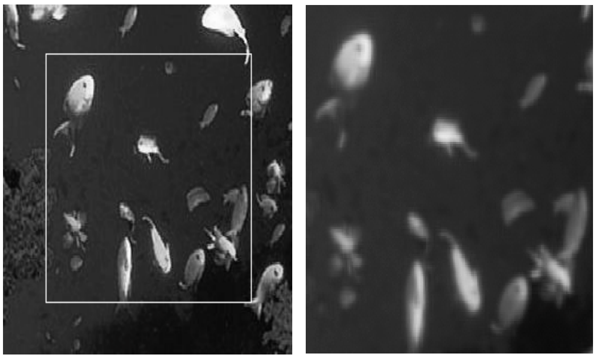
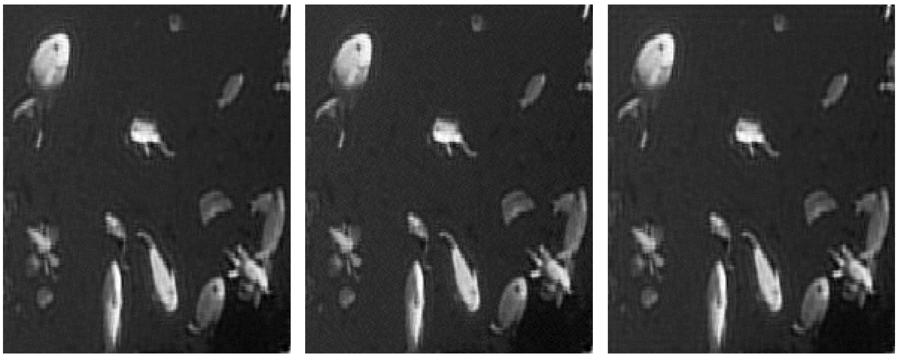
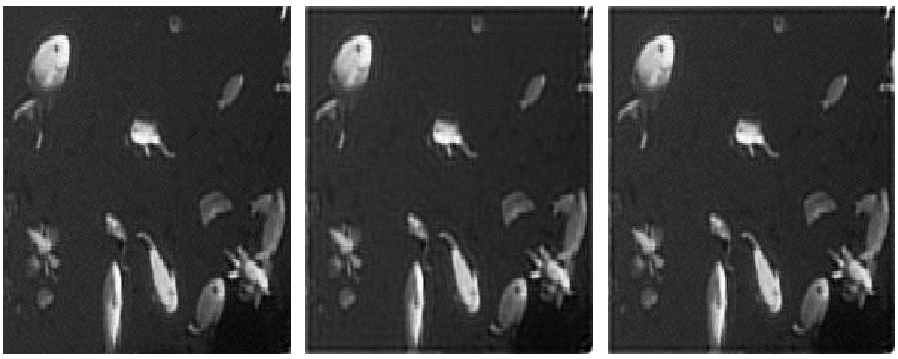


(a) True image (b) Blurred and noisy image

Figure 1: True image and blurred image with noise of Test 1.



(a) Global CGLS with the reflexive BC (b) Global GMRES with the reflexive BC (c) Global CGLS with the antireflexive BC



(d) Global GMRES with the antireflexive BC (e) Global CGLS with the zero BC (f) Global GMRES with the zero BC

Figure 2: Computed restorations with three different boundary conditions

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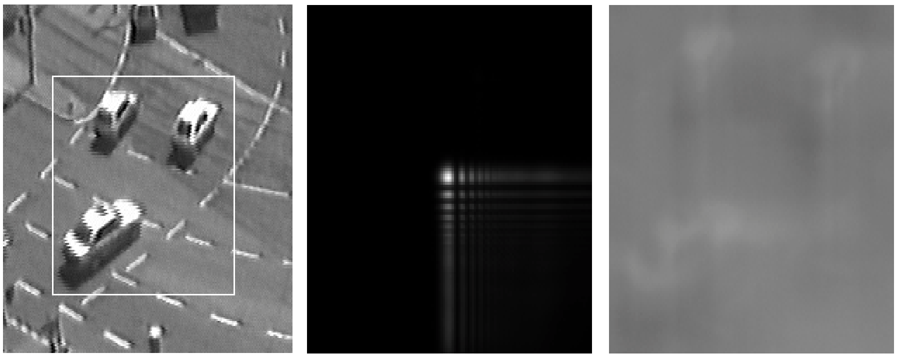

(a) True image (b) Unseparable blur (c) Blurred and noisy image

Figure 3: True image, PSF and blurred image with noise of Test 2.

Table 2: Relative errors and Number of iterations with two different BCs.

BC	Method	Error	It.s	Original error
Reflexive	Global CGLS	0.2446	91	0.2960
	Global GMRES	0.2097	11	
Antireflexive	Global CGLS	0.2096	15	
	Global GMRES	0.2310	9	

wavefront coding, where a cubic phase filter is used to improve depth of field resolution in light efficient wide aperture optical systems [35, 24, 27]. The data of Test 2 is given by Figure 3. In the true image, the FOV is also delimited by white lines. The 128-by-128 blurred and noisy image shown on the right side of Figure 3, has been cut out from the larger 256-by-256 image. In this test, 0.2% Gaussian white noise was added to the blurred pixel values. It is not difficult to see that the PSF h in this test is not separable. So we need to compute a rank-one approximation of h by computing the SVD of h, and then construct A_1 and B_1 as described in section 1 by using a method in [19]. That is,

$$h \approx b_1 a_1^T \Longrightarrow H \approx A_1 \otimes B_1.$$

In fact, one can get an optimal Kronecker product decomposition $H = A_1 \otimes B_1$ [22, 24, 27]. The results about relative errors and the number of iterations are given in Table 2 for $\alpha = 1.5$. Corresponding computed restorations are shown in Figure 4. These results clearly show the satisfactory efficiency of global CGLS and GMRES for unseparable image restoration problems.

The aim of the third test is to give evidence of the efficiency of projected restarted global methods. The third test data we use is shown in Figure 5. In this test, the symmetric separable truncated Guassian blur we consider is given

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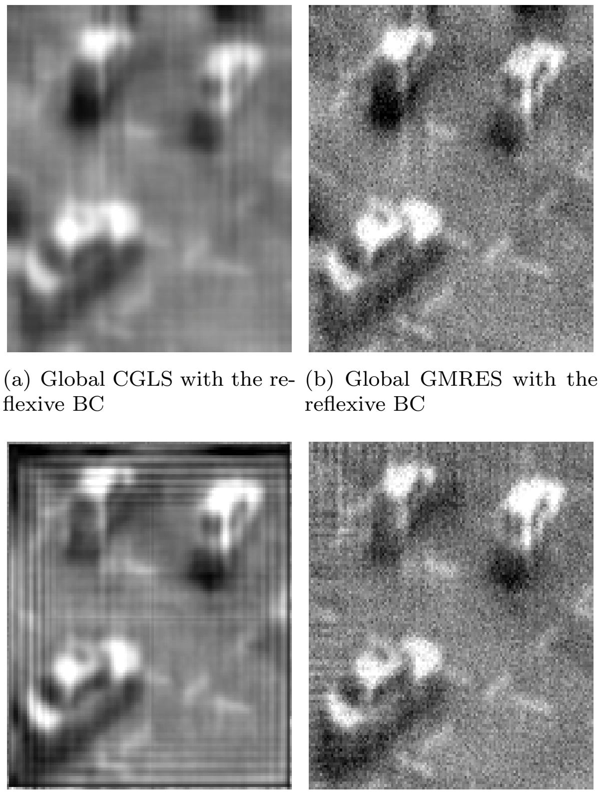
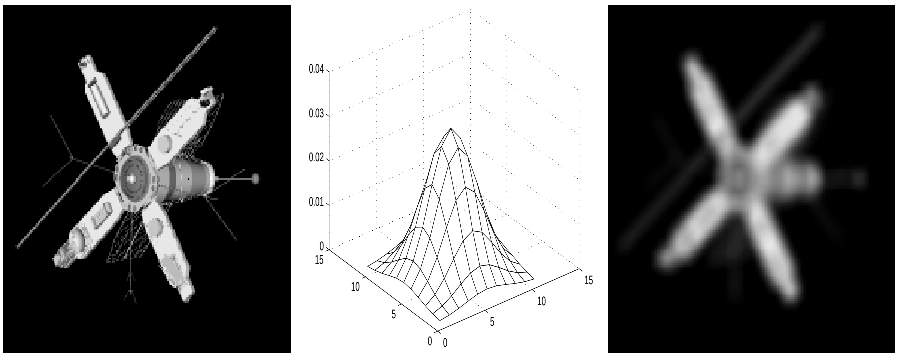


(a) Global CGLS with the reflexive BC
(b) Global GMRES with the reflexive BC
(c) Global CGLS with the antireflexive BC
(d) Global GMRES with the antireflexive BC

Figure 4: Computed restorations with two boundary conditions



(a) True image (b) Truncated Guassian blur (c) Blurred and noisy image

Figure 5: Original image, PSF and blurred image with noise of Test 3.

by

$$h_{ij} = \begin{cases} ce^{-0.1(i^2+j^2)} & \text{if } |i-j| \leq 5, \\ 0 & \text{otherwise,} \end{cases}$$

where c is the normalization constant such that $\sum_{i,j} h_{i,j} = 1$; see Figure 5. After blurring, we add 0.01 Gaussian white noise to the blur data.

In Test 3, we apply the projected restarted global methods and the restarted global methods with the maximum number of inner iterations $N = 10$, the maximum number of outer iterations $M = 7$ and $\alpha = 1.2$. The outer iterations were terminated where the discrepancy principle is satisfied or the maximum number of outer iterations is reached. Figure 6 plots the relative errors in unconstrained and nonnegative constrained computed solutions at end of each outer iteration. Figure 7 displays the computed approximate solutions using projected restarted global methods and the restarted global methods under the zero boundary condition.

In the fourth experiment, we compare the performance of the global CGLS and GMRES methods with that of the other two popular regularization methods (truncated singular value decomposition and Tikhonov regularization) in 3D image restoration under the zero BC. The true image is $128 \times 128 \times 27$ simulated MRI of a human brain, available in the the Matlab Image Processing Toolbox. Restoration of this image was used as a test problem in [36, 38]. To produce the distorted image, we build an out-of-focus PSF using the function *psfDefocus* with $dim = 11$ and $R = 5$ in [3], and convolve it with the MRI image, then add 5% Gaussian noise to the result. The test data is shown in Figure 8. The original relative error of the blurred and noisy image for Test 4 is 0.3555.

Since the PSF used in Test 4 is not separable, we construct the approximate Kronecker product decomposition $H = A_1 \otimes B_1$. We build the two matrices A_1 and B_1 under the zero BC. A comparison of relative errors and CPU time with four different methods based on the Kronecker product decomposition is

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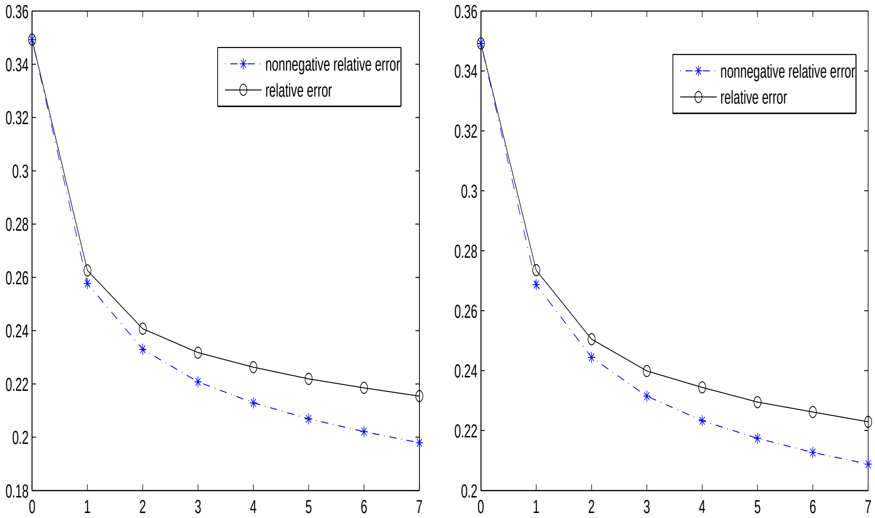

(a) Global GMRES (b) Global CGLS

Figure 6: Comparisons of relative errors of the projected restarted global methods and the restarted global methods.

Table 3: Relative errors and CPU time with different methods for Test 4.

Method	rel. error	CPU time
Global CGLS	0.2750	0.1866 (s)
Global GMRES	0.2750	0.1312 (s)
TSVD	0.2895	0.2351 (s)
Tikhonov	0.2956	0.5053 (s)

given in Table 3. We choose $\alpha = 1$ in the global methods. The regularization parameters of the TSVD and Tikhonov regularization methods are chosen by the generalized cross validation method. It is not difficult to see that the relative errors of our global methods are smaller than these of the TSVD and Tikhonov regularization methods. It cost less CPU time for the global methods obtaining the approximate restored images than the two other widely used regularization methods. It is mainly because that the main operations of in the global methods is the matrix-vector product which theoretically require $O(n^2 \log n)$ floating point operations while the TSVD and Tikhonov regularization methods need the singular value decompositions of two smaller matrices A_1 and B_1 which require $O(n^3)$ floating point operations.

The corresponding restored images using the four different methods under the zero BC are shown in Figure 9. The restored image in Figure 9(a) is determined after 10 steps of the global CGLS method and the restored image in

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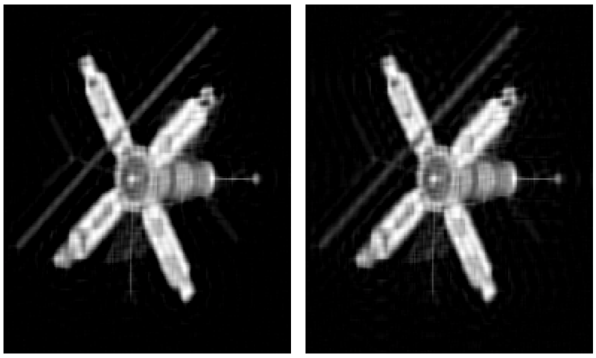
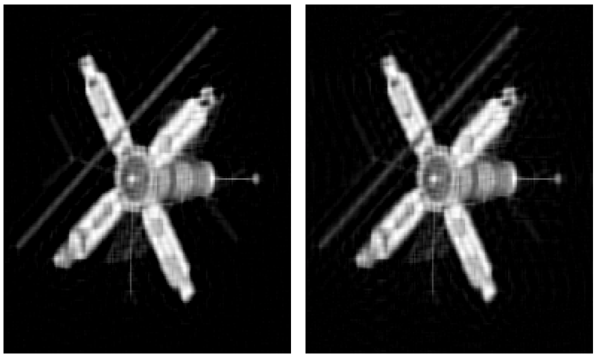
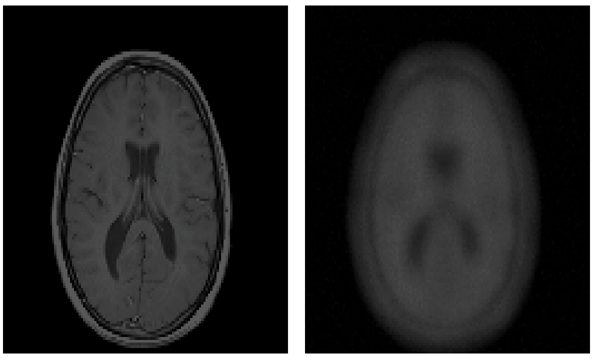


(a) CGLS restored image with nonnegative constraint (b) CGLS restored image



(c) GMRES restored image with nonnegative constraint (d) GMRES restored image

Figure 7: Restarted restored images of Test 3.



(a) True image (b) Blurred and noisy image

Figure 8: True image and blurred image with noise of Test 4.

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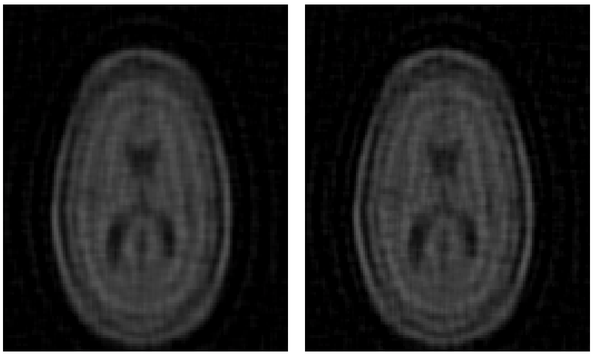
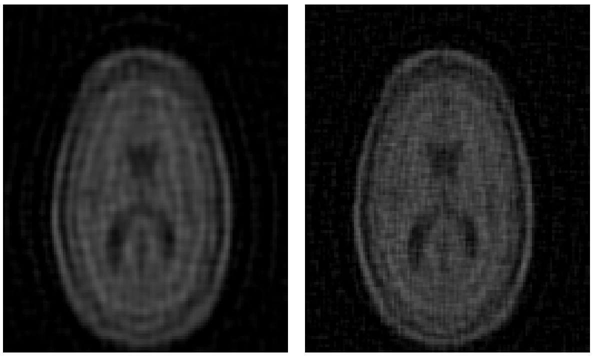


(a) Global CGLS (b) Global GMRES



(c) TSVD (d) Tikhonov regularization

Figure 9: Restored images using the zero BC for Test 4.

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Figure 9(b) is determined after 9 steps of the global GMRES method. The two restored images are obtained with a stopping rule based on the discrepancy principle.

6 Conclusion

In this paper, we present the global CGLS and GMRES methods for computing approximate solutions of large-scale ill-posed problems arising from image restoration. For constrained problems, we propose the projected restarted global methods. The global iterative algorithms presented exhibit semiconvergence on unregularized problems. Thus, regularization can be achieved by early termination of the iterations. Several numerical examples are used to show that the global methods are very effective for image restoration problems.

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A note on the symmetric properties for the second kind twisted (h, q)-Euler polynomials

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Abstract : In [9], we studied the second kind twisted (h, q)-Euler numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the the second kind twisted Euler polynomials.

Key words : the second kind Euler numbers and polynomials, the second kind twisted Euler numbers and polynomials, twisted Euler numbers and polynomials, the second kind twisted (h, q)-Euler numbers and polynomials, alternating sums

1 Introduction

Euler numbers, Euler polynomials, q-Euler numbers, q-Euler polynomials, the second kind Euler number and the second kind Euler polynomials were studied by many authors (see for details [1-9]). Euler numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second twisted (h, q)-Euler polynomials and the power sums.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by

$g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x)-g(y)}{x-y}$$

have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For $g \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows(see [1,2]):

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

In [9], we introduced the second kind twisted (h,q)-Euler numbers $E_{n,q,\omega}^{(h)}$ and polynomials $E_{n,q,\omega}^{(h)}(x)$ and investigate their properties.

Let us define the second kind twisted (h,q)-Euler numbers $E_{n,q,\omega}^{(h)}$ and polynomials $E_{n,q,\omega}^{(h)}(x)$ as follows:

$$I_{-1}(\phi_\omega(y) q^{hy} e^{(2y+1)t}) = \int_{\mathbb{Z}_p} \phi_\omega(y) q^{hy} e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,\omega}^{(h)} \frac{t^n}{n!}, \tag{1.3}$$

$$\begin{aligned} I_{-1}(\phi_\omega(y) q^{hy} e^{(2y+1+x)t}) &= \int_{\mathbb{Z}_p} \phi_\omega(y) q^{hy} e^{(2y+1+x)t} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} E_{n,q,\omega}^{(h)}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.4}$$

The following elementary properties of the second kind twisted (h,q)-Euler numbers $E_{n,q,\omega}^{(h)}$ and polynomials $E_{n,q,\omega}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4)(see, for details, [9]). We, therefore, choose to omit details involved. By (1.3) and (1.4), we obtain the following Witt's formula.

Theorem 1. For $\omega \in T_p$ and $h \in \mathbb{Z}$, we have

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^{hx} (2x+1)^n d\mu_{-1}(x) = E_{n,q,\omega}^{(h)},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^{hy} (2y+1+x)^n d\mu_{-1}(y) = E_{n,q,\omega}^{(h)}(x).$$

Theorem 2. For any positive integer n, we have

$$E_{n,q,\omega}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q,\omega}^{(h)} x^{n-k}.$$

2

2 The alternating sums of powers of consecutive (h,q)-odd integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$ and $h \in \mathbb{Z}$. Let ω be the p^N-th root of unity. By using (1.4), we give the alternating sums of powers of consecutive (h,q)-integers as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q,\omega}^{(h)} \frac{t^n}{n!} &= \frac{2e^t}{\omega q^h e^{2t}+1} \\ &= 2\sum_{n=0}^{\infty}(-1)^n \omega^n q^{nh} e^{(2n+1)t}.\end{aligned}$$

From the above, we obtain

$$\begin{aligned}&-\sum_{n=0}^{\infty}(-1)^n \omega^n q^{nh} e^{(2n+2k+1)t} + \sum_{n=0}^{\infty}(-1)^{n-k}\omega^{n-k}q^{(n-k)h}e^{(2n+1)t} \\ &= \sum_{n=0}^{k-1}(-1)^{n-k}\omega^{n-k}q^{(n-k)h}e^{(2n+1)t}.\end{aligned}$$

By using (1.3)and (1.4), we obtain

$$\begin{aligned}&-\frac{1}{2}\sum_{j=0}^{\infty} E_{j,q,\omega}^{(h)}(2k)\frac{t^j}{j!} + \frac{1}{2}(-1)^{-k}\omega^{-k}q^{-kh}\sum_{j=0}^{\infty} E_{j,q,\omega}^{(h)}\frac{t^j}{j!} \\ &= \sum_{j=0}^{\infty}\left((-1)^{-k}\omega^{-k}q^{-kh}\sum_{n=0}^{k-1}(-1)^n\omega^n q^{nh}(2n+1)^j\right)\frac{t^j}{j!}.\end{aligned}$$

By comparing coefficients of $\dfrac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1}(-1)^n\omega^n q^{nh}(2n+1)^j = \frac{(-1)^{k+1}\omega^k q^{kh}E_{j,q,\omega}^{(h)}(2k) + E_{j,q,\omega}^{(h)}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$ and ω be the p^N-th root of unity. Then we obtain

$$\begin{aligned}T_{j,q,\omega}^{(h)}(k-1) &= \sum_{n=0}^{k-1}(-1)^n\omega^n q^{nh}(2n+1)^j \\ &= \frac{(-1)^{k+1}\omega^k q^{kh}E_{j,q,\omega}^{(h)}(2k) + E_{j,q,\omega}^{(h)}}{2}.\end{aligned}$$

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Remark 4. For the alternating sums of powers of consecutive odd integers, we have

$$\begin{aligned}\lim_{q\to 1} T^{(h)}_{j,q,\omega}(k-1) &= \sum_{n=0}^{k-1}(-1)^n\omega^n(2n+1)^j \\ &= \frac{(-1)^{k+1}\omega^k E_{j,\omega}(2k)+E_{j,\omega}}{2},\end{aligned}$$

where $E_{j,\omega}(x)$ and $E_{j,\omega}$ denote the second kind twisted Euler polynomials and the second kind twisted Euler numbers, respectively.

3 The symmetry property of the q-deformed fermionic integral on $\mathbb{Z}_p$

In this section, we assume that $q \in \mathbb{C}_p$ and $\omega \in T_p$. We obtain recurrence identities the second twisted (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers. By using (1.1), we have

$$I_{-1}(g_n)+(-1)^{n-1}I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k),$$

where $n \in \mathbb{N}, g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n)+I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k) \text{ (see [1], [2], [3], [5]).} \tag{3.1}$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x)+\int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k). \tag{3.2}$$

Substituting $g(x) = \omega^x q^{hx}e^{(2x+1)t}$ into the above, we obtain

$$\begin{aligned}&\int_{\mathbb{Z}_p} \omega^{x+n}q^{h(x+n)}e^{(2(x+n)+1)t}d\mu_{-1}(x)+\int_{\mathbb{Z}_p} \omega^x q^{hx}e^{(2x+1)t}d\mu_{-1}(x) \\ &= 2\sum_{j=0}^{n-1}(-1)^j\omega^j q^{hj}e^{(2j+1)t}.\end{aligned} \tag{3.3}$$

After some elementary calculations, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} \omega^x q^{hx}e^{(2x+1)t}d\mu_{-1}(x) = \frac{2e^t}{\omega q^h e^{2t}+1}, \\ &\int_{\mathbb{Z}_p} \omega^{x+n}q^{h(x+n)}e^{(2(x+n)+1)t}d\mu_{-1}(x) = \omega^n q^{hn}e^{2nt}\frac{2e^t}{\omega q^h e^{2t}+1}.\end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^{hx} e^{(2x+1)t} d\mu_{-1}(x)$$
$$= \frac{2e^t(1+\omega^n q^{hn} e^{2nt})}{\omega q^h e^{2t}+1}.$$

From the above, we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} \omega^{x+n} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^{hx} e^{(2x+1)t} d\mu_{-1}(x)\\ &= \frac{2\int_{\mathbb{Z}_p} \omega^x q^{hx} e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{hnx} e^{2ntx} d\mu_{-1}(x)}.\end{aligned} \tag{3.5}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (3.3), we obtain

$$\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p} \omega^{x+n} q^{h(x+n)} (2x+1+2n)^m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^{hx} (2x+1)^m d\mu_{-1}(x)\right)\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{n-1}(-1)^j \omega^j q^{hj}(2j+1)^m\right)\frac{t^m}{m!}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\omega^n q^{hn}\sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}\omega^x q^{hx}(2x+1)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p}\omega^x q^{hx}(2x+1)^m d\mu_{-1}(x)$$
$$= 2\sum_{j=0}^{n-1}(-1)^j\omega^j q^{hj}(2j+1)^m$$

By using (2.1), we have

$$\begin{aligned}&\omega^n q^{hn}\sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}\omega^x q^{hx}(2x+1)^k d\mu_{-1}(x)\\ &+\int_{\mathbb{Z}_p}\omega^x q^{hx}(2x+1)^m d\mu_{-1}(x) = 2T_{m,q,\omega}^{(h)}(n-1).\end{aligned} \tag{3.6}$$

By using (3.5) and (3.6), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we obtain

$$\frac{2\int_{\mathbb{Z}_p}\omega^x q^{hx} e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}\omega^{nx} q^{hnx} e^{2ntx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty}\left(2T_{m,q,\omega}^{(h)}(n-1)\right)\frac{t^m}{m!}. \tag{3.7}$$

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Let w_1 and w_2 be odd positive integers. By using (3.7), we have

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \omega^{(w_1x_1+w_2x_2)} q^{h(w_1x_1+w_2x_2)} e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t} d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} \omega^{w_1w_2x} q^{hw_1w_2x} e^{2w_1w_2xt} d\mu_{-1}(x)}$$
$$= \frac{2e^{w_1t}e^{w_2t}e^{w_1w_2xt}(\omega^{w_1w_2}q^{hw_1w_2}e^{2w_1w_2t}+1)}{(\omega^{w_1}q^{hw_1}e^{2w_1t}+1)(\omega^{w_2}q^{hw_2}e^{2w_2t}+1)} \tag{3.8}$$

By using (3.7) and (3.8), after elementary calculations, we obtain

$$\begin{aligned} a &= \left(\frac{1}{2}\int_{\mathbb{Z}_p} \omega^{w_1x_1} q^{hw_1x_1} e^{(w_1(2x_1+1)+w_1w_2x)t} d\mu_{-1}(x_1)\right) \\ &\times \left(\frac{2\int_{\mathbb{Z}_p} \omega^{w_2x_2} q^{hw_2x_2} e^{(2x_2+1)(w_2t)} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} \omega^{w_1w_2x} q^{hw_1w_2x} e^{2w_1w_2tx} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} E^{(h)}_{m,q^{w_1},\omega^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} T^{(h)}_{m,q^{w_2},\omega^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right). \end{aligned} \tag{3.9}$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E^{(h)}_{j,q^{w_1},\omega^{w_1}}(w_2x)w_1^j T^{(h)}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{3.10}$$

By using the symmetry in (3.9), we have

$$\begin{aligned} a &= \left(\frac{1}{2}\int_{\mathbb{Z}_p} \omega^{w_2x_2} q^{hw_2x_2} e^{(w_2(2x_2+1)+w_1w_2x)t} d\mu_{-1}(x_2)\right) \\ &\times \left(\frac{2\int_{\mathbb{Z}_p} \omega^{w_1x_1} q^{hw_1x_1} e^{(2x_1+1)(w_1t)} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} \omega^{w_1w_2x} q^{hw_1w_2x} e^{2w_1w_2tx} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} E^{(h)}_{m,q^{w_2},\omega^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} T^{(h)}_{m,q^{w_1},\omega^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right). \end{aligned}$$

Thus we have

$$a = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E^{(h)}_{j,q^{w_2},\omega^{w_2}}(w_1x)w_2^j T^{(h)}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \tag{3.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.10) and (3.11), we arrive at the following theorem:

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Theorem 6. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^{j}E^{(h)}_{j,q^{w_2},\omega^{w_2}}(w_1x)T^{(h)}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)$$
$$=\sum_{j=0}^{m}\binom{m}{j}w_1^{j}w_2^{m-j}E^{(h)}_{j,q^{w_1},\omega^{w_1}}(w_2x)T^{(h)}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1),$$

where $E^{(h)}_{k,q,\omega}(x)$ and $T^{(h)}_{m,q,\omega}(k)$ denote the second kind twisted (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j}x^{j-k}E^{(h)}_{k,q^{w_2},\omega^{w_2}}T^{(h)}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)$$
$$=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k}x^{j-k}E^{(h)}_{k,q^{w_1},\omega^{w_1}}T^{(h)}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1).$$

By using (3.8), we have

$$\begin{aligned}
a &= \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\\
&\quad\times\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\
&= \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\\
&\quad\times\left(2\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2hj}e^{(2j+1)(w_2t)}\right)\\
&= \sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2hj}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{hw_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2hj}E^{(h)}_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.
\end{aligned}\tag{3.12}$$

By using the symmetry property in (3.12), we also have

$$
\begin{aligned}
a &= \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\\
&\quad\times\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\
&= \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\\
&\quad\times\left(2\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1hj}e^{(2j+1)(w_1t)}\right)\\
&= \sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1hj}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{hw_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1hj}E^{(h)}_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.13}
$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.12) and (3.13), we have the following theorem.

Theorem 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$
\begin{aligned}
&\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2hj}E^{(h)}_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\\
&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1hj}E^{(h)}_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n.
\end{aligned}
\tag{3.14}
$$

Substituting $w_1 = 1$ into (3.14), we arrive at the following corollary.

Corollary 9. Let w_2 be odd positive integer. Then we obtain

$$
E^{(h)}_{n,q,\omega}(x) = w_2^n\sum_{j=0}^{w_2-1}(-1)^j\omega^j q^{hj}E^{(h)}_{n,q^{w_2},\omega^{w_2}}\left(\frac{x-w_2+(2j+1)}{w_2}\right).
$$

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A note on Hausdorff intuitionstic fuzzy metric spaces

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In this paper, we explore two types of Hausdorff intuitionistic fuzzy metrics on compact sets, and prove that they are identical. Also, we prove that if two intuitionistic fuzzy metric spaces induce the same topology, then the corresponding Hausdorff intuitionistic fuzzy metric topological spaces coincide.

Keywords: Intuitionistic fuzzy metric space, Hausdorff intuitionistic fuzzy metric space, Topology, Continuous, Uniformity.

AMS Subject Classifications: 54A40, 54E70, 54E35

1 Introduction

Motivated by the importance of obtaining an appropriate notion of a fuzzy metric space to fuzzy topological spaces, many authors have introduced several notions of fuzzy metric in different ways [4, 5, 9, 10]. In particular, George and Veeramani [5] gave a definition of fuzzy metric with the help of continuous t-norms and proved that the topology induced by this fuzzy metric is first countable and Hausdorff. On the other hand, Atanassov [1] defined and studied the notion of intuitionistic fuzzy sets, and later much progress in the study of intuitionistic fuzzy sets can be found in [1, 6, 9]. Using the idea of intuitionistic fuzzy sets due to Atanassov, Park [12] presented a notion of an intuitionistic fuzzy metric space as a generalization of a fuzzy metric space gave by George and Veeramani. Furthermore, some known results of metric spaces including Uniform limit theorem and Baire's theorem for intuitionstic fuzzy metric spaces are proven in [12]. Saadati and Park [15] studied some properties of intuitionistic

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fuzzy metric spaces as completeness, precompactness and compactness. There have been other more contributions in the study of intuitionstic fuzzy metric spaces by many authors [7, 8, 11, 13].

In [2], Beer studied three consistent types of Hausdorff metrics on compact sets. Gregori et al [7] gave a definition of Hausdorff intuitionistic fuzzy metric on compact sets, which corresponds to the first type of Hausdorff metric due to Beer [2]. It is a natural problem to explore other corresponding types of Hausdorff intuitionistic fuzzy metrics on compact sets. This is done in the present paper.

We construct another type of Hausdorff intuitionistic fuzzy metric on compact sets, which corresponds to the second type of Hausdorff metric due to Beer [2]. Also, we study some continuities on an intuitionistic fuzzy metric space. Using these continuities, we show that the two types of Hausdorff intuitionistic fuzzy metrics on compact sets coincide. Moreover, we prove that if two intuitionistic fuzzy metric spaces induce the same topology, then the corresponding Hausdorff intuitionistic fuzzy metric topological spaces coincide.

2 Preliminaries

In the section we recall some concepts and auxiliary results. Our basic reference for general topology is [3].

Definition 2.1 [16] A *continuous t-norm* is a binary operation $*: [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:
(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a * b = a \cdot b; a * b = \min\{a, b\}$.

Definition 2.2 [16] A *continuous t-conorm* is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:
(a) $\diamond$ is associative and commutative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-conorms: $a \diamond b = \min\{a + b, 1\}; a \diamond b = \max\{a, b\}$.

Definition 2.3 [12] An *intuitionstic fuzzy metric space* is 5-tuple $(X, M, N, *, \diamondsuit)$ such that X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:
(a) $M(x, y, t) + N(x, y, t) \leq 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ if and only if $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;

(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(f) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
(g) $N(x, y, t) < 1$;
(h) $N(x, y, t) = 0$ if and only if $x = y$;
(i) $N(x, y, t) = N(y, x, t)$;
(j) $N(x, y, t) \diamondsuit N(y, z, s) \geq N(x, z, t + s)$;
(k) the function $N(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.4 [12] In intuitionistic fuzzy metric space X, for every $x, y \in X$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing.

Definition 2.5 [12] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, $r \in (0, 1), t > 0$ and $x \in X$. The set

$$B_{(M,N)}(x, r, t) = \{y \in X | M(x, y, t) > 1 - r, N(x, y, t) < r\}$$

is called *the open ball with center x and radius r with respect to t.*

Obviously, $\{B_{(M,N)}(x, r, t)\}$ forms a base of a topology on X. The topology is denoted by $\tau_{(M,N)}$ and known to be metrizable (see [7]).

Example 2.6 [12] Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamondsuit b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X \times X \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \qquad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}.$$

for all $h, k, m, n \in \mathbf{R}^+$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Lemma 2.7 [15] *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ the topology on X induced by the intuitionistic fuzzy metric. Then for a sequence $\{x_n\}_{n \in \mathbf{N}}$ in X, x_n converges to x if and only if $M(x_n, x, t) \to 0 (n \to \infty)$.*

3 Hausdorff intuitionistic fuzzy metric spaces

Given an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$, we shall denote by $\mathcal{P}(X)$ and $\mathcal{K}(X)$, the set of nonempty subsets and the set of nonempty compact subsets of $(X, \tau_{(M,N)})$, respectively. For each $B \in \mathcal{P}(X)$, $a \in X$ and $t > 0$, let $M(a, B, t) := \sup_{b \in B} M(a, b, t)$ and $N(a, B, t) := \inf_{b \in B} N(a, b, t)$ (see Definition 3.2 of [15]). By conditions (d) and (i) in Definition 2.3, we observe that $M(a, B, t) = M(B, a, t)$ and $N(a, B, t) = N(B, a, t)$. Let $B_{(M,N)}(B, r, t) := \bigcup_{b \in B} B_{(M,N)}(b, r, t)$.

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Definition 3.1 [7] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Define two functions $H_M, H_N : \mathcal{K}(X) \times \mathcal{K}(X) \times (0, \infty) \to [0, 1]$ as follows, respectively: for each $A, B \in \mathcal{K}(X)$ and $t > 0$,

$$H_M(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\}$$

and

$$H_N(A, B, t) = \max\{\sup_{a\in A} N(a, B, t), \sup_{b\in B} N(A, b, t)\}.$$

It was proved in [7] that $(\mathcal{K}(X), H_M, H_N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space. (H_M, H_N) is called *the Hausdorff intuitionistic fuzzy metric on* $\mathcal{K}(X)$.

By Proposition 1 of [14], the next lemma can be obtained immediately.

Lemma 3.2 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then M and N are both continuous functions on $X \times X \times (0, \infty)$.*

Lemma 3.3 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, $a \in X$ and $B \in \mathcal{P}(X)$. Then $t \mapsto M(a, B, t)$ and $t \mapsto N(a, B, t)$ are a continuous nondecreasing and a continuous increasing function on $(0, \infty)$, respectively.*

Proof We will verify that $t \mapsto M(a, B, t)$ is a continuous nondecreasing function on $(0, \infty)$.

By Lemma 2.4, for every $b \in B$ and $t_2 \geq t_1 > 0$, we have $M(a, b, t_2) \geq M(a, b, t_1)$. So $M(a, B, t_2) \geq M(a, B, t_1)$, which means that $t \mapsto M(a, B, t)$ is a nondecreasing function on $(0, \infty)$.

Next, we are now going to prove that $t \mapsto M(a, B, t)$ is continuous on $(0, \infty)$.

Let $t_0 \in (0, \infty)$ and $\{t_i\}_{i\in\mathbf{N}}$ be a nondecreasing sequence in $(0, \infty)$ with $\lim_{i\to\infty} t_i = t_0$. Put $M(a, B, t_i) \to r_0 (i \to \infty)$. Observe that $t_i \leq t_0$ for every $i \in \mathbf{N}$. It follows that $M(a, B, t_i) \leq M(a, B, t_0)$. Hence $r_0 = \lim_{i\to\infty} M(a, B, t_i) \leq M(a, B, t_0)$. On the other hand, since $\lim_{i\to\infty} t_i = t_0$, we can choose an $i_n > n$ such that $t_{i_n} > t_0 - \frac{1}{n}$ for every $n > [\frac{1}{t_0}] + 1$. Note that for every $b \in B$, $t \mapsto M(a, b, t)$ is continuous on $(0, \infty)$. Then, for each $\varepsilon > 0$, there exists an $m > [\frac{1}{t_0}] + 1$ such that $M(a, b, t_0 - \frac{1}{m}) > M(a, b, t_0) - \varepsilon$. Hence

$$M(a, B, t_{i_m}) \geq M(a, B, t_0 - \frac{1}{m}) \geq M(a, b, t_0 - \frac{1}{m}) > M(a, b, t_0) - \varepsilon.$$

Therefore, passing to the limit as $\varepsilon \to 0$, we obtain

$$r_0 = \lim_{i_m\to\infty} M(a, B, t_{i_m}) = \lim_{m\to\infty} M(a, B, t_{i_m}) \geq M(a, b, t_0).$$

Hence $r_0 \geq \sup_{b\in B} M(a, b, t_0) = M(a, B, t_0)$. Analogously, we can obtained the case of a nonincreasing sequence $\{t_i\}_{i\in\mathbf{N}}$ in $(0, \infty)$ with $\lim_{i\to\infty} t_i = t_0$.

A similar argument as above show that $t \mapsto N(a, B, t)$ is a continuous increasing function on $(0, \infty)$. We are done.

Lemma 3.4 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, $A \in \mathcal{P}(X)$ and $t > 0$. Then $x \mapsto M(x, A, t)$ and $x \mapsto N(x, A, t)$ are both continuous functions on X.*

Proof We only need to verify that $x \mapsto M(x, A, t)$ is continuous on X. The proof of the continuity of $x \mapsto N(x, A, t)$ is similar.

Let $x_0 \in X$, $t > 0$ and $\varepsilon \in (0, t)$, and let $\{x_n\}_{n \in \mathbf{N}}$ be a sequence in X with x_n converges to x_0. Since $\{M(x_n, A, t)\}_{n \in \mathbf{N}}$ is a sequence in (0,1], there is a subsequence $\{x_{n_m}\}_{m \in \mathbf{N}}$ of $\{x_n\}_{n \in \mathbf{N}}$ such that the sequence $\{M(x_{n_m}, A, t)\}_{m \in \mathbf{N}}$ converges to some point of [0,1]. Then

$$M(x_{n_m}, A, t) \geq M(x_{n_m}, a, t) \geq M(x_{n_m}, x_0, \varepsilon) * M(x_0, a, t - \varepsilon)$$

for every $a \in A$. By the continuity of $*$, we obtain

$$\begin{aligned} M(x_{n_m}, A, t) &\geq M(x_{n_m}, x_0, \varepsilon) * \sup\nolimits_{a \in A} M(x_0, a, t - \varepsilon) \\ &\geq M(x_{n_m}, x_0, \varepsilon) * M(x_0, A, t - \varepsilon). \end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned} \lim_{m \to \infty} M(x_{n_m}, A, t) &\geq \lim_{m \to \infty} M(x_{n_m}, x_0, \varepsilon) * M(x_0, A, t - \varepsilon) \\ &= 1 * M(x_0, A, t - \varepsilon) = M(x_0, A, t - \varepsilon). \end{aligned}$$

So, according to Lemma 3.3, we get $\lim\limits_{m \to \infty} M(x_{n_m}, A, t) \geq M(x_0, A, t)$. On the other hand, we have

$$M(x_0, A, t + \varepsilon) \geq M(x_0, a, t + \varepsilon) \geq M(x_0, x_{n_m}, \varepsilon) * M(x_{n_m}, a, t).$$

Thus, by the continuity of $*$, we obtain

$$\begin{aligned} M(x_0, A, t + \varepsilon) &\geq M(x_0, x_{n_m}, \varepsilon) * \sup\nolimits_{a \in A} M(x_{n_m}, a, t) \\ &= M(x_0, x_{n_m}, \varepsilon) * M(x_{n_m}, A, t). \end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned} M(x_0, A, t + \varepsilon) &\geq \lim_{m \to \infty} M(x_0, x_{n_m}, \varepsilon) * \lim_{m \to \infty} M(x_{n_m}, A, t) \\ &= 1 * \lim_{m \to \infty} M(x_{n_m}, A, t) = \lim_{m \to \infty} M(x_{n_m}, A, t). \end{aligned}$$

Due to Lemma 3.3, we get $M(x_0, A, t) \geq \lim\limits_{m \to \infty} M(x_{n_m}, A, t)$. Hence

$$\lim_{m \to \infty} M(x_{n_m}, A, t) = M(x_0, A, t).$$

We finish the proof.

Lemma 3.5 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then, for every $a \in X$, $t > 0$ and $B \in \mathcal{K}(X)$, there exist $b_1, b_2 \in B$ such that $M(a, B, t) = M(a, b_1, t)$ and $N(a, B, t) = N(a, b_2, t)$, respectively.*

Proof By Lemma 3.2, we deduce that $\{M(a, b, t) | b \in B\}$ and $\{N(a, b, t) | b \in B\}$ are both closed subsets of [0,1]. Hence $\sup\limits_{b \in B} M(a, b, t) \in \{M(a, b, t) | b \in B\}$

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and $\sup_{b\in B} N(a,b,t) \in \{M(a,b,t)|b \in B\}$. Thus, there exist $b_1, b_2 \in B$ such that $\sup_{b\in B} M(a,b,t) = M(a,b_1,t)$ and $\sup_{b\in B} N(a,b,t) = N(a,b_2,t)$, that is, $M(a,B,t) = M(a,b_1,t)$ and $N(a,B,t) = N(a,b_2,t)$. We are done.

Theorem 3.6 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then, for every $A, B \in \mathcal{K}(X)$ and $t > 0$, there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $H_M(A,B,t) = M(a_1,b_1,t)$ and $H_N(A,B,t) = N(a_2,b_2,t)$, respectively.*

Proof Without loss of generality, we assume that $H_M(A,B,t) = \inf_{a\in A} M(a,B,t)$. According to Lemma 3.4, we conclude that $\{M(a,B,t)|a \in A)\}$ is a compact subset of [0,1]. Then there exists an $a_1 \in A$ such that $M(a_1,B,t) = \inf_{a\in A} M(a,B,t)$. Also, by Lemma 3.5, we can find a $b_1 \in B$ such that $M(a_1,b_1,t) = M(a_1,B,t)$. Hence $H_M(A,B,t) = M(a_1,b_1,t)$. To prove that there exist $a_2 \in A$ and $b_2 \in B$ such that $H_N(A,B,t) = N(a_2,b_2,t)$, we use the similar argument as above. The proof is finished.

Proposition 3.7 [7] *If $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space, then so is $(X, M_N, N, \star, \diamondsuit)$, where M_N is defined on $X\times X\times(0,\infty)$ by $M_N(x,y,t) = 1-N(x,y,t)$ and $\star$ is the continuous $t-$norm defined by $a\star b = 1-[(1-a)\diamondsuit(1-b)]$.*

Definition 3.8 Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. We define two functions $H'_M, H'_N : \mathcal{K}(X) \times \mathcal{K}(X) \times (0,\infty) \to [0,1]$ as follows, respectively: for every $A, B \in \mathcal{K}(X)$ and $t > 0$,

$$H'_M(A,B,t) = 1 - \inf\{r|B \subset B_{(M,N)}(A,r,t), A \subset B_{(M,N)}(B,r,t)\}$$

and

$$H'_N(A,B,t) = \inf\{r|B \subset B_{(M_N,N)}(A,r,t), A \subset B_{(M_N,N)}(B,r,t)\}.$$

Theorem 3.9 *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then $H_M(A,B,t) = H'_M(A,B,t), H_N(A,B,t) = H'_N(A,B,t)$.*

Proof Now, we are going to prove $H_M(A,B,t) = H'_M(A,B,t)$. Put $H'_M(A,B,t) = 1-r_0$. Take $r \in (r_0, 1)$. Then we get $B \subset B_{(M,N)}(A,r,t)$ and $A \subset B_{(M,N)}(B,r,t)$. This show that $M(A,b,t) > 1-r$ and $M(a,B,t) > 1-r$ for every $b \in B$ and $a \in A$. Hence $\inf_{b\in B} M(A,b,t) \geq 1-r$ and $\inf_{a\in A} M(a,B,t) \geq 1-r$. So $\min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\} \geq 1-r$, i.e., $H_M(A,B,t) \geq 1-r$. By taking the limits when $r \to r_0$, we obtain $H_M(A,B,t) \geq 1-r_0 = H'_M(A,B,t)$. Assume that $H_M(A,B,t) > H'_M(A,B,t)$. Put $H_M(A,B,t) = 1-r_1$. Then we can choose an $r_2 \in (r_1, r_0)$, which implies that $1-r_1 > 1-r_2 > 1-r_0$. Since $\inf_{a\in A} M(a,B,t) \geq 1-r_1$ and $\inf_{b\in B} M(A,b,t) \geq 1-r_1$, we have $M(a,B,t) \geq 1-r_1 > 1-r_2$ and $M(A,b,t) \geq 1-r_1 > 1-r_2$ for all $b \in B$ and $a \in A$. According to Lemma 3.5, we can find $b_0 \in B$ and $a_0 \in A$ such that $M(a,b_0,t) = M(a,B,t)$ and $M(a_0,b,t) = M(A,b,t)$. Hence $N(a,B,t) \leq N(a,b_0,t) \leq 1-M(a,b_0,t) <$

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r_2 and $N(A,b,t) \le N(a_0,b,t) \le 1 - M(a_0,b,t) < r_2$. Thus $B \subset B_{(M,N)}(A,r_2,t)$ and $A \subset B_{(M,N)}(B,r_2,t)$. So $H'_M(A,B,t) \ge 1 - r_2 > 1 - r_0 = H'_M(A,B,t)$, which is a contradiction. Consequencely, $H_M(A,B,t) = H'_M(A,B,t)$.

A similar argument shows that $H_N(A,B,t) = H'_N(A,B,t)$. We are done.

Remark 3.10 Obviously, $(\mathcal{K}(X), H'_M, H'_N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Theorem 3.11 *In two intuitionistic fuzzy metric spaces $(X, M_i, N_i, *_i, \diamondsuit_i)(i = 1,2)$, the topology induced by the uniformity $\mathcal{U}_i$ on X coincides with $\tau_{(M_i,N_i)}$, where i=1,2. If $\tau_{(M_1,N_1)} = \tau_{(M_2,N_2)}$, then $(\mathcal{K}(X), \tau_{(H_{M_1},H_{N_1})})$ coincides with $(\mathcal{K}(X), \tau_{(H_{M_2},H_{N_2})})$.*

Proof For each $n \in \mathbf{N}$, we define

$$U_n = \{(x,y) \in X \times X | M_1(x,y,\frac{1}{n}) > 1 - \frac{1}{n}, N_1(x,y,\frac{1}{n}) < \frac{1}{n}\}$$

and

$$V_n = \{(x,y) \in X \times X | M_2(x,y,\frac{1}{n}) > 1 - \frac{1}{n}, N_2(x,y,\frac{1}{n}) < \frac{1}{n}\}.$$

Due to Lemma 2.6 of [15], $\{U_n | n \in \mathbf{N}\}$ and $\{V_n | n \in \mathbf{N}\}$ are a base for $\mathcal{U}_1$ and a base for $\mathcal{U}_2$, respectively. Since $\tau_{(M_1,N_1)} = \tau_{(M_2,N_2)}$, we deduce that for each $n \in \mathbf{N}$, there exist $k,l \in \mathbf{N}$ such that $V_k \subset U_n$ and $U_l \subset V_n$. Therefore, for every $C \in \mathcal{P}(X)$ and $n \in \mathbf{N}$, there exist $k,l \in \mathbf{N}$ such that $B_{(M_2,N_2)}(C,\frac{1}{k},\frac{1}{k}) \subset B_{(M_1,N_1)}(C,\frac{1}{n},\frac{1}{n})$ and $B_{(M_1,N_1)}(C,\frac{1}{l},\frac{1}{l}) \subset B_{(M_2,N_2)}(C,\frac{1}{n},\frac{1}{n})$. Let $A \in \mathcal{K}(X)$ and $n \in \mathbf{N}$. Then there exists a $k \in \mathbf{N}$ such that

$$B_{(M_2,N_2)}(A,\frac{1}{k},\frac{1}{k}) \subset B_{(M_1,N_1)}(A,\frac{1}{n+1},\frac{1}{n+1}).$$

Hence $A \in B_{(H_{M_2},H_{N_2})}(A,\frac{1}{k},\frac{1}{k}) \subset B_{(H_{M_1},H_{N_1})}(A,\frac{1}{n},\frac{1}{n})$. In fact, take $B \in B_{(H_{M_2},H_{N_2})}(A,\frac{1}{k},\frac{1}{k})$. Then, by Theorem 3.9, we have

$$H_{M_2}(A,B,\frac{1}{k}) = 1 - \inf\{r | A \subset B_{(M_2,N_2)}(B,r,\frac{1}{k}), B \subset B_{(M_2,N_2)}(A,r,\frac{1}{k})\} > 1 - \frac{1}{k}.$$

So $A \subset B_{(M_2,N_2)}(B,\frac{1}{k},\frac{1}{k})$ and $B \subset B_{(M_2,N_2)}(A,\frac{1}{k},\frac{1}{k})$. Hence $A \subset B_{(M_1,N_1)}(B,\frac{1}{n+1},\frac{1}{n+1})$ and $B \subset B_{(M_1,N_1)}(A,\frac{1}{n+1},\frac{1}{n+1})$. Thus $H_{M_1}(A,B,\frac{1}{n+1}) \ge 1 - \frac{1}{n+1}$. It follows that

$$H_{M_1}(A,B,\frac{1}{n}) \ge H_{M_1}(A,B,\frac{1}{n+1}) \ge 1 - \frac{1}{n+1} > 1 - \frac{1}{n}.$$

Moreover, By Remark 3.10, we get $H_{N_1}(A,B,\frac{1}{n}) \le 1 - H_{M_1}(A,B,\frac{1}{n}) < \frac{1}{n}$. So

$$A \in B_{(H_{M_2},H_{N_2})}(A,\frac{1}{k},\frac{1}{k}) \subset B_{(H_{M_1},H_{N_1})}(A,\frac{1}{n},\frac{1}{n}).$$

Similarly, we can obtain that for every $D \in \mathcal{K}(X)$ and $n \in \mathbf{N}$, there exists an $l \in \mathbf{N}$ such that

$$D \in B_{(H_{M_1},H_{N_1})}(D,\frac{1}{l},\frac{1}{l}) \subset B_{(H_{M_2},H_{N_2})}(D,\frac{1}{n},\frac{1}{n}).$$

We finish the proof.

4 Conclusion

We have proven that two types of Hausdorff intuitionistic fuzzy metrics on compact sets are identical. Moreover, we have proven that if two intuitionistic fuzzy metric spaces induce the same topology, then the corresponding Hausdorff intuitionistic fuzzy metric topological spaces coincide.

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Integral inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions

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Abstract

In the paper, the authors introduce a notion "(α, m)-GA-convex functions" and establish some integral inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions.

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1 Introduction

In [10, 15], the concepts of m-convex functions and (α, m)-convex functions were introduced as follows.

Definition 1.1 ([15])**.** A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex for $m \in (0, 1]$ if the inequality

$$f(\alpha x + m(1-\alpha)y) \le \alpha f(x) + m(1-\alpha)f(y) \tag{1.1}$$

holds for all $x, y \in [0, b]$ and $\alpha \in [0, 1]$.

Definition 1.2 ([10])**.** For $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.2}$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

Hereafter, a few of inequalities of Hermite-Hadamard type for the m-convex and (α, m)-convex functions were presented, some of them can be recited as following theorems.

Theorem 1.1 ([3, Theorems 2.2]). *Let $I \supset \mathbb{R}_0 = [0,\infty)$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$, where $L([a,b])$ denotes the set of all Lebesgue integrable functions on $[a,b]$. If $|f'(x)|^q$ is m-convex on $[a,b]$ for some given numbers $m \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le \frac{b-a}{4}\min\left\{\left[\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right]^{1/q}, \left[\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right]^{1/q}\right\}.\end{aligned} \tag{1.3}$$

Theorem 1.2 ([3, Theorem 3.1]). *Let $I \supset [0,\infty)$ be an open interval and let $f : I \to (-\infty,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α,m)-convex on $[a,b]$ for some given numbers $m,\alpha \in (0,1]$, and $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \\ \times \min\left\{\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}, \left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned} \tag{1.4}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right). \tag{1.5}$$

For more information on Hermite-Hadamard type inequalities for various kinds of convex functions, please refer to the monograph [5], the recently published papers [1, 2, 4, 6, 8, 9, 13, 14, 16, 17, 18], and closely related references therein.

In this paper, we will introduce a new concept "(α,m)-geometric-arithmetically convex function" (simply speaking, (α,m)-GA-convex function) and establish some integral inequalities of Hermite-Hadamard type for (α,m)-GA-convex functions.

2 A definition and a lemma

Now we introduce the so-called (α,m)-GA-convex functions.

Definition 2.1. Let $f : (0,b] \to \mathbb{R}$ and $(\alpha,m) \in (0,1]^2$. If

$$f\big(x^\lambda y^{m(1-\lambda)}\big) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y) \tag{2.1}$$

for all $x,y \in (0,b]$ and $\lambda \in [0,1]$, then $f(x)$ is said to be a (α,m)-geometric-arithmetically convex function or, simply speaking, an (α,m)-GA-convex function. If (2.1) is reversed, then $f(x)$ is said to be a (α,m)-geometric-arithmetically concave function or, simply speaking, a (α,m)-GA-concave function.

Remark 2.1. When $m = \alpha = 1$, the (α,m)-GA-convex (concave) function defined in Defintion 2.1 becomes a GA-convex (concave) function defined in [11, 12].

To establish some new Hermite-Hadamard type inequalities for (α, m)-GA-convex functions, we need the following lemma.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then*

$$\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \,\mathrm{d}\, x = \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} f'\big(a^{1-t} b^t\big) \,\mathrm{d}\, t. \tag{2.2}$$

Proof. Let $x = a^{1-t} b^t$ for $0 \le t \le 1$. Then

$$(\ln b - \ln a) \int_0^1 a^{3(1-t)} b^{3t} f'\big(a^{1-t} b^t\big) \,\mathrm{d}\, t = \int_a^b x^2 f'(x) \,\mathrm{d}\, x = b^2 f(b) - a^2 f(a) - 2 \int_a^b x f(x) \,\mathrm{d}\, x.$$

Lemma 2.1 is thus proved. □

3 Inequalities of Hermite-Hadamard type

Now we turn our attention to establish inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions.

Theorem 3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (α, m)-GA-convex on $\big(0, \max\{a^{1/m}, b\}\big]$ for $(\alpha, m) \in (0, 1]^2$ and $q \ge 1$, then*

$$\begin{aligned} \left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \,\mathrm{d}\, x\right| &\le \frac{\ln b - \ln a}{2} \big[L\big(a^3, b^3\big)\big]^{1-1/q} \\ &\quad \times \big\{m\big[L\big(a^3, b^3\big) - G(\alpha, 3)\big] \big|f'\big(a^{1/m}\big)\big|^q + G(\alpha, 3) |f'(b)|^q\big\}^{1/q}, \end{aligned} \tag{3.1}$$

where

$$G(\alpha, \ell) = \int_0^1 t^\alpha a^{\ell(1-t)} b^{\ell t} \,\mathrm{d}\, t \quad \text{and} \quad L(x, y) = \frac{y - x}{\ln y - \ln x} \tag{3.2}$$

for $\ell \ge 0$ and for $x, y > 0$ with $x \ne y$.

Proof. Making use of the (α, m)-GA-convexity of $|f'|^q$ on $\big(0, \max\big\{a^{1/m}, b\big\}\big]$, Lemma 2.1, and Hölder inequality yields

$$\begin{aligned} &\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \,\mathrm{d}\, x\right| \le \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} \big|f'\big(a^{1-t} b^t\big)\big| \,\mathrm{d}\, t \\ &\le \frac{\ln b - \ln a}{2} \left[\int_0^1 a^{3(1-t)} b^{3t} \,\mathrm{d}\, t\right]^{1-1/q} \left[\int_0^1 a^{3(1-t)} b^{3t} \left|f'\Big(\big(a^{1/m}\big)^{m(1-t)} b^t\Big)\right|^q \,\mathrm{d}\, t\right]^{1/q} \\ &\le \frac{\ln b - \ln a}{2} \left(\frac{b^3 - a^3}{\ln b^3 - \ln a^3}\right)^{1-1/q} \left[\int_0^1 a^{3(1-t)} b^{3t} \big(t^\alpha |f'(b)|^q + m(1 - t^\alpha) \big|f'\big(a^{1/m}\big)\big|^q\big) \,\mathrm{d}\, t\right]^{1/q} \\ &= \frac{\ln b - \ln a}{2} \big[L\big(a^3, b^3\big)\big]^{1-1/q} \big\{m\big[L\big(a^3, b^3\big) - G(\alpha, 3)\big] \big|f'\big(a^{1/m}\big)\big|^q + G(\alpha, 3) |f'(b)|^q\big\}^{1/q}. \end{aligned}$$

As a result, the inequality (3.1) follows. The proof of Theorem 3.1 is complete. □

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Corollary 3.1. *Under the conditions of Theorem 3.1, if $q=1$, then*

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right| \\ \le \frac{\ln b-\ln a}{2}\big\{m\big[L\big(a^3,b^3\big)-G(\alpha,3)\big]\big|f'\big(a^{1/m}\big)\big|+G(\alpha,3)|f'(b)|\big\}. \quad (3.3)$$

Corollary 3.2. *Under the conditions of Theorem 3.1, if $\alpha=1$, then*

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right| \le \frac{\big(b^3-a^3\big)^{1-1/q}}{6} \\ \times\Big\{m\big[L\big(a^3,b^3\big)-a^3\big]\big|f'\big(a^{1/m}\big)\big|^q+\big[b^3-L\big(a^3,b^3\big)\big]|f'(b)|^q\Big\}^{1/q}. \quad (3.4)$$

Proof. This follows from the fact that

$$G(1,3)=\int_0^1 ta^{3(1-t)}b^{3t}\,\mathrm{d}\,t=\frac{b^3-L\big(a^3,b^3\big)}{3(\ln b-\ln a)}.$$

The proof of Corollary 3.2 is complete. □

Corollary 3.3. *Under the conditions of Theorem 3.1, we have*

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right| \le \frac{\ln b-\ln a}{2}\left(\frac{1}{\alpha+1}\right)^{1/q} \\ \times\big[L\big(a^3,b^3\big)\big]^{1-1/q}\big\{m\big[(\alpha+1)L\big(a^3,b^3\big)-b^3\big]\big|f'\big(a^{1/m}\big)\big|^q+b^3|f'(b)|^q\big\}^{1/q} \quad (3.5)$$

and

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right| \le \frac{\ln b-\ln a}{2}L\big(a^3,b^3\big)|f'(b)|. \quad (3.6)$$

Proof. Using $\big(\frac{b}{a}\big)^{3t}\le\big(\frac{b}{a}\big)^3$ for $t\in[0,1]$ in (3.2) gives

$$G(\alpha,3)=a^3\int_0^1 t^\alpha\left(\frac{b}{a}\right)^{3t}\mathrm{d}\,t\le\frac{b^3}{\alpha+1}.$$

Substituting this inequality into (3.1) yields (3.5).

Utilizing $t^\alpha\le 1$ for $t\in[0,1]$ in (3.2) reveals

$$G(\alpha,3)\le\int_0^1 a^{3(1-t)}b^{3t}\,\mathrm{d}\,t=L\big(a^3,b^3\big).$$

Combining this inequality with (3.1) yields (3.6). Corollary 3.3 is thus proved. □

Theorem 3.2. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ with $0<a<b<\infty$. If $|f'|^q$ is (α,m)-GA-convex on $\big(0,\max\big\{a^{1/m},b\big\}\big]$ for $(\alpha,m)\in(0,1]^2$ and $q>1$, then*

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$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right|\le\frac{\ln b-\ln a}{2}\left(\frac{1}{\alpha+1}\right)^{1/q}\\ \times\left[L\left(a^{3q/(q-1)},b^{3q/(q-1)}\right)\right]^{1-1/q}\left[|f'(b)|^q+\alpha m\left|f'\left(a^{1/m}\right)\right|^q\right]^{1/q},\quad(3.7)$$

where L is defined as in (3.2).

Proof. Since $|f'|^q$ is (α,m)-GA-convex on $\left(0,\max\left\{a^{1/m},b\right\}\right]$, from Lemma 2.1 and Hölder inequality, we have

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right|\le\frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|\mathrm{d}\,t$$
$$\le\frac{\ln b-\ln a}{2}\left[\int_0^1 a^{3q/(q-1)(1-t)}b^{3q/(q-1)t}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left|f'\left(\left(a^{1/m}\right)^{m(1-t)}b^t\right)\right|^q\mathrm{d}\,t\right]^{1/q}$$
$$\le\frac{\ln b-\ln a}{2}\left[\frac{b^{3q/(q-1)}-a^{3q/(q-1)}}{\ln b^{3q/(q-1)}-\ln a^{3q/(q-1)}}\right]^{1-1/q}\left[\int_0^1\left(t^\alpha|f'(b)|^q+m(1-t^\alpha)\left|f'\left(a^{1/m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}$$
$$=\frac{\ln b-\ln a}{2}\left[L\left(a^{3q/(q-1)},b^{3q/(q-1)}\right)\right]^{1-1/q}\left[\frac{1}{\alpha+1}|f'(b)|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(a^{1/m}\right)\right|^q\right]^{1/q}.$$

The proof of Theorem 3.2 is complete. □

Corollary 3.4. *Under the conditions of Theorem 3.2, if $\alpha=1$, then*

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right|\le\frac{\ln b-\ln a}{2^{1+1/q}}\\ \times\left[L\left(a^{3q/(q-1)},b^{3q/(q-1)}\right)\right]^{1-1/q}\left[|f'(b)|^q+m\left|f'\left(a^{1/m}\right)\right|^q\right]^{1/q}.\quad(3.8)$$

Theorem 3.3. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|^q$ is (α,m)-GA-convex on $\left(0,\max\left\{a^{1/m},b\right\}\right]$ for $q>1$ and $(\alpha,m)\in(0,1]^2$, then*

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right|\le\frac{\ln b-\ln a}{2}\\ \times\left\{m\left[L\left(a^{3q},b^{3q}\right)-G(\alpha,3q)\right]\left|f'\left(a^{1/m}\right)\right|^q+G(\alpha,3q)|f'(b)|^q\right\}^{1/q},\quad(3.9)$$

where G and L are respectively defined as in (3.2).

Proof. Since $|f'|^q$ is (α,m)-GA-convex on $\left(0,\max\left\{a^{1/m},b\right\}\right]$, from Lemma 2.1 and Hölder inequality, we have

$$\left|\frac{b^2f(b)-a^2f(a)}{2}-\int_a^b xf(x)\,\mathrm{d}\,x\right|$$
$$\le\frac{\ln b-\ln a}{2}\left(\int_0^1 1\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 a^{3q(1-t)}b^{3qt}\left|f'\left(\left(a^{1/m}\right)^{m(1-t)}b^t\right)\right|^q\mathrm{d}\,t\right]^{1/q}$$
$$\le\frac{\ln b-\ln a}{2}\left[mL\left(a^{3q},b^{3q}\right)\left|f'\left(a^{1/m}\right)\right|^q+G(\alpha,3q)\left(|f'(b)|^q-m\left|f'\left(a^{1/m}\right)\right|^q\right)\right]^{1/q}.$$

The proof of Theorem 3.3 is complete. □

Corollary 3.5. *Under the conditions of Theorem 3.3, if $\alpha = 1$, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\,\mathrm{d}\,x\right| \le \frac{(\ln b - \ln a)^{1-1/q}}{2}\left(\frac{1}{3q}\right)^{1/q} \times \left\{m\left[L\left(a^{3q}, b^{3q}\right) - a^{3q}\right]\left|f'\left(a^{1/m}\right)\right|^q + \left[b^{3q} - L\left(a^{3q}, b^{3q}\right)\right]|f'(b)|^q\right\}^{1/q}. \quad (3.10)$$

Proof. From

$$G(1, 3q) = \int_0^1 t a^{3q(1-t)} b^{3qt}\,\mathrm{d}\,t = \frac{b^{3q} - L\left(a^{3q}, b^{3q}\right)}{\ln b^{3q} - \ln a^{3q}},$$

Corollary 3.5 follows. □

Theorem 3.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (α, m)-GA-convex on $\left(0, \max\left\{a^{1/m}, b\right\}\right]$ for $q > 1$, $q > p > 0$, and $(\alpha, m) \in (0, 1]^2$, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\,\mathrm{d}\,x\right| \le \frac{\ln b - \ln a}{2}\left[L\left(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}\right)\right]^{1-1/q} \times \left\{m\left[L\left(a^{3p}, b^{3p}\right) - G(\alpha, 3p)\right]\left|f'\left(a^{1/m}\right)\right|^q + G(\alpha, 3p)|f'(b)|^q\right\}^{1/q}, \quad (3.11)$$

where G and L are respectively defined as in (3.2).

Proof. Since $|f'|^q$ is (α, m)-GA-convex on $\left(0, \max\left\{a^{1/m}, b\right\}\right]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\,\mathrm{d}\,x\right| &\le \frac{\ln b - \ln a}{2}\left[\int_0^1 a^{3(q-p)/(q-1)(1-t)} b^{3(q-p)/(q-1)t}\,\mathrm{d}\,t\right]^{1-1/q} \\
&\quad \times \left[\int_0^1 a^{3p(1-t)} b^{3pt}\left|f'\left(\left(a^{1/m}\right)^{m(1-t)} b^t\right)\right|^q \mathrm{d}\,t\right]^{1/q} \\
&\le \frac{\ln b - \ln a}{2}\left[\frac{b^{3(q-p)/(q-1)} - a^{3(q-p)/(q-1)}}{\ln b^{3(q-p)/(q-1)} - \ln a^{3(q-p)/(q-1)}}\right]^{1-1/q} \\
&\quad \times \left[\int_0^1 a^{3p(1-t)} b^{3pt}\left(t^\alpha |f'(b)|^q + m(1 - t^\alpha)\left|f'\left(a^{1/m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q} \\
&= \frac{\ln b - \ln a}{2}\left[L\left(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}\right)\right]^{1-1/q} \\
&\quad \times \left[mL\left(a^{3p}, b^{3p}\right)\left|f'\left(a^{1/m}\right)\right|^q + G(\alpha, 3p)\left(|f'(b)|^q - m\left|f'\left(a^{1/m}\right)\right|^q\right)\right]^{1/q}.
\end{aligned}$$

The proof of Theorem 3.4 is complete. □

Corollary 3.6. *Under the conditions of Theorem 3.4, if $\alpha = 1$, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\,\mathrm{d}\,x\right| \le \frac{(\ln b - \ln a)^{1-1/q}}{2}\left(\frac{1}{3p}\right)^{1/q}\Big[L\Big(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}\Big)\Big]^{1-1/q}\left\{m\left[L\left(a^{3p}, b^{3p}\right) - a^{3p}\right]\left|f'\left(a^{1/m}\right)\right|^q + \left[b^{3p} - L\left(a^{3p}, b^{3p}\right)\right]|f'(b)|^q\right\}^{1/q}.$$

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Proof. By

$$G(1,3p)=\int_0^1 ta^{3p(1-t)}b^{3pt}\,\mathrm{d}\,t=\frac{b^{3p}-L\big(a^{3p},b^{3p}\big)}{\ln b^{3p}-\ln a^{3p}},$$

Corollary 3.6 can be proved easily. □

Theorem 3.5. *Let $f,g:\mathbb{R}_+\to\mathbb{R}_0$ and $fg\in L([a,b])$ for $0<a<b<\infty$. If f^q is (α_1,m_1)-GA-convex on $\big(0,\max\big\{a^{1/m_1},b\big\}\big]$ and g^q is (α_2,m_2)-GA-convex on $\big(0,\max\big\{a^{1/m_2},b\big\}\big]$ for $q\ge1$, $(\alpha_1,m_1)\in(0,1]^2$, and $(\alpha_2,m_2)\in(0,1]^2$, then*

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x &\le(\ln b-\ln a)[L(a,b)]^{1-1/q}\big\{m_1m_2[L(a,b)-G(\alpha_1,1)-G(\alpha_2,1)\\&\quad+G(\alpha_1+\alpha_2,1)]f^q\big(a^{1/m_1}\big)g^q\big(a^{1/m_2}\big)+m_1[G(\alpha_2,1)-G(\alpha_1+\alpha_2,1)]f^q\big(a^{1/m_1}\big)g^q(b)\\&\quad+m_2[G(\alpha_1,1)-G(\alpha_1+\alpha_2,1)]f^q(b)g^q\big(a^{1/m_2}\big)+G(\alpha_1+\alpha_2,1)f^q(b)g^q(b)\big\}^{1/q},\end{aligned}$$

where G and L are respectively defined as in (3.2).

Proof. Using the (α_1,m_1)-GA-convexity of f^q and the (α_2,m_2)-GA-convexity of g^q yields

$$f^q\big(a^{1-t}b^t\big)\le t^{\alpha_1}f^q(b)+m_1(1-t^{\alpha_1})f^q\big(a^{1/m_1}\big)$$

and

$$g^q\big(a^{1-t}b^t\big)\le t^{\alpha_2}g^q(b)+m_2(1-t^{\alpha_2})g^q\big(a^{1/m_2}\big)$$

for $0\le t\le1$. Letting $x=a^{1-t}b^t$ for $0\le t\le1$ and using Hölder's inequality figure out

$$\begin{aligned}&\int_a^b f(x)g(x)\,\mathrm{d}\,x=(\ln b-\ln a)\int_0^1 a^{1-t}b^tf\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)\,\mathrm{d}\,t\\&\le(\ln b-\ln a)\bigg(\int_0^1 a^{1-t}b^t\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg\{\int_0^1 a^{1-t}b^t\big[f\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)\big]^q\,\mathrm{d}\,t\bigg\}^{1/q}\\&\le(\ln b-\ln a)\bigg(\int_0^1 a^{1-t}b^t\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg\{\int_0^1 a^{1-t}b^t\big[t^{\alpha_1}f^q(b)\\&\quad+m_1(1-t^{\alpha_1})f^q\big(a^{1/m_1}\big)\big]\big[t^{\alpha_2}g^q(b)+m_2(1-t^{\alpha_2})g^q\big(a^{1/m_2}\big)\big]\,\mathrm{d}\,t\bigg\}^{1/q}\\&=(\ln b-\ln a)[L(a,b)]^{1-1/q}\bigg\{\int_0^1 a^{1-t}b^t\big[t^{\alpha_1+\alpha_2}f^q(b)g^q(b)\\&\quad+m_1t^{\alpha_2}(1-t^{\alpha_1})f^q\big(a^{1/m_1}\big)g^q(b)+m_2t^{\alpha_1}(1-t^{\alpha_2})f^q(b)g^q\big(a^{1/m_2}\big)\\&\quad+m_1m_2(1-t^{\alpha_1})(1-t^{\alpha_2})f^q\big(a^{1/m_1}\big)g^q\big(a^{1/m_2}\big)\big]\,\mathrm{d}\,t\bigg\}^{1/q}\\&=(\ln b-\ln a)[L(a,b)]^{1-1/q}\big\{m_1m_2[L(a,b)-G(\alpha_1,1)-G(\alpha_2,1)\\&\quad+G(\alpha_1+\alpha_2,1)]f^q\big(a^{1/m_1}\big)g^q\big(a^{1/m_2}\big)+m_1[G(\alpha_2,1)-G(\alpha_1+\alpha_2,1)]f^q\big(a^{1/m_1}\big)g^q(b)\\&\quad+m_2[G(\alpha_1,1)-G(\alpha_1+\alpha_2,1)]f^q(b)g^q\big(a^{1/m_2}\big)+G(\alpha_1+\alpha_2,1)f^q(b)g^q(b)\big\}^{1/q}.\end{aligned}$$

The proof of Theorem 3.5 is complete. □

Corollary 3.7. *Under the conditions of Theorem 3.5, if $q=1$, then*

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le (\ln b-\ln a)\big\{&m_1m_2[L(a,b)-G(\alpha_1,1)-G(\alpha_2,1)\\&+G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g\big(a^{1/m_2}\big)+m_1[G(\alpha_2,1)-G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g(b)\\&+m_2[G(\alpha_1,1)-G(\alpha_1+\alpha_2,1)]f(b)g\big(a^{1/m_2}\big)+G(\alpha_1+\alpha_2,1)f(b)g(b)\big\};\end{aligned}\tag{3.12}$$

if $q=1$ and $\alpha_1=\alpha_2=m_1=m_2=1$, then

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le \frac{1}{\ln b-\ln a}\{&[2L(a,b)-a(\ln b-\ln a)-2a]f(a)g(a)+[a+b\\&-2L(a,b)][f(a)g(b)+f(b)g(a)]+[2L(a,b)+b(\ln b-\ln a)-2b]f(b)g(b)\};\end{aligned}\tag{3.13}$$

if $\alpha_1=\alpha_2=m_1=m_2=1$, then

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le \frac{[L(a,b)]^{1-1/q}}{(\ln b-\ln a)^{2/q-1}}\big\{&[2L(a,b)-a(\ln b-\ln a)-2a]f^q(a)g^q(a)+[a+b\\&-2L(a,b)][f^q(a)g^q(b)+f^q(b)g^q(a)]+[2L(a,b)+b(\ln b-\ln a)-2b]f^q(b)g^q(b)\big\}^{1/q}.\end{aligned}$$

Theorem 3.6. *Let $f,g:\mathbb{R}_+\to\mathbb{R}_0$ and $fg\in L([a,b])$ for $0<a<b<\infty$. If f^q is (α_1,m_1)-GA-convex on $\big(0,\max\big\{a^{1/m_1},b\big\}\big]$ and $g^{q/(q-1)}$ is (α_2,m_2)-GA-convex on $\big(0,\max\big\{a^{1/m_2},b\big\}\big]$ for $q>1$, $(\alpha_1,m_1)\in(0,1]^2$, and $(\alpha_2,m_2)\in(0,1]^2$, then*

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le (\ln b-\ln a)&\big\{m_1f^q\big(a^{1/m_1}\big)L(a,b)+G(\alpha_1,1)\big[f^q(b)-m_1f^q\big(a^{1/m_1}\big)\big]\big\}^{1/q}\\&\times\big\{m_2g^{q/(q-1)}\big(a^{1/m_2}\big)L(a,b)+G(\alpha_2,1)\big[g^{q/(q-1)}(b)-m_2g^{q/(q-1)}\big(a^{1/m_2}\big)\big]\big\}^{1-1/q},\end{aligned}\tag{3.14}$$

where G and L are respectively defined as in (3.2).

Proof. By the (α_1,m_1)-GA-convexity of f^q and the (α_2,m_2)-GA-convexity of $g^{q/(q-1)}$, it follows that

$$f^q\big(a^{1-t}b^t\big)\le t^{\alpha_1}f^q(b)+m_1(1-t^{\alpha_1})f^q\big(a^{1/m_1}\big)$$

and

$$g^{q/(q-1)}\big(a^{1-t}b^t\big)\le t^{\alpha_2}g^{q/(q-1)}(b)+m_2(1-t^{\alpha_2})g^{q/(q-1)}\big(a^{1/m_2}\big)$$

for $t\in[0,1]$. Letting $x=a^{1-t}b^t$ for $0\le t\le 1$ and employing Hölder's inequality yield

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}\,x &\le \left[\int_a^b f^q(x)\,\mathrm{d}\,x\right]^{1/q}\left[\int_a^b g^{q/(q-1)}(x)\,\mathrm{d}\,x\right]^{1-1/q}\\
&=(\ln b-\ln a)\left[\int_0^1 a^{1-t}b^tf^q\big(a^{1-t}b^t\big)\,\mathrm{d}\,t\right]^{1/q}\left[\int_0^1 a^{1-t}b^tg^{q/(q-1)}\big(a^{1-t}b^t\big)\,\mathrm{d}\,t\right]^{1-1/q}\\
&\le(\ln b-\ln a)\left[\int_0^1 a^{1-t}b^t\big[t^{\alpha_1}f^q(b)+m_1(1-t^{\alpha_1})f^q\big(a^{1/m_1}\big)\big]\,\mathrm{d}\,t\right]^{1/q}\end{aligned}$$

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\begin{aligned}
&\quad \times \left[\int_0^1 a^{1-t}b^t\big[t^{\alpha_2}g^{q/(q-1)}(b)+m_2(1-t^{\alpha_2})g^{q/(q-1)}\big(a^{1/m_2}\big)\big]\,\mathrm{d}\,t\right]^{1-1/q}\\
&=(\ln b-\ln a)\big\{m_1f^q\big(a^{1/m_1}\big)L(a,b)+G(\alpha_1,1)\big[f^q(b)-m_1f^q\big(a^{1/m_1}\big)\big]\big\}^{1/q}\\
&\quad \times \big\{m_2g^{q/(q-1)}\big(a^{1/m_2}\big)L(a,b)+G(\alpha_2,1)\big[g^{q/(q-1)}(b)-m_2g^{q/(q-1)}\big(a^{1/m_2}\big)\big]\big\}^{1-1/q}.
\end{aligned}
$$

The proof of Theorem 3.6 is complete. □

Corollary 3.8. *Under the conditions of Theorem 3.6, if $\alpha_1=\alpha_2=m_1=m_2=1$, then*

$$
\begin{aligned}
\int_a^b f(x)g(x)\,\mathrm{d}\,x &\le \{f^q(a)[L(a,b)-a]+[b-L(a,b)]f^q(b)\}^{1/q}\\
&\quad \times \big\{g^{q/(q-1)}(a)[L(a,b)-a]+[b-L(a,b)]g^{q/(q-1)}(b)\big\}^{1-1/q}. \quad (3.15)
\end{aligned}
$$

Theorem 3.7. *Let $f,g:\mathbb{R}_+\to\mathbb{R}_0$ and $fg\in L([a,b])$ for $0<a<b<\infty$. If f is (α_1,m_1)-GA-concave on $\big(0,\max\{a^{1/m_1},b\}\big]$ and g is (α_2,m_2)-GA-concave on $\big(0,\max\{a^{1/m_2},b\}\big]$ for $(\alpha_1,m_1)\in(0,1]^2$ and $(\alpha_2,m_2)\in(0,1]^2$, then*

$$
\begin{aligned}
\int_a^b f(x)g(x)\,\mathrm{d}\,x &\ge (\ln b-\ln a)\big\{m_1m_2[L(a,b)-G(\alpha_1,1)-G(\alpha_2,1)\\
&\quad +G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g\big(a^{1/m_2}\big)+m_1[G(\alpha_2,1)-G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g(b)\\
&\quad +m_2[G(\alpha_1,1)-G(\alpha_1+\alpha_2,1)]f(b)g\big(a^{1/m_2}\big)+G(\alpha_1+\alpha_2,1)f(b)g(b)\big\}, \quad (3.16)
\end{aligned}
$$

where G and L are respectively defined as in (3.2).

Proof. Since f is (α_1,m_1)-GA-concave on $\big(0,\max\{a^{1/m_1},b\}\big]$ and g is (α_2,m_2)-GA-concave on $\big(0,\max\{a^{1/m_2},b\}\big]$, we have

$$
f\big(a^{1-t}b^t\big)\ge t^{\alpha_1}f(b)+m_1(1-t^{\alpha_1})f\big(a^{1/m_1}\big)\quad\text{and}\quad g\big(a^{1-t}b^t\big)\ge t^{\alpha_2}g(b)+m_2(1-t^{\alpha_2})g\big(a^{1/m_2}\big)
$$

for $t\in[0,1]$. Further letting $x=a^{1-t}b^t$ for $0\le t\le 1$ and utilizing Hölder's inequality reveal

$$
\begin{aligned}
&\int_a^b f(x)g(x)\,\mathrm{d}\,x=(\ln b-\ln a)\int_0^1 a^{1-t}b^tf\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)\,\mathrm{d}\,t\\
&\ge(\ln b-\ln a)\left\{\int_0^1 a^{1-t}b^t\big[t^{\alpha_1}f(b)+m_1(1-t^{\alpha_1})f\big(a^{1/m_1}\big)\big]\big[t^{\alpha_2}g(b)+m_2(1-t^{\alpha_2})g\big(a^{1/m_2}\big)\big]\,\mathrm{d}\,t\right\}\\
&=(\ln b-\ln a)\int_0^1 a^{1-t}b^t\big[t^{\alpha_1+\alpha_2}f(b)g(b)+m_1(1-t^{\alpha_1})t^{\alpha_2}f\big(a^{1/m_1}\big)g(b)\\
&\quad +m_2t^{\alpha_1}(1-t^{\alpha_2})f(b)g\big(a^{1/m_2}\big)+m_1m_2(1-t^{\alpha_1})(1-t^{\alpha_2})g\big(a^{1/m_2}\big)f\big(a^{1/m_1}\big)\big]\mathrm{d}\,t\\
&=(\ln b-\ln a)\big\{m_1m_2[L(a,b)-G(\alpha_1,1)-G(\alpha_2,1)\\
&\quad +G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g\big(a^{1/m_2}\big)+m_1[G(\alpha_2,1)-G(\alpha_1+\alpha_2,1)]f\big(a^{1/m_1}\big)g(b)\\
&\quad +m_2[G(\alpha_1,1)-G(\alpha_1+\alpha_2,1)]f(b)g\big(a^{1/m_2}\big)+G(\alpha_1+\alpha_2,1)f(b)g(b)\big\}.
\end{aligned}
$$

The proof of Theorem 3.7 is complete. □

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Corollary 3.9. *Under the conditions of Theorem 3.7, if $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$, we have*

$$\int_a^b f(x)g(x)\,\mathrm{d}\,x \ge \frac{1}{\ln b - \ln a}\{[2L(a,b) - a(\ln b - \ln a) - 2a]f(a)g(a) + [a + b - 2L(a,b)][f(a)g(b) + f(b)g(a)] + [2L(a,b) + b(\ln b - \ln a) - 2b]f(b)g(b)\}. \quad (3.17)$$

Remark 3.1. This paper is a slightly revised version of the preprint [7].

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Existence of Nonoscillatory Solutions for System of Higher Order Neutral Differential Equations with Distributed Delay

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Abstract

This article contains some sufficient conditions for the existence of nonoscillatory solutions for the systems of higher order neutral differential equations. The Banach contraction principle is used to prove results and examples are given to illustrate applicability of results.

Keywords: Neutral equations, Fixed point, Higher-order system, Nonoscillatory solution.

1. Introduction

In this article, we consider the following systems of higher-order neutral differential equations with distributed deviating arguments

$$[\mathbf{x}(t) + P(t)\mathbf{x}(t-\tau)]^{(n)} + (-1)^{n+1}\left[\int_{a_1}^{b_1} Q_1(t,\xi)\mathbf{x}(t-\xi)d\xi - \int_{a_2}^{b_2} Q_2(t,\xi)\mathbf{x}(t-\xi)d\xi\right] = \mathbf{0}, \tag{1}$$

$$[\mathbf{x}(t) + B\mathbf{x}(t-\tau)]^{(n)} + (-1)^{n+1}\left[\int_{a_1}^{b_1} Q_1(t,\xi)\mathbf{x}(t-\xi)d\xi - \int_{a_2}^{b_2} Q_2(t,\xi)\mathbf{x}(t-\xi)d\xi\right] = \mathbf{0}, \tag{2}$$

$$\begin{aligned}&\left[\mathbf{x}(t) + \int_{a_3}^{b_3} \tilde{P}(t,\xi)\mathbf{x}(t-\xi)d\xi\right]^{(n)}\\ &+(-1)^{n+1}\left[\int_{a_1}^{b_1} Q_1(t,\xi)\mathbf{x}(t-\xi)d\xi - \int_{a_2}^{b_2} Q_2(t,\xi)\mathbf{x}(t-\xi)d\xi\right] = \mathbf{0}\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\left[\mathbf{x}(t) + B\int_{a_3}^{b_3} \mathbf{x}(t-\xi)d\xi\right]^{(n)}\\ &+(-1)^{n+1}\left[\int_{a_1}^{b_1} Q_1(t,\xi)\mathbf{x}(t-\xi)d\xi - \int_{a_2}^{b_2} Q_2(t,\xi)\mathbf{x}(t-\xi)d\xi\right] = \mathbf{0},\end{aligned} \tag{4}$$

where $n \geqslant 1$ is an integer, $\tau \in (0,\infty)$, $b_i > a_i \geqslant 0$, $i = 1,2,3.$, $P \in C([t_0,\infty),\mathbf{R})$, $\tilde{P} \in C([t_0,\infty) \times [a_3,b_3],\mathbf{R})$, $\mathbf{x} \in \mathbf{R}^n$, Q_i is continuous $n \times n$ matrix on $[t_0,\infty) \times [a_i,b_i]$, $i = 1,2.$, and B is nonsingular constant $n \times n$ matrix.

In recent years, the problem of the existence of nonoscillatory solutions of neutral differential equations has been studied by many authors [1–8]. However, there has been a few studies concerning with the problem of the existence of nonoscillatory solutions for system of neutral delay equations [9,10]. In this article, we extend the results of Candan [10] to distributed deviating arguments case. We refer the reader to [11–16] for related books. Four theorems for (1), two theorems for (2) and (3), and one theorem for (4) are given according to the value of $P(t)$, $\|B\|$ and $\|B^{-1}\|$.

Let $m_1 = \max\{\tau, b_1, b_2\}$. By a solution of (1) (or (2)) we mean a function $\mathbf{x} \in C([t_1 - m_1,\infty),\mathbf{R}^n)$, for some $t_1 \geqslant t_0$, such that $\mathbf{x} + P(t)\mathbf{x}(t-\tau)$ (or $\mathbf{x} + B\mathbf{x}(t-\tau)$) are n times continuously differentiable on $[t_1,\infty)$ such that (1) (or (2)) are satisfied for $t \geqslant t_1$, respectively. Similarly, let $m_2 = \max\{b_1, b_2, b_3\}$. By a solution of (3) (or (4)) we mean a function $\mathbf{x} \in C([t_1 - m_2,\infty),\mathbf{R}^n)$, for some $t_1 \geqslant t_0$, such that $\mathbf{x}(t) + \int_{a_3}^{b_3} \tilde{P}(t,\xi)\mathbf{x}(t-\xi)d\xi$

(or $\mathbf{x}(t)+B\int_{a_3}^{b_3}\mathbf{x}(t-\xi)d\xi$) are n times continuously differentiable on $[t_1,\infty)$ such that (3) (or (4)) are satisfied for $t \geqslant t_1$, respectively.

As is customary, a solution of (1)-(4) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

2. Main Results

Theorem 1. *Assume that* $0 \leqslant P(t) \leqslant p < \frac{1}{2}$ *and*

$$\int_{t_0}^{\infty} s^{n-1}\int_{a_i}^{b_i}\|Q_i(s,\xi)\|d\xi ds < \infty, \quad i=1,2. \tag{5}$$

Then (1) has a bounded nonoscillatory solution.

Proof. It follows from (5) that we can choose a $t_1 > t_0$,

$$t_1 \geqslant t_0 + \max\{\tau, b_1, b_2\} \tag{6}$$

sufficiently large such that for $t \geqslant t_1$

$$\frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right)ds \leqslant \frac{\|\mathbf{b}\|-pC_2-C_1}{C_2}, \tag{7}$$

where $\mathbf{b}$ is a constant vector, C_1 and C_2 are positive constants such that

$$pC_2 + C_1 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_2 + C_1.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0,\infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x}\in\Omega : C_1 \leqslant \|\mathbf{x}(t)\| \leqslant C_2, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - P(t)\mathbf{x}(t-\tau) + & \\ \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2}Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. For $t \geqslant t_1$ and $\mathbf{x} \in A$, using (7) we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &= \Bigg\|\mathbf{b} - P(t)\mathbf{x}(t-\tau) \\ &+ \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2}Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds\Bigg\| \\ &\leqslant \|\mathbf{b}\| + p\|\mathbf{x}(t-\tau)\| \\ &+ \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\mathbf{x}(s-\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\mathbf{x}(s-\xi)\|d\xi\right)ds \\ &\leqslant \|\mathbf{b}\| + pC_2 + \frac{C_2}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right)ds \\ &\leqslant C_2 \end{aligned}$$

and

$$\begin{aligned}
\|(T\mathbf{x})(t)\| &= \Bigg\|\mathbf{b} - P(t)\mathbf{x}(t-\tau) \\
&+ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds\Bigg\| \\
&\geqslant \|\mathbf{b}\| - p\|\mathbf{x}(t-\tau)\| \\
&- \frac{1}{(n-1)!}\left\|\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds\right\| \\
&\geqslant \|\mathbf{b}\| - pC_2 - \frac{C_2}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right)ds \\
&\geqslant C_1,
\end{aligned}$$

which shows that T maps A into itself. Next, we show that T is a contraction mapping on A. For $\mathbf{x}_1, \mathbf{x}_2 \in A$ and $t \geqslant t_1$,

$$\begin{aligned}
\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| &\leqslant P(t)\|\mathbf{x}_1(t-\tau) - \mathbf{x}_2(t-\tau)\| \\
&+ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|\|\mathbf{x}_1(s-\xi) - \mathbf{x}_2(s-\xi)\|d\xi\right)ds \\
&+ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_2}^{b_2} \|Q_2(s,\xi)\|\|\mathbf{x}_1(s-\xi) - \mathbf{x}_2(s-\xi)\|d\xi\right)ds
\end{aligned}$$

or using (7), we obtain

$$\begin{aligned}
|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)| &\leqslant ||\mathbf{x}_1 - \mathbf{x}_2|| \\
&\times \left(p + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right)ds\right) \\
&\leqslant q_1||\mathbf{x}_1 - \mathbf{x}_2||,
\end{aligned}$$

where $q_1 < 1$. This implies with the sup norm that

$$\|T\mathbf{x}_1 - T\mathbf{x}_2\| \leqslant q_1\|\mathbf{x}_1 - \mathbf{x}_2\|,$$

i.e., T is a contraction mapping on A. Therefore there exists a unique solution $\mathbf{x} \in A$ with $\|\mathbf{x}\| > 0$ of $T\mathbf{x} = \mathbf{x}$. The theorem is proved. □

Theorem 2. *Assume (5) holds and $2 < p \leqslant P(t) \leqslant p_0 < \infty$. Then (1) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$,

$$t_1 + \tau \geqslant t_0 + \max\{b_1, b_2\} \tag{8}$$

sufficiently large such that

$$\begin{aligned}
&\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right)ds \\
&\qquad\qquad \leqslant \frac{\|\mathbf{b}\| - C_4 - p_0C_3}{C_4}, \quad t \geqslant t_1
\end{aligned} \tag{9}$$

where $\mathbf{b}$ is a constant vector, C_3 and C_4 are positive constants such that

$$p_0C_3 + C_4 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant p_0C_3 + pC_4.$$

3

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{x \in \Omega : C_3 \leqslant \|\mathbf{x}(t)\| \leqslant C_4, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{P(t+\tau)} \Bigg\{ \mathbf{b} - \mathbf{x}(t+\tau) + \\ \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} \left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi \right) ds \Bigg\}, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. For $t \geqslant t_1$ and $\mathbf{x} \in A$, using (9) we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\leqslant \frac{1}{p}\Bigg(\Bigg\|\mathbf{b} - \mathbf{x}(t+\tau) \\ &+ \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right) ds\Bigg\|\Bigg) \\ &\leqslant \frac{1}{p}\Bigg(\|\mathbf{b}\| + \|\mathbf{x}(t+\tau)\| \\ &+ \frac{1}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\mathbf{x}(s-\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\mathbf{x}(s-\xi)\|d\xi\right) ds\Bigg) \\ &\leqslant \frac{1}{p}\left(\|\mathbf{b}\| + C_4 + \frac{C_4}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right) ds\right) \\ &\leqslant C_4 \end{aligned}$$

and

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geqslant \frac{1}{p_0}\Bigg(\Bigg\|\mathbf{b} - \mathbf{x}(t+\tau) \\ &+ \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right) ds\Bigg\|\Bigg) \\ &\geqslant \frac{1}{p_0}\Bigg(\|\mathbf{b}\| - \|\mathbf{x}(t+\tau)\| \\ &- \frac{1}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left\|\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right\| ds\Bigg) \\ &\geqslant \frac{1}{p_0}\left(\|\mathbf{b}\| - C_4 - \frac{C_4}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right) ds\right) \\ &\geqslant C_3, \end{aligned}$$

which shows that T maps A into itself. Next, we show that T is a contraction mapping on A. For $\mathbf{x}_1, \mathbf{x}_2 \in A$ and $t \geqslant t_1$,

$$\begin{aligned} \|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| &\leqslant \frac{1}{p}\Bigg(\|\mathbf{x}_1(t+\tau) - \mathbf{x}_2(t+\tau)\| \\ &+ \frac{1}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|\|\mathbf{x}_1(s-\xi) - \mathbf{x}_2(s-\xi)\|d\xi\right) ds \\ &+ \frac{1}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}\left(\int_{a_2}^{b_2} \|Q_2(s,\xi)\|\|\mathbf{x}_1(s-\xi) - \mathbf{x}_2(s-\xi)\|d\xi\right) ds\Bigg) \end{aligned}$$

or using (9), we obtain

$$\begin{aligned}|(T\mathbf{x}_1)(t)-(T\mathbf{x}_2)(t)| &\leqslant \frac{\|\mathbf{x}_1-\mathbf{x}_2\|}{p} \\ &\times \left(1+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi+\int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right)ds\right) \\ &\leqslant q_1||\mathbf{x}_1-\mathbf{x}_2||,\end{aligned}$$

where $q_1 < 1$. This implies with the sup norm that

$$\|T\mathbf{x}_1 - T\mathbf{x}_2\| \leqslant q_1\|\mathbf{x}_1-\mathbf{x}_2\|,$$

i.e., T is a contraction mapping on A. Therefore there exists a unique solution $\mathbf{x}\in A$ with $\|\mathbf{x}\|>0$ of $T\mathbf{x}=\mathbf{x}$. The proof is complete. □

Theorem 3. *Assume that (5) holds and $-\frac{1}{2} < p \leqslant P(t) < 0$. Then (1) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (6) such that

$$\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi+\int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right)ds \leqslant \frac{\|\mathbf{b}\|-|p|C_6-C_5}{C_6}, \quad t\geqslant t_1$$

where $\mathbf{b}$ is a constant vector, C_5 and C_6 are positive constants such that

$$|p|C_6 + C_5 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_6 + C_5.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0,\infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x}\in\Omega : C_5 \leqslant \|\mathbf{x}(t)\| \leqslant C_6, \quad t\geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - P(t)\mathbf{x}(t-\tau) + \\ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds, & t\geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0\leqslant t\leqslant t_1.\end{cases}$$

Clearly $T\mathbf{x}$ is continuous. The proof is similar to that of Theorem 1, and therefore we omit the remaining part of the proof. Thus the proof is complete. □

Theorem 4. *Assume that (5) holds and $-\infty < p_0 \leqslant P(t) \leqslant p < -2$. Then (1) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (8) such that

$$\begin{aligned}\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}&\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi+\int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right)ds \\ &\leqslant \frac{\|\mathbf{b}\|-C_8-|p_0|C_7}{C_8}, \quad t\geqslant t_1,\end{aligned}$$

where $\mathbf{b}$ is a constant vector, C_7 and C_8 are positive constants such that

$$|p_0|C_7 + C_8 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant |p_0|C_7 + |p|C_8.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0,\infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{x\in\Omega : C_7 \leqslant \|\mathbf{x}(t)\| \leqslant C_8, \quad t\geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{P(t+\tau)} \Bigg\{ \mathbf{b} - \mathbf{x}(t+\tau) + \\ \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} \left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi \right) ds \Bigg\}, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. The remaining part of the proof is similar to that of Theorem 2, and therefore it is omitted. Thus the theorem is proved. □

Theorem 5. *Assume that (5) holds and* $0 < \|B\| < \frac{1}{2}$. *Then (2) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (6) such that

$$\frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\| d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\| d\xi \right) ds \leqslant \frac{\|\mathbf{b}\| - \|B\|C_2^* - C_1^*}{C_2^*}, \quad t \geqslant t_1$$

where $\mathbf{b}$ is a constant vector, C_1^* and C_2^* are positive constants such that

$$\|B\|C_2^* + C_1^* < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_2^* + C_1^*.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x} \in \Omega : C_1^* \leqslant \|\mathbf{x}(t)\| \leqslant C_2^*, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - B\mathbf{x}(t-\tau) + \\ \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi \right) ds, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. The remaining part of the proof is similar to that of Theorem 1, therefore it is omitted. □

Theorem 6. *Assume that (5) holds and* $0 < \|B^{-1}\| < \frac{1}{2}$. *Then (2) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (8) such that

$$\begin{aligned} \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\| d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\| d\xi \right) ds \\ \leqslant \frac{\|B^{-1}\mathbf{b}\| - C_3^* - C_4^*\|B^{-1}\|}{C_4^*\|B^{-1}\|}, \quad t \geqslant t_1 \end{aligned} \tag{10}$$

where $\mathbf{b}$ is a constant vector, C_3^* and C_4^* are positive constants such that

$$C_4^*\|B^{-1}\| + C_3^* < \|B^{-1}\mathbf{b}\| < 2\|B^{-1}\mathbf{b}\| \leqslant C_4^* + C_3^*.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{x \in \Omega : C_3^* \leqslant \|\mathbf{x}(t)\| \leqslant C_4^*, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} B^{-1}\Big\{\mathbf{b} - \mathbf{x}(t+\tau) + \\ \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right) ds\Big\}, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. For $t \geqslant t_1$ and $\mathbf{x} \in A$, using (10) we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\leqslant \|B^{-1}\mathbf{b}\| \\ &+ \|B^{-1}\|\left(C_4^* + \frac{C_4^*}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right) ds\right) \\ &\leqslant C_4^* \end{aligned}$$

and

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geqslant \|B^{-1}\mathbf{b}\| \\ &- \|B^{-1}\|\left(C_4^* + \frac{C_4^*}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right) ds\right) \\ &\geqslant C_3^*, \end{aligned}$$

which shows that T maps A into itself. Next, we show that T is a contraction mapping on A. For $\mathbf{x}_1, \mathbf{x}_2 \in A$ and $t \geqslant t_1$, and by using (10)

$$\begin{aligned} \|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| &\leqslant \|B^{-1}\|\|\mathbf{x}_1 - \mathbf{x}_2\| \\ &\times \left(1 + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right) ds\right) \\ &\leqslant q_1\|\mathbf{x}_1 - \mathbf{x}_2\|, \end{aligned}$$

where $q_1 < 1$. This implies with the sup norm that

$$\|T\mathbf{x}_1 - T\mathbf{x}_2\| \leqslant q_1\|\mathbf{x}_1 - \mathbf{x}_2\|,$$

i.e., T is a contraction mapping on A. Therefore there exists a unique solution $\mathbf{x} \in A$ with $\|\mathbf{x}\| > 0$ of $T\mathbf{x} = \mathbf{x}$. The theorem is proved. $\square$

Theorem 7. *Assume that (5) holds and* $0 \leqslant \int_{a_3}^{b_3}\tilde{p}(t,\xi)d\xi \leqslant p < \frac{1}{2}$. *Then (3) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (6) such that

$$\frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}\left(\int_{a_1}^{b_1}\|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2}\|Q_2(s,\xi)\|d\xi\right) ds \leqslant \frac{\|\mathbf{b}\| - pC_6^* - C_5^*}{C_6^*}, \quad t \geqslant t_1$$

where $\mathbf{b}$ is a constant vector, C_5^* and C_6^* are positive constants such that

$$pC_6^* + C_5^* < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_6^* + C_5^*.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x} \in \Omega : C_5^* \leqslant \|\mathbf{x}(t)\| \leqslant C_6^*, \quad t \geqslant t_0\}.$$

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Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - \int_{a_3}^{b_3} \tilde{p}(t,\xi)x(t-\xi)d\xi + \\ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. Since the proof is similar to that of Theorem 1, we omit the remaining part of the proof. Thus the proof is complete. □

Theorem 8. *Assume that (5) holds and* $-\frac{1}{2} < p \leqslant \int_{a_3}^{b_3} \tilde{p}(t,\xi)d\xi < 0$. *Then (3) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (6) such that

$$\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right)ds \leqslant \frac{\|\mathbf{b}\| - |p|C_8^* - C_7^*}{C_8^*}, \quad t \geqslant t_1$$

where $\mathbf{b}$ is a constant vector, C_7^* and C_8^* are positive constants such that

$$|p|C_8^* + C_7^* < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_8^* + C_7^*.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x} \in \Omega : C_7^* \leqslant \|\mathbf{x}(t)\| \leqslant C_8^*, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - \int_{a_3}^{b_3} \tilde{p}(t,\xi)x(t-\xi)d\xi + \\ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds, & t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. Since the proof is similar to that of Theorem 1, we omit the remaining part of the proof. Thus the proof is complete. □

Theorem 9. *Assume that (5) holds and* $0 < (b_3 - a_3)\|B\| < \frac{1}{2}$. *Then (4) has a bounded nonoscillatory solution.*

Proof. It follows from (5) that we can choose a $t_1 > t_0$ sufficiently large satisfying (6) such that for $t \geqslant t_1$

$$\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} \|Q_1(s,\xi)\|d\xi + \int_{a_2}^{b_2} \|Q_2(s,\xi)\|d\xi\right)ds \leqslant \frac{\|\mathbf{b}\| - (b_3-a_3)\|B\|C_{10}^* - C_9^*}{C_{10}^*},$$

where $\mathbf{b}$ is a constant vector, C_9^* and C_{10}^* are positive constants such that

$$(b_3 - a_3)\|B\|C_{10}^* + C_9^* < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leqslant C_{10}^* + C_9^*.$$

Let Ω denote the Banach space of all continuous bounded vector functions defined on $[t_0, \infty)$ with the sup norm. We consider the subset A of Ω as

$$A = \{\mathbf{x} \in \Omega : C_9^* \leqslant \|\mathbf{x}(t)\| \leqslant C_{10}^*, \quad t \geqslant t_0\}.$$

Obviously, A is a bounded, closed, convex subset of Ω. Define the operator T on A as

$$(T\mathbf{x})(t) = \begin{cases} \mathbf{b} - B\int_{a_3}^{b_3} x(t-\xi)d\xi + \\ \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left(\int_{a_1}^{b_1} Q_1(s,\xi)\mathbf{x}(s-\xi)d\xi - \int_{a_2}^{b_2} Q_2(s,\xi)\mathbf{x}(s-\xi)d\xi\right)ds, \ t \geqslant t_1 \\ (T\mathbf{x})(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly $T\mathbf{x}$ is continuous. The remaining part of the proof is similar to that of Theorem 1, therefore it is omitted. □

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Example 1. *Consider the neutral differential equation system*

$$\left(\boldsymbol{x}(t) - \frac{1}{5}\boldsymbol{x}(t-\tau)\right)^{(3)} + \int_{7/30}^{11/15} Q_1(t,\xi)\boldsymbol{x}(t-\xi)d\xi - \int_{1}^{e^{1/15}} Q_2(t,\xi)\boldsymbol{x}(t-\xi)d\xi = \boldsymbol{0}, \tag{11}$$

where

$$Q_1(t,\xi) = \frac{(6 + 6\ln t + 2(\ln t)^2)\ln(t-\xi)}{t^3(\ln(t-\xi)+1)(\ln t)^4}\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{5}{3} \end{pmatrix}$$

and

$$Q_2(t,\xi) = \frac{(6 + 6\ln(t-\tau) + 2(\ln(t-\tau))^2)\ln(t-\xi)}{\xi(t-\tau)^3(\ln(t-\xi)+1)(\ln(t-\tau))^4}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

It is easy to verify that the assumptions of Theorem 3 are satisfied. In fact,

$$\boldsymbol{x} = \begin{pmatrix} 1 + \frac{1}{\ln t} \\ 1 + \frac{1}{\ln t} \end{pmatrix}, \quad t > 1 + \max\{\tau, e^{1/15}\}$$

is a nonoscillatory solution of (11).

Example 2. *Consider the neutral differential equation system*

$$(\boldsymbol{x}(t) + B\boldsymbol{x}(t-\tau))^{(7)} + \int_{2}^{4} Q_1(t,\xi)\boldsymbol{x}(t-\xi)d\xi - \int_{3/2}^{9/2} Q_2(t,\xi)\boldsymbol{x}(t-\xi)d\xi = \boldsymbol{0}, \tag{12}$$

where

$$B = \begin{pmatrix} \frac{2e^{-\tau}}{15} & \frac{e^{-\tau}}{15} \\ \frac{e^{-\tau}}{25} & \frac{4e^{-\tau}}{25} \end{pmatrix}$$

$$Q_1(t,\xi) = \frac{\xi}{\alpha e^t + e^\xi}\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

and

$$Q_2(t,\xi) = \frac{\xi}{\alpha e^t + e^\xi}\begin{pmatrix} \frac{4}{9} & \frac{16}{45} \\ \frac{12}{65} & \frac{8}{13} \end{pmatrix}.$$

It is easy to verify that the assumptions of Theorem 5 are satisfied. In fact,

$$\boldsymbol{x} = \begin{pmatrix} \alpha + e^{-t} \\ \alpha + e^{-t} \end{pmatrix}, \quad \alpha \in \boldsymbol{R}, \quad t > \max\{\tau, \frac{9}{2}\}$$

is a nonoscillatory solution of (12).

Example 3. *Consider the neutral differential equation system*

$$\left(\boldsymbol{x}(t) + \frac{1}{6}\int_0^2 \boldsymbol{x}(t-\xi)d\xi\right)^{(2)} - \int_{4}^{10} Q_1(t,\xi)\boldsymbol{x}(t-\xi)d\xi + \int_{7/2}^{5} Q_2(t,\xi)\boldsymbol{x}(t-\xi)d\xi = \boldsymbol{0}, \tag{13}$$

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where

$$Q_1(t,\xi) = \frac{(t-\xi)(2\ln t - \frac{t\ln t}{6} + \frac{t}{6} - 3)}{t^3(t-\xi+\ln(t-\xi))}\begin{pmatrix} \frac{1}{8} & \frac{1}{24} \\ \frac{1}{3} & \frac{-1}{6} \end{pmatrix}$$

and

$$Q_2(t,\xi) = \frac{(t-\xi)(1-\ln(t-2))}{(t-2)^2(t-\xi+\ln(t-\xi))}\begin{pmatrix} \frac{1}{9} & 0 \\ 1 & \frac{-8}{9} \end{pmatrix}.$$

It is easy to verify that the assumptions of Theorem 7 are satisfied. In fact,

$$\boldsymbol{x} = \begin{pmatrix} 1+\frac{\ln t}{t} \\ 1+\frac{\ln t}{t} \end{pmatrix}, \quad t > 11$$

is a nonoscillatory solution of (13).

Example 4. *Consider the neutral differential equation system*

$$\left(\boldsymbol{x}(t) + B\int_0^1 \boldsymbol{x}(t-\xi)d\xi\right)^{(5)} + \int_{\ln 2}^{\ln 6} Q_1(t,\xi)\boldsymbol{x}(t-\xi)d\xi - \int_{1/15}^{1/6} Q_2(t,\xi)\boldsymbol{x}(t-\xi)d\xi = \boldsymbol{0}, \tag{14}$$

where

$$B = \begin{pmatrix} \frac{1}{30} & \frac{2}{15} \\ \frac{1}{8} & \frac{1}{24} \end{pmatrix}$$

$$Q_1(t,\xi) = \frac{e^{\xi}(1+t-\xi)(29-t)}{(1+t)^6}\begin{pmatrix} \frac{1}{8} & \frac{7}{8} \\ \frac{7}{10} & \frac{3}{10} \end{pmatrix}$$

and

$$Q_2(t,\xi) = -\frac{(1+t-\xi)}{t^5}\begin{pmatrix} 4 & 36 \\ 16 & 24 \end{pmatrix}.$$

It is easy to verify that the assumptions of Theorem 9 are satisfied. In fact,

$$\boldsymbol{x} = \begin{pmatrix} \frac{1}{1+t} \\ \frac{1}{1+t} \end{pmatrix}, \quad t > \ln 6$$

is a nonoscillatory solution of (14).

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On the stability of septic and octic functional equations

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Abstract: In this paper, we establish the general solutions of the septic and octic functional equations on commutative groups, respectively. Moreover, we prove some stability results concerning these two types of functional equations in normed linear spaces.

Keywords: Hyers-Ulam stability; Additive symmetric function;Difference operator; Septic functional equation; Octic functional equation

1 Introduction

The study of stability problems for functional equations originated from a question proposed by Ulam [33] concerning the stability of group homomorphisms. Afterwards, Hyers [13] gave the first significant partial solution in Banach spaces for the above-mentioned question in 1941. In 1978, Rassias [28] generalized the result of Hyers for approximately linear mappings by allowing the Cauchy difference to be unbounded. Later, Găvruţa [12] provided a generalization of Rassias's Theorem for approximately additive mappings, in which a more general function is employed to characterized the error of approximation. Subsequently, the stability problems for several types of functional equations in various spaces have been extensively studied by many authors [15, 16, 21, 24, 29, 30, 31, 32]. Different with the direct proof used before, Radu [25] proposed a novel method for studying the stability of the Cauchy functional equation based on a fixed point result in generalized metric spaces. Such a method simplified the proof of the original results, and then it also further stimulated the study of the stability of functional equations. Until now, there are many related results obtained by this method [4, 5, 6, 18, 34].

In 1984, Cholewa [8] initiated the study of the stability of the following functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1}$$

The equation (1) is said to be a quadratic functional equation since the function $f(x)=x^2$ is a solution. Meantime, each solution of the quadratic functional equation is called a quadratic mappings. In the following, further researches concerning the stability problems of the quadratic functional equation have been done by various authors [2, 3, 10, 11, 14, 19, 22]. It is worth noting that the fixed point method has been repeatedly used in several papers mentioned above.

Several years later, Rassias [27] proposed the following functional equation

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y) \tag{2}$$

and considered the solution and the stability problem of this equation. Note that the equation (2) is called a cubic functional because the function $f(x)=x^3$ satisfies this equation. Accordingly, each solution of the cubic functional equation is called a cubic mapping. Afterwards, the stability problems of generalized cubic functional equations were investigated by Jung and Chang [7, 17].

In 1999, Rassias [26] introduced the quartic functional equation, which is given by

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+f(x-y)+24f(y). \tag{3}$$

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It is easy to see that the function $f(x) = x^4$ is a solution of the equation (3). Each solution of the quartic functional equation is called a quartic mapping. Hereafter, Lee et al. [20] further estabilished the general solution of the equation (3) and proved the stability of this equation in real normed linear spaces. Soon after, Najati [23] proved the generalized Hyers-Ulam stability of the equation (3).

Recently, Xu et al. [34] achieved the general solutions of the quintic and sextic functional equations, and then proved the stability of these two types of equations in quasi-β-normed spaces based on the fixed point method. The corresponding equations are respectively given as follows:

$$f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) - f(x-2y) = 120f(y), \tag{4}$$

$$\begin{aligned} f(x+3y)-&6f(x+2y) + 15f(x+y) - 20f(x) + 15f(x-y) \\ &- 6(x-2y) + f(x-3y) = 720f(y). \end{aligned} \tag{5}$$

Since $f(x) = x^5$ is a solution of the equation (4), it is called a quintic functional equation. Similarly, $f(x) = x^6$ is a solution of the equation (5), it it called a sextic functional equation. Simultaneously, each solution of the quintic or sextic functional equation is called a quintic and sextic mapping, respectively.

The principal purpose of this paper is to estabilish the general solutions and further investigate the stability of the following two functional equations.

$$\begin{aligned} f(x+4y)-&7f(x+3y) + 21f(x+2y) - 35f(x+y) - 21f(x-y) \\ &+ 7f(x-2y) - f(x-3y) + 35f(x) = 5040f(y), \end{aligned} \tag{6}$$

$$\begin{aligned} f(x+4y) - 8f(x+3y) + 28f(x+2y) - 56f(x+y) - 56f(x-y) \\ + 28f(x-2y) - 8f(x-3y) + f(x-4y) + 70f(x) = 40320f(y) \end{aligned} \tag{7}$$

Note that the function $f(x) = x^7$ satisfies (6), we say that the equation (6) is a septic functional equation. Analogously, the function $f(x) = x^8$ satisfies (7), we say that the equation (7) is an octic functional equation. Correspondingly, each solution of the septic and octic functional equation is said to be a septic and an octic mapping, respectively.

2 Preliminaries

For the sake of completeness, in this section, some related notions and results from [1, 9, 35] are summarized below. Let $\mathbb{N}$ and $\mathbb{Q}$ denote the set of natural numbers and the set of rational numbers, respectively.

Let X and Y be real linear spaces. A function $A : X \to Y$ is said to be *additive* if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$. It is well known that $A(rx) = rA(x)$ for all $x \in X$ and for every $r \in \mathbb{Q}$. More generally, a function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is called *symmetric* if $A_n(x_1, x_2, \cdots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$ of $\{1, 2, \cdots, n\}$. For an n-additive symmetric function $A_n(x_1, x_2, \cdots, x_n)$, its diagonal is the function $A_n(x, x, \cdots, x)$ for $x \in X$ and denoted by $A^n(x)$. Evidently, $A^n(rx) = r^nA^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a *monomial function* of degree n if $A^n \not\equiv 0$. Moreover, the resulting function after substitution $x_1 = x_2 = \cdots = x_l = x$ and $x_{l+1} = x_{l+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p : X \to Y$ is called a *generalized polynomial function* of degree n provided that there exist $A^0(x) = A^0 \in Y$ and i-additive symmetric functions $A_i : X^i \to Y$ $(1 \le i \le n)$ such that

$$p(x) = \sum_{i=0}^{n} A^i(x)$$

for all $x \in X$ and $A^n \not\equiv 0$.

Let $f : X \to Y$. The difference operator $\triangle_h$ is defined as follows:

$$\triangle_h f(x) = f(x+h) - f(x)$$

for $h \in X$. In fact, a difference opertor can be extended to an n-order difference operator in the usual composition way by induction. For each $h \in X$ and $n \in \mathbb{N} \cup \{0\}$, define

$$\triangle_h^{n+1} f(x) = \triangle_h \circ \triangle_h^n f(x)$$

with the conventions $\triangle_h^0 f(x) = f(x)$ and $\triangle_h^1 f(x) = \triangle_h f(x)$. Furthermore, a more general difference operator, which was used in the Fréchet functional equation, can be defined as

$$\triangle_{h_1,h_2,\cdots,h_{n+1}} f(x) = \triangle_{h_{n+1}} \circ \triangle_{h_n} \circ \cdots \circ \triangle_{h_1} f(x),$$

where $x, h_1, h_2, \cdots, h_{n+1} \in X$.

Lemma 2.1 ([9, 35])**.** *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Assume that the multiplication by $n!$ is bijective in S, i.e., for every element $b \in S$, the equation $n!a = b$ has a unique solution in S. Then, the function $f : G \to S$ is a solution of Fréchet functional equation*

$$\triangle_{x_1,x_2,\cdots,x_{n+1}} f(x_0) = 0 \tag{8}$$

for all $x_0, x_1, \cdots, x_{n+1} \in G$ if and only if f is a polynomial of degree at most n.

Lemma 2.2 ([9, 35])**.** *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i $(i = 1, 2, \cdots, n+1)$ additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G)$. If the functions $f, f_i : G \to S$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{9}$$

then f satisfies Fréchet functional equation $\triangle_{x_1,x_2,\cdots,x_{n+1}} f(x_0) = 0$.

3 General solutions of the septic and octic functional equations on comomutative groups

In this section, we establish the general solutions of the functional equations (6) and (7) on commutative groups. Throughout this section, G and S will denote commutative groups.

Theorem 3.1. *A function $f : G \to S$ is a solution of the functional equation (6) if and only if f is of the form $f(x) = A^7(x)$ for all $x \in X$, where $A^7(x)$ is the diagonal of the 7-additve symmetric function $A_7 : G^7 \to S$.*

Proof. Necessity: Assume that f satisfies the functional equation (6). Putting $x = y = 0$ in (6), one gets $f(0) = 0$. Substituting (x, y) with $(0, x)$ and $(x, -x)$ in (6), respectively, and adding the two resulting equations, we can obtain $f(-x) = -f(x)$, that is to say, f is an odd function. Replacing (x, y) by $(4x, x)$ and $(0, 2x)$, respectively, and subtracting the two resulting equations, we get

$$7f(7x) - 27f(6x) + 35f(5x) - 21f(4x) + 21f(3x) - 5061f(2x) + 5041f(x) = 0. \tag{10}$$

Replacing (x, y) by $(3x, x)$ in (6), we obtain that

$$f(7x) - 7f(6x) + 21f(5x) - 35f(4x) + 35f(3x) - 21f(2x) - 5033f(x) = 0. \tag{11}$$

Multiplying (11) by 7, and then subtracting (10) from the resulting equation, we get

$$11f(6x) - 56f(5x) + 112f(4x) - 112f(3x) - 2457f(2x) - 20136f(x) = 0. \tag{12}$$

Replacing (x, y) by $(2x, x)$ in (6), it follows that

$$f(6x) - 7f(5x) + 21f(4x) - 35f(3x) + 35f(2x) - 5060f(x) = 0. \tag{13}$$

Multiplying (13) by 11, and then subtracting (12) from the resulting equation, we find

$$3f(5x) - 17f(4x) + 39f(3x) - 406f(2x) + 10828f(x) = 0. \tag{14}$$

Replacing (x, y) by (x, x) in (6), we have

$$f(5x) - 7f(4x) + 21f(3x) - 34f(2x) - 5012f(x) = 0. \tag{15}$$

Mutiplying (15) by 3, and then subtracting (14) from the resulting equation, we obtain

$$f(4x) - 6f(3x) - 76f(2x) + 6466f(x) = 0. \tag{16}$$

Replacing (x, y) by $(0, x)$ in (6), we get

$$f(4x) - 6f(3x) + 14f(2x) - 5054f(x) = 0. \tag{17}$$

Subtracting (16) and (17), we can obtain

$$f(2x) = 128f(x) = 2^7 f(x) \tag{18}$$

for all $x \in G$.

Moreover, the functional equation (6) can be written as

$$\begin{aligned} f(x)+&\frac{1}{35}f(x+4y) - \frac{1}{5}f(x+3y) + \frac{3}{5}f(x+2y) - f(x+y) \\ &- \frac{3}{5}f(x-y) + \frac{1}{5}f(x-2y) - \frac{1}{35}f(x-3y) - 144f(y) = 0. \end{aligned} \tag{19}$$

Therefore, it follows from lemmas 2.1 and 2.2 that f is a generalized polynomial function of degree at most 7, this is to say, f is of the form

$$f(x) = \sum_{n=0}^{7} A^n(x) \tag{20}$$

for all $x \in G$, where $A^0(x) = A^0$ is an arbitrary element in S and $A^n(x)$ is the diagonal of the n-additive symmetric function $A_n : G^n \to S$ for $n = 1, 2, \cdots, 7$. Since f is odd and $f(0) = 0$, we know that $A^0(x) = A^0 = 0$ and $A^2(x) = A^4(x) = A^6(x) = 0$. Thus, the expression (20) can be simplified into

$$f(x) = A^7(x) + A^5(x) + A^3(x) + A^1(x). \tag{21}$$

By (18) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we can obtain $2^7A^5(x) + 2^7A^3(x) + 2^7A^1(x) = 2^5A^5(x) + 2^3A^3(x) + 2A^1(x)$. Therefore $A^1(x) = -\frac{4}{21}(4A^5(x) + 5A^3(x))$ for all $x \in G$, and hence $A^5(x) = A^3(x) = A^1(x) = 0$. So $f(x) = A^7(x)$.

Sufficiency: Assume that $f(x) = A^7(x)$ for all $x \in G$, where $A^7(x)$ is the diagonal of the 7-additive symmetric function $A_7 : G^7 \to S$. According to the definition of additive function, we know that

$$\begin{aligned} A^7(x+y) =& A^7(x) + A^7(y) + 7A^{6,1}(x,y) + 21A^{5,2}(x,y) + 35A^{4,3}(x,y) \\ &+ 35A^{3,4}(x,y) + 21A^{2,5}(x,y) + 7A^{1,6}(x,y) \end{aligned} \tag{22}$$

and $A^7(rx) = r^7A^7(x)$, $A^{s,t}(x, ry) = r^tA^{s,t}(x, y)$ $(s, t = 1, 2, \cdots, 6, s+t = 7)$ whenever $x, y \in G$ and $r \in \mathbb{Q}$. Letting (22) and the above equalities into (6), we find that f satisfies (6). The proof of the theorem is now completed. □

Theorem 3.2. *A function $f : G \to S$ is a solution of the functional equation (7) if and only if f is of the form $f(x) = A^8(x)$ for all $x \in X$, where $A^8(x)$ is the diagonal of the 8-additve symmetric function $A_8 : G^8 \to S$.*

Proof. Necessity: Assume that f satisfies the functional equation (7). Putting $x = y = 0$ in (7), one gets $f(0) = 0$. Substituting (x, y) with $(0, x)$ and $(0, -x)$ in (7), respectively, and substracting the two resulting equations, we can obtain $f(-x) = f(x)$, which means that f is an even function. Replacing (x, y) by $(4x, x)$ and $(0, 2x)$, respectively, and subtracting the two resulting equations, we have

$$4f(7x) - 18f(6x) + 28f(5x) - 21f(4x) + 28f(3x) - 10122f(2x) + 20164f(x) = 0. \tag{23}$$

Replacing (x, y) by $(3x, x)$ in (7), we get

$$f(7x) - 8f(6x) + 28f(5x) - 56f(4x) + 70f(3x) - 56f(2x) + 40291f(x) = 0. \tag{24}$$

Multiplying (24) by 4, and then subtracting (23) from the resulting equation, we obtain

$$2f(6x) - 12f(5x) + 29f(4x) - 36f(3x) - 1414f(2x) + 25904f(x) = 0. \tag{25}$$

Replacing (x, y) by $(2x, x)$ in (7), we have

$$f(6x) - 8f(5x) + 28f(4x) - 56f(3x) + 71f(2x) - 40384f(x) = 0. \tag{26}$$

Multiplying (26) by 2, and then subtracting (25) from the resulting equation, it follows that

$$4f(5x) - 27f(4x) + 76f(3x) - 1556f(2x) + 106672f(x) = 0. \tag{27}$$

Replacing (x, y) by (x, x) in (7), we get

$$f(5x) - 8f(4x) + 29f(3x) - 64f(2x) - 40222f(x) = 0. \tag{28}$$

Multiplying (28) by 4, and then subtracting (27) from the resulting equation, we obtain

$$f(4x) - 8f(3x) - 260f(2x) + 53512f(x) = 0. \tag{29}$$

Replacing (x, y) by $(0, x)$ in (7), we have

$$f(4x) - 8f(3x) + 28f(2x) - 20216f(x) = 0. \tag{30}$$

Subtracting (29) and (30), we get

$$f(2x) = 256f(x) = 2^8 f(x) \tag{31}$$

for all $x \in G$.

Moreover, the functional equation (7) can be written as

$$\begin{aligned} f(x) + \frac{1}{70}f(x+4y) - \frac{4}{35}f(x+3y) + \frac{2}{5}f(x+2y) - \frac{4}{5}f(x+y) - \frac{4}{5}f(x-y) \\ + \frac{2}{5}f(x-2y) - \frac{4}{35}f(x-3y) + \frac{1}{70}f(x-4y) - 576f(y) = 0. \end{aligned} \tag{32}$$

Therefore, it follows from lemmas 2.1 and 2.2 that f is a generalized polynomial function of degree at most 8, this is to say, f is of the form

$$f(x) = \sum_{n=0}^{8} A^n(x) \tag{33}$$

for all $x \in G$, where $A^0(x) = A^0$ is an arbitrary element in S and $A^n(x)$ is the diagonal of the n-additive symmetric function $A_n : G^n \to S$ for $n = 1, 2, \cdots, 8$. Since f is even and $f(0) = 0$, we know that $A^0(x) = A^0 = 0$ and $A^1(x) = A^3(x) = A^5(x) = A^7(x) = 0$. Thus, the expression (33) can be simplified into

$$f(x) = A^8(x) + A^6(x) + A^4(x) + A^2(x). \tag{34}$$

By (31) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we infer that $2^8A^6(x) + 2^8A^4(x) + 2^8A^2(x) = 2^6A^6(x) + 2^4A^4(x) + 2^2A^2(x)$. Therefore $A^2(x) = -\frac{4}{21}(4A^6(x) + 5A^4(x))$ for all $x \in G$, and hence $A^6(x) = A^4(x) = A^2(x) = 0$. So $f(x) = A^8(x)$.

Sufficiency: Assume that $f(x) = A^8(x)$ for all $x \in G$, where $A^8(x)$ is the diagonal of the 8-additive symmetric function $A_8 : G^8 \to S$. According to the definition of additive function, we know that

$$\begin{aligned} A^8(x+y) = A^8(x) + A^8(y) + 8A^{7,1}(x,y) + 28A^{6,2}(x,y) + 56A^{5,3}(x,y) \\ + 70A^{4,4}(x,y) + 56A^{3,5}(x,y) + 28A^{2,6}(x,y) + 8A^{1,7}(x,y) \end{aligned} \tag{35}$$

and $A^8(rx) = r^8A^8(x)$, $A^{s,t}(x, ry) = r^tA^{s,t}(x,y)$ $(s,t = 1,2,\cdots,8, s+t = 8)$ whenever $x, y \in G$ and $r \in \mathbb{Q}$. Letting (35) and the above equalities into (7), we conclude that f satisfies (7). This completes the proof. □

4 Stability of the septic functional equation (6)

Throughout this section, unless otherwise stated, X will denote a real linear space and $(Y, \|\cdot\|_Y)$ is a real Banach space. Here we will investigate the stability of the septic functional equation in real normed spaces. For notational convenience, define the difference operator

$$\begin{aligned} D_s f(x,y) = f(x+4y) - 7f(x+3y) + 21f(x+2y) - 35f(x+y) \\ - 21f(x-y) + 7f(x-2y) - f(x-3y) + 35f(x) - 5040f(y). \end{aligned} \tag{36}$$

As a special case of Lemma 3.1 in [34], we can easily obtain the following theorem.

Theorem 4.1. *Let $\varphi : X \to [0, +\infty)$ be a function such that there exists an $L < 1$ with $\varphi(2^j x) \le 2^{jr} L\varphi(x)$ for all $x, y \in X$, where $j = \pm 1$, $r \in \mathbb{N}$. If $f : X \to Y$ is a mapping satisfying*

$$\|f(2x) - 2^r f(x)\|_Y \le \varphi(x) \tag{37}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(2x) = 2^r F(x)$ and

$$\|f(x) - F(x)\|_Y \le \frac{1}{2^r|1 - L^j|}\varphi(x) \tag{38}$$

for all $x \in X$.

Theorem 4.2. *Let $\varphi : X\times X \to [0, +\infty)$ be a function such that there exists an $L < 1$ with $\varphi(2^j x, 2^j y) \le 2^{7j} L\varphi(x, y)$ for all $x, y \in X$, where $j = \pm 1$. If $f : X \to Y$ is a mapping satisfying*

$$\|D_s f(x, y)\|_Y \le \varphi(x, y) \tag{39}$$

for all $x, y \in X$, then there exists a unique septic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{2^7|1 - L^j|}\widetilde{\varphi}(x) \tag{40}$$

for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}(x) = &\frac{1}{2520}[\frac{1}{10080}(\varphi(0, 6x) + \varphi(6x, -6x)) + \frac{1}{1440}(\varphi(0, 4x) + \varphi(4x, -4x))\\ &+ \frac{1}{180}(\varphi(0, 3x) + \varphi(3x, -3x)) + \frac{13}{288}(\varphi(0, 2x) + \varphi(2x, -2x))\\ &+ \frac{373}{2520}(\varphi(0, x) + \varphi(x, -x)) + \frac{1}{2}\varphi(4x, x) + \frac{7}{2}\varphi(3x, x)\\ &+ 11\varphi(2x, x) + 21\varphi(x, x) + \frac{1}{2}\varphi(0, 2x) + 28\varphi(0, x) + \frac{217}{720}\varphi(0, 0)]\end{aligned} \tag{41}$$

for all $x \in X$.

Proof. Putting $x = y = 0$ in (39), we get

$$\|f(0)\|_Y \le \frac{1}{5040}\varphi(0, 0). \tag{42}$$

Replacing (x, y) by $(0, x)$ in (39), we infer that

$$\|f(4x)-7f(3x) + 21f(2x) - 5075f(x) + 35f(0) - 21f(-x) + 7f(-2x) - f(-3x)\|_Y \le \varphi(0, x) \tag{43}$$

for all $x \in X$. Replacing (x, y) by $(x, -x)$ in (39), we have

$$\|f(-3x)-7f(-2x) - 5019f(-x) - 35f(0) + 35f(x) - 21f(2x) + 7f(3x) - f(4x)\|_Y \le \varphi(x, -x) \tag{44}$$

for all $x \in X$. Therefore, it follows from (43) and (44) that

$$\|f(x) + f(-x)\|_Y \le \frac{1}{5040}(\varphi(0, x) + \varphi(x, -x)) \tag{45}$$

for all $x \in X$. Replacing (x, y) by $(4x, x)$ in (39), we obtain

$$\|f(8x)-7f(7x) + 21f(6x) - 35f(5x) + 35f(4x) - 21f(3x) + 7f(2x) - 5041f(x)\|_Y \le \varphi(4x, x) \tag{46}$$

for all $x \in X$. Replacing (x, y) by $(0, 2x)$ in (39), we get

$$\|f(8x)-7f(6x) + 21f(4x) - 5075f(2x) + 35f(0) - 21f(-2x) + 7f(-4x) - f(-6x)\|_Y \le \varphi(0, 2x) \tag{47}$$

for all $x \in X$. By (46) and (47), we can obtain

$$\begin{aligned}&\|7f(7x) - 28f(6x) + 35f(5x) - 14f(4x) + 21f(3x) - 5082f(2x) + 5041f(x)\\ &+ 35f(0) - 21f(-2x) + 7f(-4x) - f(-6x)\|_Y \le \varphi(4x, x) + \varphi(0, 2x)\end{aligned} \tag{48}$$

for all $x \in X$. Using (42), (45) and (48), we can infer that

$$\begin{aligned}
&\|7f(7x) - 27f(6x) + 35f(5x) - 21f(4x) + 21f(3x) - 5061f(2x) + 5041f(x)\|_Y \\
&\leq \frac{1}{5040}(\varphi(0,6x) + \varphi(6x,-6x)) + \frac{1}{720}(\varphi(0,4x) + \varphi(4x,-4x)) \\
&+ \frac{1}{240}(\varphi(0,2x) + \varphi(2x,-2x)) + \varphi(4x,x) + \varphi(0,2x) + \frac{1}{144}\varphi(0,0)
\end{aligned} \tag{49}$$

for all $x \in X$. Replacing (x, y) by $(3x, x)$ in (39), we have

$$\|f(7x)-7f(6x) + 21f(5x) - 35f(4x) + 35f(3x) - 21f(2x) - 5033f(x) - f(0)\|_Y \leq \varphi(3x, x) \tag{50}$$

for all $x \in X$. By (42), we get

$$\|f(7x)-7f(6x) + 21f(5x) - 35f(4x) + 35f(3x) - 21f(2x) - 5033f(x)\|_Y \leq \varphi(3x, x) + \frac{1}{5040}\varphi(0,0) \tag{51}$$

for all $x \in X$. By (49) and (51), we can infer that

$$\begin{aligned}
&\|11f(6x) - 56f(5x) + 112f(4x) - 112f(3x) - 2457f(2x) + 20136f(x)\|_Y \\
&\leq \frac{1}{10080}(\varphi(0,6x) + \varphi(6x,-6x)) + \frac{1}{1440}(\varphi(0,4x) + \varphi(4x,-4x)) + \frac{1}{480}(\varphi(0,2x) \\
&+ \varphi(2x,-2x)) + \frac{1}{2}\varphi(4x,x) + \frac{7}{2}\varphi(3x,x) + \frac{1}{2}\varphi(0,2x) + \frac{1}{240}\varphi(0,0)
\end{aligned} \tag{52}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (39), we get

$$\|f(6x)-7f(5x) + 21f(4x) - 35f(3x) + 35f(2x) - 5061f(x) - f(-x) + 7f(0)\|_Y \leq \varphi(2x, x) \tag{53}$$

for all $x \in X$. By (42), (45) and (53), we can obtain that

$$\begin{aligned}
&\|f(6x) - 7f(5x) + 21f(4x) - 35f(3x) + 35f(2x) - 5060f(x)\|_Y \\
&\leq \varphi(2x,x) + \frac{1}{5040}(\varphi(0,x) + \varphi(x,-x)) + \frac{1}{720}\varphi(0,0)
\end{aligned} \tag{54}$$

for all $x \in X$. Therefore, it follows from (52) and (54) that

$$\begin{aligned}
&\|21f(5x) - 119f(4x) + 273f(3x) - 2842f(2x) + 75796f(x)\|_Y \\
&\leq \frac{1}{10080}(\varphi(0,6x) + \varphi(6x,-6x)) + \frac{1}{1440}(\varphi(0,4x) + \varphi(4x,-4x)) \\
&+ \frac{1}{480}(\varphi(0,2x) + \varphi(2x,-2x)) + \frac{11}{5040}(\varphi(0,x) + \varphi(x,-x)) \\
&+ \frac{1}{2}\varphi(4x,x) + \frac{7}{2}\varphi(3x,x) + 11\varphi(2x,x) + \frac{1}{2}\varphi(0,2x) + \frac{7}{360}\varphi(0,0)
\end{aligned} \tag{55}$$

for all $x \in X$. Replacing (x, y) by (x, x) in (39), we obtain

$$\|f(5x)-7f(4x) + 21f(3x) - 35f(2x) - f(-2x) - 5005f(x) + 7f(-x) - 21f(0)\|_Y \leq \varphi(x, x) \tag{56}$$

for all $x \in X$. By (42), (45) and (56), we can infer that

$$\begin{aligned}
&\|f(5x) - 7f(4x) + 21f(3x) - 34f(2x) - 5012f(x)\|_Y \\
&\leq \varphi(x,x) + \frac{1}{5040}(\varphi(0,2x) + \varphi(2x,-2x)) + \frac{1}{720}(\varphi(0,x) + \varphi(x,-x)) + \frac{1}{240}\varphi(0,0)
\end{aligned} \tag{57}$$

for all $x \in X$. Then, it follows from (55) and (57) that

$$\begin{aligned}
&\|28f(4x) - 168f(3x) - 2128f(2x) + 181048f(x)\|_Y \\
&\leq \frac{1}{10080}(\varphi(0,6x) + \varphi(6x,-6x)) + \frac{1}{1440}(\varphi(0,4x) + \varphi(4x,-4x)) \\
&+ \frac{1}{160}(\varphi(0,2x) + \varphi(2x,-2x)) + \frac{79}{2520}(\varphi(0,x) + \varphi(x,-x)) + \frac{1}{2}\varphi(4x,x) \\
&+ \frac{7}{2}\varphi(3x,x) + 11\varphi(2x,x) + 21\varphi(x,x) + \frac{1}{2}\varphi(0,2x) + \frac{77}{720}\varphi(0,0)
\end{aligned} \tag{58}$$

for all $x \in X$. By (42), (43) and (45), we get

$$\begin{aligned}
&\|f(4x) - 6f(3x) + 14f(2x) - 5054f(x)\|_Y \\
&\leq \frac{1}{5040}(\varphi(0,3x) + \varphi(3x,-3x)) + \frac{1}{720}(\varphi(0,2x) + \varphi(2x,-2x)) \\
&+ \frac{1}{240}(\varphi(0,x) + \varphi(x,-x)) + \varphi(0,x) + \frac{1}{144}\varphi(0,0)
\end{aligned} \tag{59}$$

for all $x \in X$. By (58) and (59), we conclude that

$$\begin{aligned}
&\|f(2x) - 2^7 f(x)\|_Y \\
&\leq \frac{1}{2520}[\frac{1}{10080}(\varphi(0,6x) + \varphi(6x,-6x)) + \frac{1}{1440}(\varphi(0,4x) + \varphi(4x,-4x)) \\
&+ \frac{1}{180}(\varphi(0,3x) + \varphi(3x,-3x)) + \frac{13}{288}(\varphi(0,2x) + \varphi(2x,-2x)) \\
&+ \frac{373}{2520}(\varphi(0,x) + \varphi(x,-x)) + \frac{1}{2}\varphi(4x,x) + \frac{7}{2}\varphi(3x,x) \\
&+ 11\varphi(2x,x) + 21\varphi(x,x) + \frac{1}{2}\varphi(0,2x) + 28\varphi(0,x) + \frac{217}{720}\varphi(0,0)] \\
&= \widetilde{\varphi}(x)
\end{aligned} \tag{60}$$

for all $x \in X$. According to Theorem 4.1, there exists a unique mapping $H : X \to Y$ such that $H(2x) = 2^7 H(x)$ and

$$\|f(x) - H(x)\|_Y \leq \frac{1}{2^7|1 - L^j|}\widetilde{\varphi}(x) \tag{61}$$

for all $x \in X$. Moreover, it remains to show that H is a septic mapping. By (39), we can obtain

$$\|2^{-7jn}D_s f(2^{jn}x, 2^{jn}y)\|_Y \leq 2^{-7jn}\varphi(2^{jn}x, 2^{jn}y) \leq 2^{-7jn}(2^{7j}L)^n\varphi(x,y) = L^n\varphi(x,y) \tag{62}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. From (61) and $H(2x) = 2^7H(x)$ and also from the properties of φ, we infer that $\lim_{n\to\infty} 2^{-7jn} f(2^{jn}x)$ exists and is equal $H(x)$. After that we can count that

$$D_s H(x,y) = 0, \tag{63}$$

which means that $H\colon X \to Y$ is septic. This completes the proof. □

Corollary 4.3. *Let $(X, \|\cdot\|_X)$ be a real normed space and $(Y, \|\cdot\|_Y)$ a real Banach space. Let ϵ, p, q be positive real numbers with $p + q \neq 7$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\|D_s f(x,y)\|_Y \leq \epsilon\|x\|_X^p\|y\|_X^q \tag{64}$$

for all $x, y \in X$. Then there exists a unique septic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{\epsilon\omega_{p,q}}{|2^{p+q} - 2^7|}\|x\|_X^{p+q}, \tag{65}$$

for all $x \in X$, where

$$\begin{aligned}
\omega_{p,q} =&\frac{1}{2520 \times 10080}(213172 + 3465 \times 2^{5+p} + 455 \times 2^{p+q} + 3920 \times 3^{2+p} \\
&+ 56 \times 3^{p+q} + 315 \times 4^{2+p} + 7 \times 4^{p+q} + 6^{p+q}).
\end{aligned} \tag{66}$$

Corollary 4.4. *Let $(X, \|\cdot\|_X)$ be a real normed space and $(Y, \|\cdot\|_Y)$ a real Banach space. Let ϵ, δ, p be positive real numbers with $p \neq 7$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\|D_s f(x,y)\|_Y \leq \epsilon\|x\|_X^p + \delta\|y\|_X^p \tag{67}$$

for all $x, y \in X$. Then there exists a unique septic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{\omega_{p,\epsilon,\delta}}{|2^p - 2^7|}\|x\|_X^p, \tag{68}$$

for all $x \in X$*, where*

$$\begin{aligned}\omega_{p,\epsilon,\delta} =&\frac{1}{2520 \times 10080}[(213172 + 111335 \times 2^p + 35336 \times 3^p + 5047 \times 4^p \\ &+ 6^p)\epsilon + 2(324052 + 2975 \times 2^p + 56 \times 3^p + 7 \times 4^p + 6^p)\delta].\end{aligned} \tag{69}$$

Corollary 4.5. *Let* $(X, \|\cdot\|_X)$ *be a real normed space and* $(Y, \|\cdot\|_Y)$ *a real Banach space. Let* ϵ, p, q *be positive real numbers with* $p + q \neq 7$*. Suppose that* $f : X \to Y$ *is a mapping fulfilling*

$$\|D_s f(x, y)\|_Y \leq \epsilon(\|x\|_X^p \|y\|_X^q + (\|x\|_X^{p+q} + \|y\|_X^{p+q})) \tag{70}$$

for all $x, y \in X$*. Then there exists a unique septic mapping* $H : X \to Y$ *such that*

$$\|f(x) - H(x)\|_Y \leq \frac{\epsilon\omega_{p,q} + \omega_{p+q,\epsilon,\epsilon}}{|2^{p+q} - 2^7|}\|x\|_X^{p+q}, \tag{71}$$

for all $x \in X$*, where* $\omega_{p,q}$ *and* $\omega_{p+q,\epsilon,\epsilon}$ *are defined as in Corollaries 4.3 and 4.4.*

5 Stability of the octic functional equation (7)

As stated in the previous section, in this section, we further investigate the stability of the octic functional equation in real normed spaces. Similarly, we define the difference operator

$$\begin{aligned}D_o f(x, y) =&f(x + 4y) - 8f(x + 3y) + 28f(x + 2y) - 56f(x + y) - 56f(x - y) \\ &+ 28f(x - 2y) - 8f(x - 3y) + f(x - 4y) + 70f(x) - 40320f(y).\end{aligned} \tag{72}$$

Theorem 5.1. *Let* $\varphi : X\times X \to [0, +\infty)$ *be a function such that there exists an* $L < 1$ *with* $\varphi(2^j x, 2^j y) \leq 2^{8j} L\varphi(x, y)$ *for all* $x, y \in X$*, where* $j = \pm 1$*. If* $f : X \to Y$ *is a mapping satisfying*

$$\|D_o f(x, y)\|_Y \leq \varphi(x, y) \tag{73}$$

for all $x, y \in X$*, then there exists a unique octic mapping* $G : X \to Y$ *such that*

$$\|f(x) - G(x)\|_Y \leq \frac{1}{2^8|1 - L^j|}\widetilde{\varphi}(x) \tag{74}$$

for all $x \in X$*, where*

$$\begin{aligned}\widetilde{\varphi}(x) =&\frac{1}{288 \times 70}[\frac{1}{80640}(\varphi(8x, 8x) + \varphi(8x, -8x)) \\ &+ \frac{1}{10080}(\varphi(6x, 6x) + \varphi(6x, -6x)) + \frac{7}{960}(\varphi(4x, 4x) + \varphi(4x, -4x)) \\ &+ \frac{1}{15}(\varphi(3x, 3x) + \varphi(3x, -3x)) + \frac{139}{480}(\varphi(2x, 2x) + \varphi(2x, -2x)) \\ &+ \frac{417}{560}(\varphi(x, x) + \varphi(x, -x)) + \varphi(4x, x) + 8\varphi(3x, x) + 28\varphi(2x, x) \\ &+ 56\varphi(x, x) + 35\varphi(0, x) + \frac{851}{672}\varphi(0, 0)]\end{aligned} \tag{75}$$

for all $x \in X$*.*

Proof. Putting $x = y = 0$ in (73), we get

$$\|f(0)\|_Y \leq \frac{1}{40320}\varphi(0, 0). \tag{76}$$

Replacing (x, y) by $(x, -y)$ in (73), we obtain

$$\begin{aligned}&\|f(x + 4y) - 8f(x + 3y) + 28f(x + 2y) - 56f(x + y) - 56f(x - y) \\ &+ 28f(x - 2y) - 8f(x - 3y) + f(x - 4y) + 70f(x) - 40320f(-y)\|_Y \\ &\leq \varphi(x, -y)\end{aligned} \tag{77}$$

for all $x \in X$. By (73) and (77), we can infer that

$$\|f(x) - f(-x)\|_Y \leq \frac{1}{40320}(\varphi(x,x) + \varphi(x,-x)) \tag{78}$$

for all $x \in X$. Replacing (x, y) by $(0, 2x)$ in (73), we have

$$\begin{aligned}&\|f(8x) - 8f(6x) + 28f(4x) - 40376f(2x) + 70f(0) - 56f(-2x)\\&+ 28f(-4x) - 8f(-6x) + f(-8x)\|_Y \leq \varphi(0, 2x)\end{aligned} \tag{79}$$

for all $x \in X$. Then, it follows from (76), (78) and (79) that

$$\begin{aligned}&\|f(8x) - 8f(6x) + 28f(4x) - 20216f(2x)\|_Y\\&\leq \frac{1}{80640}(\varphi(8x,8x) + \varphi(8x,-8x)) + \frac{1}{10080}(\varphi(6x,6x) + \varphi(6x,-6x))\\&+ \frac{1}{2880}(\varphi(4x,4x) + \varphi(4x,-4x)) + \frac{1}{1440}(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{1}{1152}\varphi(0,0)\end{aligned} \tag{80}$$

for all $x \in X$. Replacing (x, y) by $(4x, x)$ in (73), we get

$$\|f(8x) - 8f(7x) + 28f(6x) - 56f(5x) + 70f(4x) - 56f(3x) + 28f(2x) - 40328f(x) + f(0)\|_Y \leq \varphi(4x, x) \tag{81}$$

for all $x \in X$. Further, we can infer from (76), (80) and (81) that

$$\begin{aligned}&\|8f(7x) - 36f(6x) + 56f(5x) - 42f(4x) + 56f(3x) - 20244f(2x) + 40328f(x))\|_Y\\&\leq \frac{1}{80640}(\varphi(8x,8x) + \varphi(8x,-8x)) + \frac{1}{10080}(\varphi(6x,6x) + \varphi(6x,-6x))\\&+ \frac{1}{2880}(\varphi(4x,4x) + \varphi(4x,-4x)) + \frac{1}{1440}(\varphi(2x,2x) + \varphi(2x,-2x))\\&+ \varphi(4x,x) + \frac{1}{1120}\varphi(0,0)\end{aligned} \tag{82}$$

for all $x \in X$. Replacing (x, y) by $(3x, x)$ in (73), we can obtain

$$\|f(7x) - 8f(6x) + 28f(5x) - 56f(4x) + 70f(3x) - 56f(2x) - 40292f(x) + f(-x) - 8f(0)\|_Y \leq \varphi(3x, x) \tag{83}$$

for all $x \in X$. By (76) and (78), we get

$$\begin{aligned}&\|f(7x) - 8f(6x) + 28f(5x) - 56f(4x) + 70f(3x) - 56f(2x) - 40291f(x)\|_Y\\&\leq \frac{1}{40320}(\varphi(x,x) + \varphi(x,-x)) + \varphi(3x,x) + \frac{1}{5040}\varphi(0,0)\end{aligned} \tag{84}$$

for all $x \in X$. Then, it follows from (82) and (84) that

$$\begin{aligned}&\|2f(6x) - 12f(5x) + 29f(4x) - 36f(3x) - 1414f(2x) + 25904f(x)\|_Y\\&\leq \frac{1}{14}[\frac{1}{80640}(\varphi(8x,8x) + \varphi(8x,-8x)) + \frac{1}{10080}(\varphi(6x,6x) + \varphi(6x,-6x))\\&+ \frac{1}{2880}(\varphi(4x,4x) + \varphi(4x,-4x)) + \frac{1}{1440}(\varphi(2x,2x) + \varphi(2x,-2x))\\&+ \frac{1}{5040}(\varphi(x,x) + \varphi(x,-x)) + \varphi(4x,x) + 8\varphi(3x,x) + \frac{5}{2016}\varphi(0,0)]\end{aligned} \tag{85}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (73), we have

$$\|f(6x) - 8f(5x) + 28f(4x) - 56f(3x) + 70f(2x) + f(-2x) - 40376f(x) - 8f(-x) + 28f(0)\|_Y \leq \varphi(2x, x) \tag{86}$$

for all $x \in X$. Using (76), (78) and (86), we obtain

$$\begin{aligned}&\|f(6x) - 8f(5x) + 28f(4x) - 56f(3x) + 71f(2x) - 40384f(x)\|_Y\\&\leq \frac{1}{5040}(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{1}{630}(\varphi(x,x) + \varphi(x,-x))\\&+ \varphi(2x,x) + \frac{1}{180}\varphi(0,0)\end{aligned} \tag{87}$$

for all $x \in X$. Then, we can infer from (85) and (87) that

$$\begin{aligned} &\|4f(5x) - 27f(4x) + 76f(3x) - 1556f(2x) + 106672f(x)\|_Y \\ &\le \frac{1}{14}[\frac{1}{80640}(\varphi(8x, 8x) + \varphi(8x, -8x)) + \frac{1}{10080}(\varphi(6x, 6x) + \varphi(6x, -6x)) \\ &+ \frac{1}{2880}(\varphi(4x, 4x) + \varphi(4x, -4x)) + \frac{1}{160}(\varphi(2x, 2x) + \varphi(2x, -2x)) \\ &+ \frac{5}{112}(\varphi(x, x) + \varphi(x, -x)) + \varphi(4x, x) + 8\varphi(3x, x) + 28\varphi(2x, x) + \frac{177}{1120}\varphi(0, 0)] \end{aligned} \tag{88}$$

for all $x \in X$. Replacing (x, y) by (x, x) in (73), we get

$$\begin{aligned} &\|f(5x) - 8f(4x) + 28f(3x) + f(-3x) - 56f(2x) - 8f(-2x) \\ &- 40250f(x) + 28f(-x) - 56f(0)\|_Y \le \varphi(x, x) \end{aligned} \tag{89}$$

for all $x \in X$. By (76), (78) and (89), we have

$$\begin{aligned} &\|f(5x) - 8f(4x) + 29f(3x) - 64f(2x) - 40222f(x)\|_Y \\ &\le \frac{1}{5040}(\varphi(3x, 3x) + \varphi(3x, -3x)) + \frac{1}{630}(\varphi(2x, 2x) + \varphi(2x, -2x)) \\ &+ \frac{1}{180}(\varphi(x, x) + \varphi(x, -x)) + \varphi(x, x) + \frac{1}{90}\varphi(0, 0) \end{aligned} \tag{90}$$

for all $x \in X$. Therefore, it follows from (88) and (90) that

$$\begin{aligned} &\|f(4x) - 8f(3x) - 260f(2x) + 53512f(x)\|_Y \\ &\le \frac{1}{70}[\frac{1}{80640}(\varphi(8x, 8x) + \varphi(8x, -8x)) + \frac{1}{10080}(\varphi(6x, 6x) + \varphi(6x, -6x)) \\ &+ \frac{1}{2880}(\varphi(4x, 4x) + \varphi(4x, -4x)) + \frac{1}{90}(\varphi(3x, 3x) + \varphi(3x, -3x)) \\ &+ \frac{137}{1440}(\varphi(2x, 2x) + \varphi(2x, -2x)) + \frac{1793}{5040}(\varphi(x, x) + \varphi(x, -x)) \\ &+ \varphi(4x, x) + 8\varphi(3x, x) + 28\varphi(2x, x) + 56\varphi(x, x) + \frac{1573}{2016}\varphi(0, 0)] \end{aligned} \tag{91}$$

for all $x \in X$. Replacing (x, y) by $(0, x)$ in (73), we obtain

$$\begin{aligned} &\|f(4x) + f(-4x) - 8f(3x) - 8f(-3x) + 28f(2x) + 28f(-2x) \\ &- 40376f(x) - 56f(-x) + 70f(0)\|_Y \le \varphi(0, x) \end{aligned} \tag{92}$$

for all $x \in X$. Using (76), (78) and (92), we get

$$\begin{aligned} &\|f(4x) - 8f(3x) + 28f(2x) - 20216f(x)\|_Y \\ &\le \frac{1}{2}[\frac{1}{5040}(\varphi(4x, 4x) + \varphi(4x, -4x)) + \frac{1}{630}(\varphi(3x, 3x) + \varphi(3x, -3x)) \\ &+ \frac{1}{180}(\varphi(2x, 2x) + \varphi(2x, -2x)) + \frac{1}{90}(\varphi(x, x) + \varphi(x, -x)) + \varphi(0, x) + \frac{1}{72}\varphi(0, 0)] \end{aligned} \tag{93}$$

for all $x \in X$. By (91) and (93), we conclude that

$$\begin{aligned} &\|f(2x) - 2^8 f(x)\|_Y \\ &\le \frac{1}{288 \times 70}[\frac{1}{80640}(\varphi(8x, 8x) + \varphi(8x, -8x)) + \frac{1}{10080}(\varphi(6x, 6x) + \varphi(6x, -6x)) \\ &+ \frac{7}{960}(\varphi(4x, 4x) + \varphi(4x, -4x)) + \frac{1}{15}(\varphi(3x, 3x) + \varphi(3x, -3x)) \\ &+ \frac{139}{480}(\varphi(2x, 2x) + \varphi(2x, -2x)) + \frac{417}{560}(\varphi(x, x) + \varphi(x, -x)) + \varphi(4x, x) \\ &+ 8\varphi(3x, x) + 28\varphi(2x, x) + 56\varphi(x, x) + 35\varphi(0, x) + \frac{851}{672}\varphi(0, 0)] \\ &= \widetilde{\varphi}(x) \end{aligned} \tag{94}$$

for all $x \in X$. According to Theorem 4.1, there exists a unique mapping $G : X \to Y$ such that $G(2x) = 2^8 G(x)$ and

$$\|f(x) - G(x)\|_Y \leq \frac{1}{2^8|1 - L^j|}\widetilde{\varphi}(x) \tag{95}$$

for all $x \in X$. In addition, it suffices to show that H is an octic mapping. By (73), it follows that

$$\|2^{-8jn} D_o f(2^{jn}x, 2^{jn}y)\|_y \leq 2^{-8jn}\varphi(2^{jn}x, 2^{jn}y) \leq 2^{-8jn}(2^{8j}L)^n\varphi(x, y) = L^n\varphi(x, y) \tag{96}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. From (95) and $G(2x) = 2^8 G(x)$ and also from the properties of φ, we infer that $\lim_{n\to\infty} 2^{-8jn} f(2^{jn}x)$ exists and is equal $G(x)$. After that we can count that

$$D_o G(x, y) = 0, \tag{97}$$

which means that $G\colon X \to Y$ is octic. This completes the proof. □

Corollary 5.2. *Let $(X, \|\cdot\|_X)$ be a real normed space and $(Y, \|\cdot\|_Y)$ a real Banach space. Let ϵ, p, q be positive real numbers with $p + q \neq 8$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\|D_o f(x, y)\|_Y \leq \epsilon\|x\|_X^p\|y\|_X^q \tag{98}$$

for all $x, y \in X$. Then there exists a unique octic mapping $G : X \to Y$ such that

$$\|f(x) - G(x)\|_Y \leq \frac{\epsilon\omega_{p,q}}{|2^{p+q} - 2^8|}\|x\|_X^{p+q}, \tag{99}$$

for all $x \in X$, where

$$\begin{aligned}\omega_{p,q} =&\frac{1}{288\times 70}\Big(\frac{16097}{280} + 7\times 2^{p+2} + \frac{139}{15}\times 2^{p+q-4} + \frac{7}{15}\times 2^{2p+2q-5}\\ &\frac{1}{315}2^{3p+3q-7} + 8\times 3^p + \frac{1}{35}2^{p+q-4}3^{p+q-2} + \frac{2}{5}\times 3^{p+q-1} + 4^p\Big).\end{aligned} \tag{100}$$

Corollary 5.3. *Let $(X, \|\cdot\|_X)$ be a real normed space and $(Y, \|\cdot\|_Y)$ a real Banach space. Let ϵ, δ, p be positive real numbers with $p \neq 8$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\|D_o f(x, y)\|_Y \leq \epsilon\|x\|_X^p + \delta\|y\|_X^p \tag{101}$$

for all $x, y \in X$. Then there exists a unique octic mapping $G : X \to Y$ such that

$$\|f(x) - G(x)\|_Y \leq \frac{\omega_{p,\epsilon,\delta}}{|2^p - 2^8|}\|x\|_X^p, \tag{102}$$

for all $x \in X$, where

$$\begin{aligned}\omega_{p,\epsilon,\delta} =&\frac{1}{288\times 70\times 40320}[(2317968 + 144039\times 2^{p+3} + 2^{p+3}3^p + 109312\times 3^{p+1}\\ &+ 10227\times 4^{p+1} + 8^p)\epsilon + (5221008 + 2919\times 2^{p+3} + 2^{p+3}3^p\\ &+ 1792\times 3^{p+1} + 10227\times 4^{p+1} + 8^p)\delta].\end{aligned} \tag{103}$$

Corollary 5.4. *Let $(X, \|\cdot\|_X)$ be a real normed space and $(Y, \|\cdot\|_Y)$ a real Banach space. Let ϵ, p, q be positive real numbers with $p + q \neq 8$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\|D_o f(x, y)\|_Y \leq \epsilon(\|x\|_X^p\|y\|_X^q + (\|x\|_X^{p+q} + \|y\|_X^{p+q})) \tag{104}$$

for all $x, y \in X$. Then there exists a unique octic mapping $G : X \to Y$ such that

$$\|f(x) - G(x)\|_Y \leq \frac{\epsilon\omega_{p,q} + \omega_{p+q,\epsilon,\epsilon}}{|2^{p+q} - 2^8|}\|x\|_X^{p+q}, \tag{105}$$

for all $x \in X$, where $\omega_{p,q}$ and $\omega_{p+q,\epsilon,\epsilon}$ are defined as in Corollaries 5.2 and 5.3.

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Approximation properties of a kind of q-Beta operators

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Abstract. In this paper, we introduce a new kind of q-Beta operators based on the concept of q-integers. We estimate the moments of these operators and investigate the weighted statistical approximation properties. We also establish a local approximation theorem and obtain the estimation of the convergence rate of the operators.

Keywords: q-integers, q-Beta operators, weighted statistical approximation, rate of convergence.

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1 Introduction

In recent years, the applications of q-calculus in the approximation theory became a hot reseach topic. Since the q-Bernstein polynomials were introduced by Phillips [13] in 1997, many researches have been presented on this topic [1, 2, 4, 5, 12, 13].

In 1991, Mazhar [11] defined and studied some approximation properties of the following sequence of linear positive operators:

$$L_n(f;x) = \frac{(2n)!x^{n+1}}{n!(n-1)!}\int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)dt, \quad n>1, \quad x>0. \tag{1}$$

The sequence of operators (1) is a kind of Beta type operators. Though the application of q-calculus in approximation theory is an active topic, it seems there are few papers mentioned about the q analogue of these operators defined in (1). Inspired by Aral and Gupta [1], who defined a generalization of q-Baskakov type operators using q-Beta integral and obtained some important approximation properties, we propose the q analogue of this kind of Beta type operators.

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Before introducing the operators, we mention certain definitions based on q-integers, details can be found in [8, 9]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ...; \\ 1, & k = 0, \end{cases}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \geq k \geq 0).$$

The q-improper integrals are defined as (see [10])

$$\int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0,$$

provided the sums converge absolutely.
The q-Beta integral is defined by

$$B_q(t;s) = K(A;t)\int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_qx, \tag{2}$$

where $K(x;t)=\frac{1}{x+1}x^t\left(1+\frac{1}{x}\right)_q^t(1+x)_q^{1-t}$ and $(a+b)_q^\tau = \prod_{j=0}^{\tau-1}(a+q^jb), \quad \tau > 0.$
In particular for any positive integer m, n

$$K(x,n) = q^{\frac{n(n-1)}{2}}, \quad K(x,0) = 1 \quad and \quad B_q(m;n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}, \tag{3}$$

where $\Gamma_q(t)$ is the q-Gamma function satisfing the following functional equations:

$$\Gamma_q(t+1) = [t]_q\Gamma_q(t), \qquad \Gamma_q(1) = 1.$$

(see [4]).

For $f \in C[0,\infty)$, $q \in (0,1)$ and $n > 1,\ n \in \mathbb{N}$, we introduce a kind of q-Beta operators $L_{n,q}(f;x)$ as

$$L_{n,q}(f;x) = \frac{[2n]_q!\,(q^nx)^{n+1}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\int_0^{\infty/A} \frac{t^{n-1}}{(q^nx+t)_q^{2n+1}} f(t)d_qt. \tag{4}$$

Note that for $q \to 1^-$, $L_{n,1^-}(f;x)$ become the operators defined in (1).

2 Some preliminary results

In order to obtain the approximation properties of the operators $L_{n,q}$, we need the following lemmas:

Lemma 2.1. *For any $k \in \mathbb{N}$, $k \leq n$ and $q \in (0,1)$, we have*

$$L_{n,q}(t^k;x) = \frac{[n+k-1]_q![n-k]_q!}{q^{\frac{k(k-1)}{2}}[n]_q![n-1]_q!}x^k. \tag{5}$$

Proof. Using the properties of q-Beta integral, we have

$$\begin{aligned}
L_{n,q}(t^k;x) &= \frac{[2n]_q!\,(q^n x)^{n+1}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\int_0^{\infty/A}\frac{t^{n+k-1}}{(q^n x+t)_q^{2n+1}}d_q t \\
&= \frac{[2n]_q!\,(q^n x)^{k}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\int_0^{\infty/A}\frac{\left(\frac{t}{q^n x}\right)^{n+k-1}}{\left(1+\frac{t}{q^n x}\right)_q^{2n+1}}d_q\left(\frac{t}{q^n x}\right) \\
&= \frac{[2n]_q!\,(q^n x)^{k}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\frac{B_q(n+k;n-k+1)}{K(A;n+k)} \\
&= \frac{[2n]_q!\,(q^n x)^{k}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\frac{[n+k-1]_q![n-k]_q!}{[2n]_q!q^{\frac{(n+k)(n+k-1)}{2}}} \\
&= \frac{[n+k-1]_q![n-k]_q!}{q^{\frac{k(k-1)}{2}}[n]_q![n-1]_q!}x^k.
\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2. *The following equalities hold:*

$$L_{n,q}(1;x) = 1, \quad L_{n,q}(t;x) = x, \quad L_{n,q}(t^2;x) = \frac{[n+1]_q}{q[n-1]_q}x^2, \tag{6}$$

$$L_{n,q}\left((t-x)^2;x\right) = \frac{1+q^n}{q[n-1]_q}x^2, \tag{7}$$

Proof. From Lemma 2.1, take $k = 0,1,2$, we get (6). Since $L_{n,q}\left((t-x)^2;x\right) = L_{n,q}(t^2;x) - 2xL_{n,q}(t;x) + x^2$, using (6), we obtain (7) easily. □

Remark 2.3. *Let $n \in \mathbb{N}$ and $x \in [0,\infty)$, then for every $q \in (0,1)$, by Lemma 2.2, we have*

$$L_{n,q}(t-x;x) = 0. \tag{8}$$

3 Weighted statistical approximation properties

In this section, we present the statistical approximation properties of the operators $L_{n,q}$ by using a Korovkin-type theorem proved in [6].

Let $\mathbb{N}$ be the set of all natural numbers and let K be a subset of $\mathbb{N}$. The density of K is defined by $\delta(K) := \lim_n \frac{1}{n}\sum_{k=1}^n \chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := \{x_n\}$ is called statistically convergent to a number L, if for every $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$. Let $A := (a_{jn}), j, n = 1, 2, ...$ be

an infinite summability matrix. For a given sequence $x := \{x_n\}$, the $A-$transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{k=1}^{\infty} a_{jn}x_n$ provided the series convergence for each j. We say that A is regular if $\lim_n (Ax)_j = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = \{x_n\}$ is called A-statistically convergence to L provided that for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim_n x_n = L$ (see [7]). For $A = C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not vice versa.

A real function $\rho(x)$ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim\limits_{|x|\to\infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$. Let (see [12])

$B_\rho(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : |f(x)| \leq M_f \rho(x),\ M_f$ is a positive constant depending only on $f\}$,

$C_\rho(\mathbb{R}) := \{f \in B_\rho(\mathbb{R}) : f$ is continuous on $\mathbb{R}\}$.

Endowed with the norm $||\cdot||_\rho$, where $||f||_\rho := \sup \frac{|f(x)|}{\rho(x)}$, $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces.

Using $A-$statistical convergence, Duman and Orhan proved the following Korovkin-type theorem.

Theorem 3.1 (see [6]). *Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let L_n be a sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$, where ρ_1 and ρ_2 satisfy*

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0,$$

then

$$st_A - \lim_n ||L_n f - f||_{\rho_2} = 0 \quad for\ all\ f \in C_{\rho_1}(\mathbb{R})$$

if and only if

$$st_A - \lim_n ||L_n F_v - F_v||_{\rho_1} = 0 \quad for\ all\ v = 0, 1, 2,$$

where $F_v = \frac{x^v \rho_1(x)}{1+x^2}, v = 0, 1, 2.$

We consider the weight functions $\rho_1(x) = 1 + x^2$, $\rho_2(x) = 1 + x^{2+\alpha}$, $\alpha > 0$. Further more, we consider a sequence $q := \{q_n\}$ for $0 < q_n < 1$ satisfying

$$st - \lim_n q_n = 1, \tag{9}$$

for example, define the sequence $q = \{q_n\}$ by $q_n = \begin{cases} 1/2; & if n = m^2, (m = 1, 2, 3...) \\ e^{1/n}\left(1 - \frac{1}{n}\right); & if n \neq m^2. \end{cases}$

(see [5]). We can deduce that it satisfies the conditions (9) for statistically convergence but it does not work for ordinary case.

If $e_i = t^i$, $t \in \mathbb{R}^+$, $i = 0, 1, 2, ...$ stands for the ith monomial, then we have

Theorem 3.2. *Let $q := \{q_n\}$ be a sequence satisfying (9), then for all $f \in C_{\rho_1}(\mathbb{R}^+)$, we have*

$$st - \lim_n ||L_{n,q_n} f - f||_{\rho_2} = 0. \tag{10}$$

Proof. Obviously,

$$st - \lim_n ||L_{n,q_n}(e_0) - e_0||_{\rho_1} = 0, \quad st - \lim_n ||L_{n,q_n}(e_1) - e_1||_{\rho_1} = 0.$$

By Lemma 2.2, we have

$$\frac{|L_{n,q_n}(e_2; x) - e_2(x)|}{1 + x^2} = \frac{\frac{1+q_n^n}{q_n[n-1]_{q_n}} x^2}{1 + x^2} \leq \frac{1 + q_n^n}{q_n[n-1]_{q_n}}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \{k : ||L_{k,q_k}(e_2) - e_2||_{\rho_1} \geq \varepsilon\}, \quad U_1 := \left\{k : \frac{1 + q_k^k}{q_k[k-1]_{q_k}} \geq \varepsilon\right\}.$$

Then one can see that $U \subseteq U_1$, so we have

$$\delta\{k \leq n : ||L_{k,q_k}(e_2) - e_2||_{\rho_1} \geq \varepsilon\} \leq \delta \left\{k \leq n : \frac{1 + q_k^k}{q_k[k-1]_{q_k}} \geq \varepsilon\right\}.$$

Since $st - \lim_n q_n = 1$, we have $st - \lim_n \frac{1 + q_n^n}{q_n[n-1]_{q_n}} = 0$, which implies that the right-hand side of the above inequality is zero, we have

$$st - \lim_n ||L_{n,q_n}(e_2) - e_2||_{\rho_1} = 0.$$

Then the proof of Theorem 3.2 is obtained by Theorem 3.1 with $A = C_1$, where C_1 is a Cesàro matrix of order one. □

4 Local approximation

In this section we establish direct and local approximation theorems in connection with the operators $L_{n,q}(f; x)$.

We denote the space of all real valued continuous bounded functions f defined on the interval $[0, \infty)$ by $C_B[0, \infty)$. The norm $|| \cdot ||$ on the space $C_B[0, \infty)$ is given by $||f|| = \sup\{|f(x)| : x \in [0, \infty)\}$.

Further let us consider Peetre's $K-$functional:

$$K_2(f; \delta) = \inf_{g \in W^2} \left\{||f - g|| + \delta||g''||\right\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$.

For $f \in C_B[0, \infty)$, the modulus of continuity of second order is defined by

$$\omega_2(f; \delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)|.$$

From [3, p.177], there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \quad \delta > 0. \tag{11}$$

Our first result is a direct local approximation theorem for the operators $L_{n,q}(f;x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, we have*

$$|L_{n,q}(f;x) - f(x)| \le C\omega_2\left(f;\sqrt{\frac{1+q^n}{2q[n-1]_q}x}\right), \tag{12}$$

where C is a positive constant.

Proof. Let $g \in W^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad x,t \in [0,\infty).$$

Using (8), we get

$$L_{n,q}(g;x) = g(x) + L_{n,q}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Thus, we have

$$\begin{aligned}|L_{n,q}(g;x) - g(x)| &= \left|L_{n,q}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \le L_{n,q}\left(\left|\int_x^t (t-u)|g''(u)|du\right|;x\right)\\ &\le L_{n,q}\left((t-x)^2;x\right)||g''|| = \frac{1+q^n}{q[n-1]_q}x^2||g''||,\end{aligned} \tag{13}$$

On the other hand, using Lemma 2.2, we have

$$|L_{n,q}(f;x)| \le \frac{[2n]_q!\,(q^n x)^{n+1}\,q^{\frac{n(n-1)}{2}}}{[n]_q![n-1]_q!}\int_0^{\infty/A}\frac{t^{n-1}}{(q^n x+t)_q^{2n+1}}|f(t)|d_qt \le ||f||. \tag{14}$$

Now (13) and (14) imply

$$\begin{aligned}|L_{n,q}(f;x) - f(x)| &\le |L_{n,q}(f-g;x) - (f-g)(x)| + |L_{n,q}(g;x) - g(x)|\\ &\le 2||f-g|| + \frac{1+q^n}{q[n-1]_q}x^2||g''||.\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we get

$$|L_{n,q}(f;x) - f(x)| \le 2K_2\left(f;\frac{1+q^n}{2q[n-1]_q}x^2\right).$$

By (11), for every $q \in (0,1)$, we have

$$|L_{n,q}(f;x) - f(x)| \le C\omega_2\left(f;\sqrt{\frac{1+q^n}{2q[n-1]_q}x}\right).$$

This completes the proof of Theorem 4.1. □

Remark 4.2. *Let $q = \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$, we have $\lim_{n\to\infty}\frac{1+q_n^n}{2q_n[n-1]_{q_n}} = 0$, this gives us the point-wise rate of convergence of the operators $L_{n,q_n}(f;x)$ to $f(x)$.*

5 Rate of convergence

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where M_f is a constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $||f||_{x^2} = \sup\limits_{x\in[0,\infty)} \dfrac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of f on the closed interval $[0,a]$, $(a>0)$ by

$$\omega_a(f;\delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

Obviously, for function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f;\delta)$ tends to zero.

Theorem 5.1. *Let $f \in C_{x^2}[0,\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f;\delta)$ be the modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then we have*

$$||L_{n,q}(f)-f||_{C[0,a]} \leq \frac{12M_f a^2(1+a^2)}{q[n-1]_q} + 2\omega_{a+1}\left(f;\frac{\sqrt{2}a}{\sqrt{q[n-1]_q}}\right). \tag{15}$$

Proof. For $x \in [0,a]$ and $t > a+1$, we have

$$|f(t)-f(x)| \leq M_f(2+x^2+t^2) \leq M_f[2+3x^2+2(t-x)^2],$$

hence, we obtain

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2. \tag{16}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f;|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta), \quad \delta>0. \tag{17}$$

From (16) and (17), we get

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{18}$$

For $x \in [0,a]$ and $t \geq 0$, by Schwarz's inequality and Lemma 2.2, we have

$$\begin{aligned}
& |L_{n,q}(f;x)-f(x)| \\
\leq\ & L_{n,q}(|f(t)-f(x)|;x) \\
\leq\ & 6M_f(1+a^2)L_{n,q}((t-x)^2;x) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{L_{n,q}((t-x)^2;x)}\right) \\
\leq\ & \frac{12M_f a^2(1+a^2)}{q[n-1]_q} + \omega_{a+1}(f;\delta)\left(1+\frac{\sqrt{2}a}{\delta\sqrt{q[n-1]_q}}\right),
\end{aligned}$$

by taking $\delta = \frac{\sqrt{2}a}{\sqrt{q[n-1]_q}}$, we get the assertion of Theorem 5.1. □

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ALMOST STABILITY OF THE AGARWAL ET AL. ITERATION SCHEME INVOLVING STRICTLY HEMICONTRACTIVE MAPPINGS IN SMOOTH BANACH SPACES

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Abstract. Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X, $S : K \to K$ be nonexpansive and $T : K \to K$ be continuous strictly hemicontractive mappings. Under some conditions we obtain that the iteration scheme due to Agarwal et al. converges strongly to the common fixed point of S and T and the iteration scheme is almost common-stable on K.

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1. Introduction

Chidume [4] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Schu [19] generalized the result in [4] to both uniformly continuous strongly pseudocontractive mappings and real smooth Banach spaces. Park [17] extended the result in [4] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [18] proved that the Mann and Ishikawa iteration

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methods may exhibit different behaviors for different classes of nonlinear mappings. Harder and Hicks [7], [8] revealed the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings. Harder [6] established applications of stability results to first order differential equations. Afterwards, several generalizations have been made in various directions (see for example [2], [3], [5], [10]-[14], [20]).

Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X, $S : K \to K$ be nonexpansive and $T : K \to K$ be continuous strictly hemicontractive mappings. Under some conditions we obtain that the iteration scheme due to Agarwal et al. [1] converges strongly to the common fixed point of S and T and the iteration scheme is almost common-stable on K.

2. Preliminaries

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbol $F(T)$ stand for the set of fixed points of T (for a single-valued map $T : X \to X$, $x \in X$ is called a *fixed point* of T iff $Tx = x$). We denote by J the *normalized duality mapping* from X to 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Definition 2.1. For every ε with $0 \leq \varepsilon \leq 2$, we define the *modulus* $\delta(\varepsilon)$ *of convexity of* X by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

The space X is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$.

Definition 2.2. A Banach space X is called *uniformly smooth* if X^* is uniformly convex. It is known that if X is uniformly smooth, then J is uniformly continuous on bounded subsets of X.

Let $T : K \to K$ be a mapping.

Definition 2.3. The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|$$

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for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive* and if $0 \leq L < 1$, T is called *contraction*.

Definition 2.4. ([5], [21]) (1) The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \leqslant \| x - y + r((I - T)x - (I - T)y \|$$

for each $x, y \in K$ and for all $r > 0$.

(2) The mapping T is said to be *strongly pseudocontractive* if there exists $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

for all $x, y \in K$ and $r > 0$.

(3) The mapping T is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists $t_x > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt_x(Tx - Ty)\|$$

for all $y \in K$ and $r > 0$.

(4) The mapping T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and there exists $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|$$

for all $x \in K$, $q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive mapping is local strongly pseudocontractive.

Let K be a nonempty convex subset of a normed space X.

(A) For arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)\, x_n + \beta_n T x_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ is known as the Ishikawa iteration scheme [9].

If $\beta_n = 0$ for $n \geq 1$, then the Ishikawa iteration scheme becomes the Mann iteration scheme [15].

(B) For arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ is known as the Agarwal et al. iteration scheme [1].

Definition 2.5. ([6]-[8]) Let K be a nonempty convex subset of X and $T : K \to K$ be a mapping. Assume that $x_1 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=1}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=1}^{\infty}$ be any bounded sequence in K and put $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$.

(1) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *T-stable* on K if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$,

(2) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *almost T-stable* on K if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

It is easy to verify that an iteration scheme $\{x_n\}_{n=1}^{\infty}$ which is T-stable on K is almost T-stable on K. Osilike [16] proved that an iteration scheme which is almost T-stable on K may fail to be T-stable on K.

Definition 2.6. Let K be a nonempty convex subset of X and $T_i : K \to K$, $i = 1, 2, \ldots, k$ be a finite family of k mappings. Assume that $x_1 \in K$ and $x_{n+1} = f(T_1, T_2, \ldots, T_k, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=1}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $\{y_n\}_{n=1}^{\infty}$ be any bounded sequence in K and put $\varepsilon_n = \|y_{n+1} - f(T_1, T_2, \ldots, T_k, y_n)\|$.

(1) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T_1, T_2, \ldots, T_k, x_n)$ is said to be *common-stable* on K if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$,

(2) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T_1, T_2, \ldots, T_k, x_n)$ is said to be *almost common-stable* on K if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

We need the following results.

Lemma 2.7. ([17]) *Let X be a smooth Banach space. Suppose that one of the following holds:*

(a) J is uniformly continuous on any bounded subsets of X,

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(b) $\langle x-y, j(x)-j(y)\rangle \le \|x-y\|^2$ *for all* $x, y \in X$,

(c) for any bounded subset D *of* X, *there is a function* $c : [0,\infty) \to [0,\infty)$ *such that*

$$Re\langle x-y, j(x)-j(y)\rangle \le c(\|x-y\|)$$

for all $x, y \in D$, *where* c *satisfies* $\lim_{t\to 0^+} \frac{c(t)}{t} = 0$.
Then for any $\epsilon > 0$ *and any bounded subset* K, *there exists* $\delta > 0$ *such that*

$$\|sx + (1-s)y\|^2 \le (1-2s)\|y\|^2 + 2sRe\langle x, j(y)\rangle + 2s\epsilon \tag{2.1}$$

for all $x, y \in K$ *and* $s \in [0,\delta]$.

Remark 2.8. (1) If X is uniformly smooth, then (a) in Lemma 2.7 holds.
(2) If X is a Hilbert space, then (b) in Lemma 2.7 holds.

Lemma 2.9. ([5]) *Let* K *be a nonempty subset of a Banach space* X *and* $T : K \to X$ *be a mapping with* $F(T) \neq \emptyset$. *Then* T *is strictly hemicontractive if and only if there exists* $t > 1$ *such that for all* $x \in K$ *and* $q \in F(T)$, *there exists* $j(x-q) \in J(x-q)$ *satisfying*

$$Re\langle x - Tx, j(x-q)\rangle \ge \left(1 - t^{-1}\right)\|x-q\|^2. \tag{2.2}$$

Lemma 2.10. ([14]) *Let* K *be a nonempty subset of an arbitrary normed linear space* X *and* $T : K \to X$ *be a mapping.*

(a) If T *is a local strongly pseudocontractive mapping and* $F(T) \neq \emptyset$, *then* $F(T)$ *is a singleton and* T *is strictly hemicontractive.*

(b) If T *is strictly hemicontractive, then* $F(T)$ *is a singleton.*

Lemma 2.11. ([14]) *Let* $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ *and* $\{c_n\}_{n=1}^\infty$ *be nonnegative real sequences and* $\epsilon' > 0$ *be a constant satisfying*

$$a_{n+1} \le (1-b_n)a_n + \epsilon' b_n + c_n, \quad n \ge 1,$$

where $\sum_{n=1}^\infty b_n = \infty$, $b_n \le 1$ *for all* $n \ge 1$ *and* $\sum_{n=1}^\infty c_n < \infty$. *Then* $\limsup_{n\to\infty} a_n \le \epsilon'$.

Remark 2.12. If $c_n = 0$ for each $n \ge 1$, then Lemma 2.11 reduces to Lemma 1 of Park [17].

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3. Main Results

We now prove our main results.

Theorem 3.1. *Let X be a smooth Banach space and any one of the Axioms (a)-(c) of Lemma 2.7 holds. Let K be a nonempty closed bounded convex subset of X, $S : K \to K$ be nonexpansive and $T : K \to K$ be continuous strictly hemicontractive mappings satisfying*

$$\|x - Sy\| \leq \|Sx - Sy\| \quad \text{and} \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{C}$$

for all $x, y \in K$. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are any sequences in $[0,1]$ satisfying conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\beta_n \leq \alpha_n$,
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from arbitrary $x_1 \in K$ by

$$\begin{cases} x_{n+1} = \alpha_n T x_n + (1-\alpha_n) S y_n, \\ y_n = (1-\beta_n) x_n + \beta_n T x_n, \quad n \geq 1. \end{cases} \tag{3.1}$$

Let $\{z_n\}_{n=1}^{\infty}$ be any sequence in K and define $\{\varepsilon_n\}_{n=1}^{\infty}$ by

$$\varepsilon_n = \|z_{n+1} - p_n\|, \quad n \geq 1, \tag{3.2}$$

where

$$\begin{aligned} p_n &= \alpha_n T z_n + (1-\alpha_n) S w_n, \\ w_n &= (1-\beta_n) z_n + \beta_n T z_n, \quad n \geq 1. \end{aligned} \tag{3.3}$$

Then (a) the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point q of S and T,

(b) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} z_n = q$, so that $\{x_n\}_{n=1}^{\infty}$ is almost common-stable on K.

Proof. It follows from Lemma 2.10 that $F(T)$ is a singleton. Thus $F(S) \cap F(T) \neq \emptyset$.

Let $M = 1 + diam(K)$. For all $n \geq 1$ it is easy to verify that

$$\begin{aligned} &\max\Big\{\sup_{n\geq 1}\|x_n - q\|, \sup_{n\geq 1}\|Tx_n - q\|, \sup_{n\geq 1}\|Sy_n - q\|, \\ &\quad \sup_{n\geq 1}\|z_n - q\|, \sup_{n\geq 1}\|p_n - q\|, \varepsilon_n\Big\} \\ &\leq M. \end{aligned} \tag{3.4}$$

Consider

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - (1-\beta_n)x_n - \beta_n Tx_n\| \\ &= \beta_n \|x_n - Tx_n\| \\ &\leq 2M\beta_n \\ &\to 0 \end{aligned}$$

as $n \to \infty$, and the continuity of S and T imply that

$$\lim_{n\to\infty}\|Tx_n - Ty_n\| = 0 = \lim_{n\to\infty}\|Sx_n - Sy_n\|. \tag{3.5}$$

Now by using condition (C), we have

$$\begin{aligned} \|Tx_n - Sy_n\| &\leq \|Tx_n - Ty_n\| + \|Ty_n - Sy_n\| \\ &\leq 2\|Tx_n - Ty_n\| + \|Sx_n - Sy_n\| \\ &\to 0 \end{aligned} \tag{3.6}$$

as $n \to \infty$, which implies that

$$\|j(Sy_n - q) - j(x_n - q)\| \to 0 \tag{3.7}$$

as $n \to \infty$ because $\|x_n - Sy_n\| \leq \|Tx_n - Sy_n\| \to 0$ as $n \to \infty$.

For given any $\epsilon > 0$ and the bounded subset K, there exists a $\delta > 0$ satisfying (2.1). Note that by (i), (ii) and (3.7) there exists an N such that

$$\begin{aligned} \alpha_n \leq \frac{1}{2(1-k)}, \quad \beta_n \leq \frac{1}{2(1-k)}, \\ \|j(Sy_n - q) - j(x_n - q)\| \leq \frac{\epsilon}{M}, \quad n \geq N, \end{aligned} \tag{3.8}$$

where $k = \frac{1}{t}$ and t satisfies (2.2). Using (3.1), Lemma 2.7 and Lemma 2.9, we infer that

$$\begin{aligned}
&\|y_n - q\|^2 \\
&= \|(1-\beta_n)x_n + \beta_n T x_n - q\|^2 \\
&= \|(1-\beta_n)(x_n - q) + \beta_n (T x_n - q)\|^2 \\
&\leq (1-2\beta_n)\|x_n - q\|^2 + 2\beta_n \mathrm{Re}(T x_n - q, j(x_n - q)) + 2\epsilon\beta_n \\
&\leq (1-2\beta_n)\|x_n - q\|^2 + 2k\beta_n \|x_n - q\|^2 + 2\epsilon\beta_n \\
&= (1-2(1-k)\beta_n)\|x_n - q\|^2 + 2\epsilon\beta_n,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\alpha_n T x_n + (1-\alpha_n) S y_n - q\|^2 \\
&= \|\alpha_n (T x_n - q) + (1-\alpha_n)(S y_n - q)\|^2 \\
&\leq (1-2\alpha_n)\|S y_n - q\|^2 + 2\alpha_n \mathrm{Re}(T x_n - q, j(S y_n - q)) + 2\epsilon\alpha_n \\
&\leq (1-2\alpha_n)\|y_n - q\|^2 + 2\alpha_n \mathrm{Re}(T x_n - q, j(x_n - q)) \\
&\quad + 2\alpha_n \mathrm{Re}(T x_n - q, j(S y_n - q) - j(x_n - q)) + 2\epsilon\alpha_n \\
&\leq (1-2\alpha_n)\|y_n - q\|^2 + 2k\alpha_n \|x_n - q\|^2 \\
&\quad + 2\alpha_n \|T x_n - q\| \|j(S y_n - q) - j(x_n - q)\| + 2\epsilon\alpha_n \\
&\leq (1-2\alpha_n)\|y_n - q\|^2 + 2k\alpha_n \|x_n - q\|^2 \\
&\quad + 2M\alpha_n \|j(S y_n - q) - j(x_n - q)\| + 2\epsilon\alpha_n,
\end{aligned} \tag{3.10}$$

and substitution of (3.9) in (3.10) yields

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\leq (1-2\alpha_n)(1-2(1-k)\beta_n)\|x_n - q\|^2 + 2k\alpha_n \|x_n - q\|^2 \\
&\quad + 2\epsilon\beta_n (1-2\alpha_n) + 2\left(M\|j(S y_n - q) - j(x_n - q)\| + \epsilon\right)\alpha_n \\
&\leq (1-2(1-k)\alpha_n)\|x_n - q\|^2 + 6\epsilon\alpha_n
\end{aligned} \tag{3.11}$$

for all $n \geq N$.

Put

$$a_n = \|x_n - q\|^2, \quad b_n = 2(1-k)\alpha_n, \quad \epsilon' = \frac{3\epsilon}{1-k} \quad \text{and} \quad c_n = 0,$$

it follows from (3.11) that

$$a_{n+1} \leq (1 - b_n)a_n + \epsilon' b_n + c_n, \quad n \geq 1.$$

Observe that $\sum_{n=1}^{\infty} b_n = \infty$ and $b_n \leq 1$ for all $n \geq 1$. It follows from Lemma 2.11 that

$$\limsup_{n\to\infty} \|x_n - q\|^2 \leq \frac{3\epsilon}{1-k}.$$

Letting $\epsilon \to 0^+$, we obtain that $\limsup_{n\to\infty} \|x_n - q\|^2 = 0$, which implies that $x_n \to q$ as $n \to \infty$.

Similarly we also have

$$\|p_n - q\|^2 \leq (1 - 2(1-k)\alpha_n) \|z_n - q\|^2 + 6\epsilon\alpha_n \tag{3.12}$$

for all $n \geq N$.

Suppose that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of (3.2) and (3.12), we infer that

$$\begin{aligned}\|z_{n+1} - q\|^2 &\leq (\|z_{n+1} - p_n\| + \|p_n - q\|)^2 \\ &\leq \|p_n - q\|^2 + 2M\varepsilon_n + \varepsilon_n^2 \\ &\leq (1 - 2(1-k)\alpha_n) \|z_n - q\|^2 + 6\epsilon\alpha_n + 3M\varepsilon_n\end{aligned} \tag{3.13}$$

for all $n \geq N$.

Put

$$a_n = \|z_n - q\|^2, \quad b_n = 2(1-k)\alpha_n, \quad \epsilon' = \frac{3\epsilon}{1-k} \quad \text{and} \quad c_n = 3M\varepsilon_n,$$

it follows from (3.13) that

$$a_{n+1} \leq (1 - b_n)a_n + \epsilon' b_n + c_n, \quad n \geq 1.$$

Observe that $\sum_{n=1}^{\infty} b_n = \infty$, $b_n \leq 1$ and $\sum_{n=1}^{\infty} c_n < \infty$ for all $n \geq 1$. It follows from Lemma 2.11 that

$$\limsup_{n\to\infty} \|z_n - q\|^2 \leq \frac{3\epsilon}{1-k}.$$

Letting $\epsilon \to 0^+$, we obtain that $\limsup_{n\to\infty} \|z_n - q\|^2 = 0$, which implies that $z_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let X be a smooth Banach space and any one of the Axioms (a)-(c) of Lemma 2.7 holds. Let K be a nonempty closed bounded convex subset of X, $S : K \to K$ be nonexpansive and $T : K \to K$ be Lipschitz strictly*

hemicontractive mappings satisfying the condition (C). Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are any sequences in $[0,1]$ satisfying the conditions (i)-(iii).

Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_1 \in K$ by (3.1)*. Let $\{z_n\}_{n=1}^{\infty}$ be any sequence in K and define $\{\varepsilon_n\}_{n=1}^{\infty}$ by* (3.2) *with* (3.3).

Then (a) the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point q of S and T,

(b) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} z_n = q$, so that $\{x_n\}_{n=1}^{\infty}$ is almost common-stable on K.

Remark 3.3. In main results, the condition (C) is not new and it is due to Liu et al. [10].

Remark 3.4. It is well known that every contractive mapping is strongly pseudocontractive, so our results are more general in comparison to the results of Agarwal et al. [1].

Remark 3.5. (1) Theorem 3.1 can also be proved for the same iterative scheme with error terms.

(2) The known results for strongly pseudocontractive mappings with fixed points are weakened by the strictly hemicontractive mappings.

(3) Our results hold in arbitrary smooth Banach spaces, where as other known results are restricted for L_p (or l_p) spaces and q-uniformly smooth Banach spaces.

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The generalized $\mathcal{S}$-convergence on fuzzy directed-complete posets

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Abstract

The concept of generalized $\mathcal{S}$-convergence of net $(x_i)_{i\in I}$ on fuzzy directed-complete posets (for short, fuzzy dcpos) is proposed and its relationship with the generalized Scott topology is studied. It is shown that for an arbitrary fuzzy dcpo, the generalized $\mathcal{S}$-convergence is topological if and only if the fuzzy dcpo is continuous.

Key words: Fuzzy dcpo, Generalized Scott topology, Generalized $\mathcal{S}$-convergence

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1. Introduction

In classical domain theory, as the main research object, domain is some kind of sets with a special order structure, elements of the sets are usually interpreted as the quantity which can make some abstract computation. And the order relation between the elements in domain is interpreted as how much computational information is contained between elements. The classical domain theory can provide semantic mathematical model for sequential programming languages and algorithms. However, with the rapid development of computer and network, more and more demands on concurrent semantics, simple two binary order can't reflect the difference between the calculated quantity. At present, quantitative domain theory can be used as attempts in the research of domain model for concurrent semantics, this research is also very active [4,7,8]. Recently, the literature [1-3] introduced fuzzy posets as the basic framework for quantitative domain theory. Then the notions of fuzzy dcpo, continuous fuzzy dcpo, generalized Scott topology and fuzzy complete lattice are introduced and some good results about this topic are obtained [9-12].

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As is well know, in classical domain theory, the Scott topology can be characterized by the $\mathcal{S}$-convergence. Therefore, based on the above work, this paper is devoted to extending $\mathcal{S}$-convergence to many-valued setting.

The main idea of fuzzy ideal and generalized Scott topology defined in [10] comes from the literature [4]. As shown in [4], this kind of definition of ideal and Scott topology is appropriate for quantitative domain theory including ordinary domain theory, metric domains, ultrametric domains and other examples, such as probabilistic domains and structure spaces, which may be useful in programming language semantics. As a result, our work is not only a generalization of classical domain theory, but also provides a reference for the studies of other quantitative domain systems.

The content of the paper is arranged as follows. In section 2, we recall some notions and properties known. In section 3, we define the generalized $\mathcal{S}$-convergence and exploit its relationship with the generalized Scott topology on fuzzy dcpos. It is shown that for an arbitrary fuzzy dcpo, the generalized $\mathcal{S}$-convergence is topological if and only if the fuzzy dcpo is continuous. Moreover, we show that the topology induced by the class of the generalized $\mathcal{S}$-convergence on a fuzzy dcpo is exactly the generalized Scott topology. Finally, some conclusions are proposed in section 4.

2. Preliminaries

Suppose that L is a complete lattice and $p, q \in L$. As defined in [5], p is said to well way below q, denoted by $p \lll q$, if for any subset $A \subseteq L$, $q \leq \vee A$ implies $p \leq r$ for some $r \in A$. The relation $\lll$ is called multiplicative if for any $p, q, r \in L$, $p \lll q$ and $p \lll r$ imply $p \lll q \wedge r$.

Suppose that L is a completely distributive lattice and $p \in L$. If $p = a \vee b$ implies $p = a$ or $p = b$ for any $a, b \in L$, then p is said to be $\vee$-irreducible.

Suppose that L is a frame (or complete Heyting algebra) and $a, b \in L$. We define $a \to b = \bigvee\{c \in L | a \wedge c \leq b\}$.

Throughout this paper, L denotes a frame. The following definitions and theorems can be found in [10-12].

2.1. Fuzzy dcpos

Definition 2.1. *A fuzzy poset is a pair (X, e) such that X is a set and $e : X \times X \longrightarrow L$ is a mapping, called a fuzzy order, that satisfies for every $x, y, z \in X$,*

(1) $e(x, x) = \top$;
(2) $e(x, y) \wedge e(y, z) \leq e(x, z)$;
(3) $e(x, y) = e(y, x) = \top$ *implies* $x = y$.

Dual to the logical correspondence in [6], let (X, e) be a fuzzy poset, $x, y \in X$, $a, b \in L$, $\{p_i | i \in I\}$ a family of elements of L, and $A \in L^X$. Then:

$$e(x,y) = [\![x \leq y]\!], \quad A(x) = [\![x \in A]\!], \quad 1 = [\![True]\!], \quad (a \wedge b) = [\![a \& b]\!],$$

$$a \to b = [\![a \Rightarrow b]\!], \quad \bigwedge_{i\in I} p_i = [\![\forall i : I.p_i]\!], \quad \bigvee_{i\in I} p_i = [\![\exists i : I.p_i]\!], \quad (a \leq b) = [\![a \vdash b]\!].$$

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Definition 2.2. *Let (X, e) be a fuzzy poset. $\varphi \in L^X$ is called a fuzzy directed set on X if*

(1) *there exists $x \in X$ such that $0 \lll \varphi(x)$;*
(2) *for any $x_1, x_2 \in X$, $a_1, a_2, a \in L$ with $a_1 \lll \varphi(x_1)$, $a_2 \lll \varphi(x_2)$ and $a \lll 1$, there is $x \in X$ such that $a \lll \varphi(x)$, $a_1 \lll e(x_1, x)$, and $a_2 \lll e(x_2, x)$.*

Definition 2.3. *Let (X, e) be a fuzzy poset. $\varphi \in L^X$ is called a fuzzy lower set if for any $x, y \in X$, $\varphi(x) \wedge e(y, x) \leq \varphi(y)$.*

If φ is a fuzzy directed set and fuzzy lower set, then we call φ a fuzzy ideal. The set of all fuzzy directed sets on X is denoted by $D_L(X)$ and the set of all fuzzy ideals on X is denoted by $I_L(X)$.

Define $\widetilde{e} : L^X \times L^X \longrightarrow L$ as $\widetilde{e}(\varphi, \psi) = \bigwedge_{x \in X} \varphi(x) \to \psi(x)$, for any $\varphi,\psi \in L^X$.

Definition 2.4. *Let (X, e) be a fuzzy poset, $x_0 \in X$ and $\varphi \in L^X$. Consider the following conditions:*

(1) *for any $x \in X, \varphi(x) \leq e(x, y)$;*
(2) *for any $y \in X, \bigwedge_{x \in X} \varphi(x) \to e(x, y) \leq e(x_0, y)$.*

x_0 is called a join of φ, denoted by $\sqcup\varphi$, if it satisfies (1) and (2); x_0 is called an upper bound of φ if it satisfies (1).

Definition 2.5. *Let (X, e_X) and (Y, e_Y) be fuzzy posets, $f : X \longrightarrow Y$ a mapping. Then f is called a fuzzy monotone mapping if for any $x, y \in X$, $e_X(x, y) \leq e_Y(f(x), f(y))$.*

Let (X, e_X) and (Y, e_Y) be fuzzy posets, $f : X \longrightarrow Y$ a mapping. Then the fuzzy forward powerset operator $\widetilde{f}^{\to} : L^X \longrightarrow L^Y$ is defined by $\widetilde{f}^{\to}(\varphi)(y) = \bigvee_{x \in X}(\varphi(x) \wedge e_Y(y, f(x)))$ for any $\varphi \in L^X$ and $y \in Y$.

Theorem 2.6. *Let (X, e_X) and (Y, e_Y) be fuzzy posets, $f : X \longrightarrow Y$ a fuzzy monotone mapping. Then the following assertions hold:*

(1) *for any $\varphi \in L^X$, $\widetilde{f}^{\to}(\varphi)$ is a fuzzy lower set on Y;*
(2) *if $\varphi \in D_L(X)$, then $\widetilde{f}^{\to}(\varphi) \in I_L(Y)$.*
(3) *if $\varphi \in D_L(X)$, then $\downarrow \varphi \in I_L(X)$.*

Definition 2.7. *A fuzzy poset (X, e) is called a fuzzy directed-complete poset (for short, fuzzy dcpo) if any fuzzy directed set on X has a join.*

Definition 2.8. *Let (X, e_X) and (Y, e_Y) be fuzzy dcpos. A mapping $f : X \longrightarrow Y$ is fuzzy Scott continuous if it is fuzzy monotone and for any $\varphi \in D_L(X)$, $f(\bigsqcup \varphi) = \bigsqcup \widetilde{f}^{\to}(\varphi)$.*

Lemma 2.9. *If (X, e) is a fuzzy dcpo, then $\downarrow \varphi = \varphi$ holds for any fuzzy lower set φ.*

Proof. For any $x \in X$, it follows $\downarrow \varphi(x) = \bigvee_{x' \in X}(\varphi(x') \wedge e(x, x')) \geq \varphi(x)$. On the other hand, $\downarrow \varphi(x) = \bigvee_{x' \in X}(\varphi(x') \wedge e(x, x')) \leq \bigvee_{x' \in X} \varphi(x) = \varphi(x)$. Hence, we have $\downarrow \varphi = \varphi$.

2.2. Generalized Scott topology

Let (X,e) be a fuzzy poset. We introduce the following notations for $x \in X$, $a \in L$, $F \subseteq X$ and $\varphi \in L^X$:

$\uparrow_a^o x = \{y \in X \mid a \lll e(x,y)\}$, $\uparrow_a x = \{y \in X \mid a \le e(x,y)\}$, $\downarrow_a^o x = \{y \in X \mid a \lll e(x,y)\}$, $\downarrow_a x = \{y \in X \mid a \le e(x,y)\}$, $\uparrow_a^o F = \bigcup\{\uparrow_a^o x \mid x \in F\}$, $\uparrow_a F = \bigcup\{\uparrow_a x \mid x \in F\}$, $\sigma_a(\varphi) = \{x \in X | a \lll \varphi(x)\}$.

Note. $\uparrow_a^o x$ and $\uparrow_a x$ are exactly $P_a^o(x)$ and $P_a(x)$ introduced in [10], respectively. In our opinion the notations of $\uparrow_a^o x$ and $\uparrow_a x$ are more intuitive.

Let (X,e) be a fuzzy dcpo and $x \in X$. $\downarrow x \in L^X$ is defined by $\downarrow x(y) = e(y,x)$ for any $y \in X$.

Definition 2.10. *Let (X,e) be a fuzzy dcpo and $U \subseteq X$.Then U is generalized Scott open if for any $\varphi \in D_L(X)$, $\bigsqcup \varphi \in U$ implies that there exist $a \lll 1$ and $x \in X$ such that $a \lll \varphi(x)$ and $\uparrow_a x \subseteq U$. The collection of all generalized Scott open subsets of X is a topology, called the generalized Scott topology and denoted by $\sigma_e(X)$ (for short, σ_e). The collection of all generalized Scott closed subsets of X denoted by $\Gamma_e(X)$ (for short, Γ_e).*

Corollary 2.11. *Let (X,e) be a fuzzy poset and $U \subseteq X$. If U is a generalized Scott open set, then for any $x \in U$, there exists $a \lll 1$ such that $\uparrow_a x \subseteq U$.*

Proof. For any $x \in U$, it is obvious that $\downarrow x \in D_L(X)$ and $\sqcup \downarrow x = x$. Since U is a generalized Scott open set, there exist $a \lll 1$ and $z \in X$ such that $a \lll e(z,x)$ and $\uparrow_a z \subseteq U$. Then for any $y \in \uparrow_a x$, we have $a \le e(z,x) \wedge e(x,y) \le e(z,y)$, which implies $\uparrow_a x \subseteq \uparrow_a z \subseteq U$.

Corollary 2.12. *Let (X,e) be a fuzzy dcpo and $1 \in L$, then $\downarrow_1 x \subseteq U$ for any $x \in U \in \Gamma_e(X)$.*

Proof. Suppose $\downarrow_1 x \nsubseteq U$, then there exists $y_0 \in \downarrow_1 x$ such that $y_0 \in X - U \in \sigma_e(X)$. Thus there exists $b \lll 1$ such that $\uparrow_b y_0 \subseteq X - U$, which implies $x \in X - U$. It is a contradiction.

Proposition 2.13. *Let L be a completely distributive lattice, (X,e) be a fuzzy poset and $x \in X$. Then for any $b \lll 1$, $\downarrow_b x$ is generalized Scott closed.*

By Proposition 2.11, we know that if $1 \lll 1$, then $\downarrow_1 x$ is generalized Scott closed.

2.3. Continuous fuzzy dcpos

Definition 2.14. *Let (X,e) be a fuzzy dcpo, $x, y \in X$ and $a \in L$. If for any $\varphi \in D_L(X)$, $a \lll e(y, \sqcup\varphi)$ implies that $a \lll \varphi(z)$ and $a \lll e(x,z)$ for some $z \in X$, then x is called L_a-way below y, denoted by $x \ll_a y$. x is said to be $L-$way below y, denoted by $x \ll_L y$, if for any $a \lll 1$, $x \ll_a y$.*

Definition 2.15. *Let (X,e) be a fuzzy dcpo and $B \subseteq X$. If for any $x \in X$, there exists $\varphi \in D_L(B)$ such that $x = \bigsqcup \widetilde{i_B^{\rightarrow}}(\varphi)$ and $\sigma_a(\widetilde{i_B^{\rightarrow}}(\varphi)) \subseteq \Downarrow_a x$ for any $a \lll 1$, then B is called a basis for X, where i_B is the embedding of B into X and $\Downarrow_a x = \{y \in X | y \ll_a x\}$.*

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Lemma 2.16. *Let (X, e) be a fuzzy dcpo and $a \in L$. For any $x \in X$ and $\varphi \in D_L(X)$ with $a \lll e(x, \sqcup\varphi)$, we have $\sqcup\varphi \in cl_{\sigma_e}(\sigma_a(\varphi))$.*

Proof. Let $x \in X$, $\varphi \in D_L(X)$ with $a \lll e(x, \sqcup\varphi)$ and U be a generalized Scott open neighborhood of $\sqcup\varphi$. Since U is a generalized Scott open set, there exist $b \lll 1$ and $z \in X$ such that $b \lll \varphi(z)$ and $\uparrow_b z \subseteq U$. Furthermore, since $\varphi \in D_L(X)$, there exists $u \in X$ such that $a \lll \varphi(u)$ and $b \lll e(z, u)$, and hence $U \cap \sigma_a(\varphi) \neq \emptyset$. By the arbitrariness of U, we have $\sqcup\varphi \in cl_{\sigma_e}(\sigma_a(\varphi))$.

Proposition 2.17. *For any $a \in L$, if (X, e) is a fuzzy dcpo and $y \in int_{\sigma_e}(\uparrow_a^o x)$, then $x \ll_a y$.*

Proof. Since $y \in int_{\sigma_e}(\uparrow_a^o x)$, there exists $b \geq a$ such that $\uparrow_b y \subseteq int_{\sigma_e}(\uparrow_a^o x) \subseteq \uparrow_a^o x$. Then for any $\varphi \in D_L(X)$ with $b \lll e(y, \sqcup\varphi)$, it follows $\sqcup\varphi \in int_{\sigma_e}(\uparrow_a^o x)$ and $\sqcup\varphi \in cl_{\sigma_e}(\sigma_b(\varphi))$ by Lemma 2.15. Thus we have $\sigma_b(\varphi) \cap int_{\sigma_e}(\uparrow_a^o x) \neq \emptyset$, i.e., there exists $z \in X$ such that $a \leq b \lll \varphi(z)$ and $a \lll e(x, z)$. Hence, $x \ll_a y$.

Definition 2.18. *A fuzzy dcpo (X, e) is said to be continuous if it has a basis.*

Theorem 2.19. *Let L be a completely distributive lattice in which 1 is $\vee-$irreducible and $\lll$ is multiplicative, (X, e) a continuous fuzzy dcpo. Then for any $0 \neq a \lll 1$ and $x \in X$, $\Uparrow_a x = \{y \in X | x \ll_a y\}$ is a generalized Scott open set and $\{\Uparrow_a x | x \in X, 0 \neq a \lll 1\}$ is a basis for generalized Scott open sets.*

Theorem 2.20. *Let (X, e_X) and (Y, e_Y) be fuzzy dcpos, $f : X \longrightarrow Y$ a fuzzy monotone mapping. Then f is fuzzy Scott continuous iff f is topologically continuous with respect to the generalized Scott topologies.*

3. Generalized Scott topology convergence

In this section, we propose the generalized $\mathcal{S}$-convergence and discuss its relationship with the generalized Scott topology.

Definition 3.1. *A net $(x_i)_{i \in I}$ in a fuzzy dcpo (X, e) is said to generalized $\mathcal{S}$-converge to an element $x \in X$ if there exists $\varphi \in D_L(X)$ such that*

(1) $e(x, \sqcup\varphi) = 1$;
(2) $\varphi(y) \leq sup_{i \in I} inf_{j \geq i} e(y, x_j)$ *for any $y \in X$.*

In this case we say that x is a $\mathcal{GS}$-limit of $(x_i)_{i \in I}$ and write briefly $x \equiv_{\mathcal{GS}} \lim x_i$. Let $\mathcal{GS}$ denote the class of those pairs $((x_i)_{i \in I}, x)$ such that $x \equiv_{\mathcal{GS}} \lim x_i$.

It is clear that for any constant net $(x_i)_{i \in I}$ in a fuzzy dcpo (X, e) with value x, we have $x \equiv_{\mathcal{GS}} \lim x_i$. If $(x_i)_{i \in I}$ generalized $\mathcal{S}$-converges to x, then it generalized $\mathcal{S}$-converges to any $y \in X$ with $e(y, x) = 1$. Thus the $\mathcal{GS}$-limits of a net are generally not unique.

The condition (1) of Definition 3.1 can be interpreted as $x \leq \sqcup\varphi$, and condition (2) can be interpreted intuitively as φ is a set of eventual lower bounds of the net $(x_i)_{i \in I}$.

Example 1 Let $X = \{T\} \cup \{x_1, x_2, ...\} \cup \{y_1, y_2, ...\} \cup \{z\}$ and $L = \{0, 1\}$. We define $e : X \times X \longrightarrow L$ as follows: $e(z, T) = e(x_i, T) = e(y_i, z) = e(z, x_i) = e(y_i, x_j) = 1$ for any $i, j = 1, 2, ...$; $e(y_i, y_j) = 1$ whenever $i \leq j$; otherwise $e(x, y) = 0$. Then we can check that (X, e) is a fuzzy dcpo. Let

$$\varphi(x) = \begin{cases} 1 & x = y_i, i = 1, 2, ... \\ 0 & otherwise \end{cases} \tag{1}$$

Then it is clear that $\sqcup\varphi = z$ and $\varphi \in \mathcal{D}_L(X)$. Since $e(y_j, x_i) = 1$ and $e(z, x_i) = 1$ for any $i, j = 1, 2, 3, ...$, we have that the net $(x_i)_{i\in I}$ generalized $\mathcal{S}$-converges to z.

Lemma 3.2. *Let (X, e) be a fuzzy poset, $\varphi, \psi \in L^X$. If $\varphi(x) \leq \psi(x)$ for any $x \in X$, then $\downarrow \varphi(x) \leq \downarrow \psi(x)$ for any $x \in X$.*

Proof. For any $x \in X$, it follows $\downarrow \varphi(x) = \bigvee_{x'\in X} \varphi(x') \wedge e(x, x') \leq \bigvee_{x'\in X} \psi(x') \wedge e(x, x') = \downarrow \psi(x)$.

Theorem 3.3. *Let $(x_i)_{i\in I}$ be a net in a fuzzy dcpo (X, e). Then $x \equiv_{\mathcal{GS}}$ lim x_i iff there exists $\varphi \in I_L(X)$ such that*

(1) $e(x, \sqcup\varphi) = 1$;
(2) $\varphi(y) \leq sup_{i\in I} inf_{j\geq i} e(y, x_j)$ *for any* $y \in X$.

Proof. Sufficiency. Obviously.

Necessity. Suppose $x \equiv_{\mathcal{GS}}$ lim x_i. Then there exists $\varphi \in D_L(X)$ such that $e(x, \sqcup\varphi) = 1$ and $\varphi(y) \leq sup_{i\in I} inf_{j\geq i} e(y, x_j)$ for any $y \in X$. Let $\psi(y) = sup_{i\in I} inf_{j\geq i} e(y, x_j)$ for any $y \in X$. Then it is easily checked that ψ is a fuzzy lower set. By Theorem 2.5 in [10] and Theorem 2.5, we have $\downarrow \varphi \in I_L(X)$ and $\sqcup\varphi = \sqcup \downarrow \varphi$. This implies $e(x, \sqcup \downarrow \varphi) = 1$ and $\downarrow \varphi(y) \leq \psi(y)$ for any $y \in X$ by Lemma 3.2.

Let $(X, \leq)$ be a dcpo. We define $e_\leq : X \times X \longrightarrow L = \{0, 1\}$ as

$$e_\leq(x, y) = \begin{cases} 1 & x \leq y \\ 0 & x \nleq y \end{cases} \tag{2}$$

for any $x, y \in X$. Then $(X, e_\leq)$ is a fuzzy dcpo by Remark made just after Proposition 3.3 in [10].

Moreover, we have the following result.

Theorem 3.4. *Let $(X, \leq)$ be a dcpo. Then $x \equiv_{\mathcal{S}}$ lim x_i (on $(X, \leq)$) iff $x \equiv_{\mathcal{GS}}$ lim x_i (on $(X, e_\leq)$).*

Proof. Necessity. Suppose $x \equiv_{\mathcal{S}}$ lim x_i. Then there exists a directed set D of eventual lower bounds of net $(x_i)_{i\in i}$ such that $x \leq supD$. By Remark made just after Proposition 3.3 in [10], we have $e_\leq(x, \sqcup\chi_D) = 1$ and $\chi_D \in D_L(X)$. Next we show $\chi_D(y) \leq sup_{i\in I} inf_{j\geq i} e_\leq(y, x_j)$ for any $y \in X$. For any $y \in X$, it follows $y \in D$ or $y \notin D$. When $y \notin D$, i.e., $\chi_D(y) = 0$, it is obvious that

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$\chi_D(y) \leq sup_{i\in I} inf_{j\geq i} e_{\leq}(y, x_j)$. If $y \in D$, i.e., $\chi_D(y) = 1$, then there exists $i \in I$ such that $y \leq x_j$ for any $j \geq i$, which implies $e_{\leq}(y, x_j) = 1$ for any $k \geq i$. Then we have $sup_{i\in I} inf_{j\geq i} e_{\leq}(y, x_j) = 1$ and hence $\chi_D(y) \leq sup_{i\in I} inf_{j\geq i} e_{\leq}(y, x_j)$ for any $y \in X$. Therefore, we have $x \equiv_{\mathcal{GS}} \lim x_i$.

Sufficiency. Suppose $x \equiv_{\mathcal{GS}} \lim x_i$. Then there exists $\varphi \in D_L(X)$ such that $e(x, \sqcup\varphi) = 1$ and $\varphi(y) \leq sup_{i\in I} inf_{j\geq i} e_{\leq}(y, x_j)$ for any $y \in X$. By Remark made just after Proposition 3.3 in [10], it follows that $D = \sigma_1(\varphi) = \{y \in X | \varphi(y) = 1\}$ is a directed set and $\sqcup\varphi = \vee D$. Then $x \leq \vee D$. Next we show that D is a set of eventual lower bounds of the net $(x_i)_{i\in I}$. Let $y \in D$, i.e., $\varphi(y) = 1 = sup_{i\in I} inf_{j\geq i} e_{\leq}(y, x_j)$, which implies there exists $i \in I$ such that $e_{\leq}(y, x_j) = 1$ for any $j \geq i$. Then it follows $y \leq x_j$ for any $j \geq i$, which implies y is an eventual lower bound of net $(x_i)_{i\in i}$. By the arbitrariness of y, D is a set of eventual lower bounds of the net $(x_i)_{i\in i}$. Therefore, it follows $x \equiv_{\mathcal{S}} \lim x_i$.

Remark 3.5. Theorem 3.4 means that the generalized $\mathcal{S}$-convergence on fuzzy dcpos generalizes the $\mathcal{S}$-convergence on dcpos.

Next, we discuss the relation between convergence and topology on fuzzy dcpos. For an arbitrary class $\mathcal{L}$ of pairs $((x_i)_{i\in I}, x)$ consisting of a net and an element of any set X, we denote

$$\mathcal{O}(\mathcal{L}) = \{U \subseteq X : \textit{ whenever } ((x_i)_{i\in I}, x) \in \mathcal{L} \textit{ and } x \in U, \textit{ then } x_i \in U \textit{ holds eventually}\}.$$

Clearly, both $\emptyset$ and X belong to $\mathcal{O}(\mathcal{L})$, which is closed under the formation of arbitrary unions and finite intersections, that is, $\mathcal{O}(\mathcal{L})$ is a topology on X.

By Definition 3.1 we know that for any $(x_i)_{i\in I} \in \mathcal{GS}$, the element x is a limit of the net $(x_i)_{i\in I}$ with respect to the topology $\mathcal{O}(\mathcal{GS})$. Since $\emptyset$ and X may be the only element of $\mathcal{O}(\mathcal{GS})$, we need to exploit it in detail.

Lemma 3.6. *Let L be a completely distributive lattice in which $\lll$ is multiplicative, (X, e) be a fuzzy dcpo and $U \subseteq X$. Then $U \in \mathcal{O}(\mathcal{GS})$ iff for any $\varphi \in D_L(X)$, $\sqcup\varphi \in U$ implies there exist $a \lll 1$ and $x \in X$ such that $a \lll \varphi(x)$ and $\uparrow_a x \subseteq U$.*

Proof. Necessity. Suppose $U \in \mathcal{O}(\mathcal{GS})$ and $\varphi \in D_L(X)$ with $\sqcup\varphi \in U$. Let $I = \{(x, a) \in X \times L | a \lll \varphi(x)\}$ and define a binary relation $\leq_I$ on I by $(x_1, a_1) \leq_I (x_2, a_2)$ iff $a_1 \leq a_2 \wedge e(x_1, x_2)$. Then $\leq_I$ is reflexive and transitive. Since $\varphi \in D_L(X)$, there exists $x \in X$ such that $0 \lll \varphi(x)$ and so $(x, 0) \in I$. Thus I is nonempty. Suppose $(x_1, a_1), (x_2, a_2) \in I$. Choose $a_1', a_2', b \lll 1$ such that $a_1' \lll \varphi(x_1)$, $a_1 \lll a_1' \wedge b$, $a_2' \lll \varphi(x_2)$ and $a_2 \lll a_2' \wedge b$. Choose $x \in X$ such that $b \lll \varphi(x)$, $a_1' \lll e(x_1, x)$ and $a_2' \lll e(x_2, x)$. Then $(x, b) \in I$ and $(x_1, a_1), (x_2, a_2) \leq (x, b)$. Hence, $(I, \leq_I)$ is a directed set. For $i = (x, a) \in I$, let $x_i = x$. Then $(x_i)_{i\in I}$ is a net in X, and obviously $e(\sqcup\varphi, \sqcup\varphi) = 1$. To see $\varphi(y) \leq sup_{i\in I} inf_{j\geq i} e(y, x_j)$ for any $y \in X$, we suppose $a \lll \varphi(y)$. Then $i = (y, a) \in I$ and $i \leq_I j$ implies $a \leq e(y, x_j)$ and so $a \leq sup_{i\in I} inf_{j\geq i} e(y, x_j)$. Thus we have $((x_i)_{i\in I}, \sqcup\varphi) \in \mathcal{GS}$. Then there exists $k \in I$ such that $x = x_i \in U$ for all $i \geq k$, this shows there exists $a \lll 1$ such that $a \lll \varphi(x)$. Let $y \in \uparrow_a x$, then $j = (y, a) \geq i = (x, a)$ and so $y \in U$. This implies $\uparrow_a x \subseteq U$.

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Sufficiency. Suppose $((x_i)_{i\in I}, x) \in \mathcal{GS}$ with $x \in U$. We must show that x_i is eventually in U. By the definition of $\mathcal{GS}$, we have that $e(x, \sqcup\varphi) = 1$ for some $\varphi \in D_L(X)$ with $\varphi(y) \leq sup_{i\in I} inf_{j\geq i} e(y, x_j)$ for any $y \in X$. Since $x \in U$ and $e(x, \sqcup\varphi) = 1$, it follows $\sqcup\varphi \in U$, which implies that there exist $a \lll 1$ and $z \in X$ such that $a \lll \varphi(z)$ and $\uparrow_a z \subseteq U$ by the assumption. Then there exists $k \in I$ such that $a \leq e(z, x_i)$ for all $i \geq k$. Therefore, $x_i \in U$ holds eventually.

Remark 3.7. By Lemma 3.6 and the definition of generalized Scott topology, we have that the topology $\mathcal{O}(\mathcal{GS})$ is exactly the generalized Scott topology on the fuzzy dcpos.

Proposition 3.8. *Let L be a completely distributive lattice in which 1 is $\vee-$irreducible and $\lll$ is multiplicative, (X, e) a continuous fuzzy dcpo. Then $x \equiv_{\mathcal{GS}}$ lim x_i iff the net $(x_i)_{i\in I} \to x$ with respect to the generalized Scott topology $\sigma_e(X)$. In particular, the generalized $\mathcal{S}$-convergence is topological.*

Proof. Since $\mathcal{O}(\mathcal{GS}) = \sigma_e(X)$, $x \equiv_{\mathcal{GS}} \lim x_i$ implies that $(x_i)_{i\in I} \to x$ with respect to $\sigma_e(X)$. Conversely, suppose that we have a convergence net $(x_i)_{i\in I} \to x$ in the $\sigma_e(X)$. Since (X, e) is continuous, there exists $\varphi \in I_L(X)$ such that $x = \sqcup\varphi$ and $\sigma_a(\varphi) \subseteq \Downarrow_a x$ for any $a \lll 1$. Let $a \lll \varphi(y)$. Then $y \in \Downarrow_a x$. Hence $(x_i)_{i\in I}$ is eventually in $\Uparrow_a y$, i.e., there exists $k \in I$ such that $x_i \in \Uparrow_a y$ for all $i \geq k$. Then $a \lll e(y, x_i)$ for all $i \geq k$, and so $a \lll sup_{k\in I} inf_{i\geq k} e(y, x_i)$. Hence, $\varphi(y) \leq sup_{k\in I} inf_{i\geq k} e(y, x_i)$. Thus we have $((x_i)_{i\in I}, x) \in \mathcal{GS}$.

Lemma 3.9. *Let $\lll$ be multiplicative on L and (X, e) be a fuzzy dcpo. If there exists $\varphi \in I_L(X)$ such that $e(x, \sqcup\varphi) = 1$ and $\sigma_a(\varphi) \subseteq \Downarrow_a x$ for any $a \lll 1$, then $\sigma_a(\varphi) = \Downarrow_a x$ and $x = \sqcup\varphi$.*

Proof. Firstly, we only need to show $\Downarrow_a x \subseteq \sigma_a(\varphi)$. Let $y \in \Downarrow_a x$. Then there exists $z \in X$ such that $a \lll \varphi(z)$ and $a \lll e(y, z)$. Then we have $a \lll \varphi(z) \wedge e(y, z) \leq \varphi(y)$, which implies $y \in \sigma_a(\varphi)$. Hence, it follows $\Downarrow_a x \subseteq \sigma_a(\varphi)$.

We further show $x = \sqcup\varphi$. Let $y \in X$ and $a \lll \varphi(y)$. Since $\sigma_a(\varphi) \subseteq \Downarrow_a x$, we have $y \in \Downarrow_a x$, which implies $a \lll e(y, x)$. By the arbitrariness of a, it follows $\varphi(y) \leq e(y, x)$, i.e., $\varphi(y) \to e(y, x) = 1$. Hence, we have $e(\sqcup\varphi, x) = \bigwedge_{y\in X} \varphi(y) \to e(y, x) = 1$ by the arbitrariness of y. Therefore, it follows that $x = \sqcup\varphi$.

Proposition 3.10. *Let L be a completely distributive lattice in which 1 is $\vee-$irreducible and $\lll$ is multiplicative, (X, e) a fuzzy dcpo. If the generalized $\mathcal{S}$-convergence is topological, then (X, e) is a continuous fuzzy dcpo.*

Proof. By Remark 3.7, the topology arising from the generalized $\mathcal{S}$-convergence is the generalized Scott topology. Thus the generalized $\mathcal{S}$-convergence is topological. Then it follows that $x \equiv_{\mathcal{GS}} \lim x_i$ iff the net $(x_i)_{i\in I} \to x$ with respect to $\sigma_e(X)$. Let $x \in X$. Define

$$I = \{(U, n, u) \in \mathcal{N}(x) \times N \times X : u \in U\},$$

where $\mathcal{N}(x)$ consists of all generalized Scott open sets containing x and define an order on I to be lexicographic order on the first two coordinates, that is,

$(U, m, u) < (V, n, v)$ iff V is a proper subset of U or $U = V$ and $m < n$. Let $x_i = u$ for $i = (U, m, u) \in I$. Then it is easy to see that $(x_i)_{i\in I}$ is a net and converges to x in the generalized Scott topology, i.e., $x = \lim x_i$. Then there exists $\varphi \in I_L(X)$ such that $e(x, \sqcup\varphi) = 1$ and $\varphi(y) \leq \bigvee_{i\in I} \bigwedge_{j\geq i} e(y, x_j)$ for any $y \in X$. Let $a \lll 1$ and $y \in X$ such that $a \lll \varphi(y)$. Then there exists $i = (U, m, u) \in I$ such that $a \lll \varphi(y) \leq e(y, x_j)$ for all $(V, n, v) = j \geq i$. In particular, we have $(U, m + 1, v) \geq (U, m, u)$ for all $v \in U$. Then for any $v \in U$, we have $v \in \uparrow_a^o y$ and so $U \subseteq \uparrow_a^o y$. Hence, $U \subseteq int_{\sigma_e}(\uparrow_a^o y)$ which implies $x \in int_{\sigma_e}(\uparrow_a^o y)$. By Proposition 2.16, it follows $y \ll_a x$ and so $\sigma_a(\varphi) \subseteq \Downarrow_a x$. By Lemma 3.9, we have $\sigma_a(\varphi) = \Downarrow_a x$ and $x = \sqcup\varphi$. Therefore, (X, e) is a continuous fuzzy dcpo.

The combination of Proposition 3.8 and Proposition 3.10 deduces the following theorem.

Theorem 3.11. *Let L be a completely distributive lattice in which 1 is $\vee-$ irreducible and $\lll$ is multiplicative, (X, e) a fuzzy dcpo. The following statements are equivalent:*

(1) *the generalized $\mathcal{S}$-convergence is topological for the generalized Scott topology $\sigma_e(X)$;*
(2) *(X, e) is a continuous fuzzy dcpo.*

Lemma 3.12. *Let L be a completely distributive lattice and $f : X \longrightarrow Y$ be a monotone mapping between the fuzzy posets (X, e_X) and (Y, e_Y). If $\varphi \in I_L(X)$, $(x_i)_{i\in I}$ is a net on (X, e_X) and $\varphi(x) \leq sup_{i\in I} inf_{j\geq i} e(x, x_j)$ for any $x \in X$, then $\widetilde{f}^{\rightarrow}(\varphi)(y) \leq sup_{i\in I} inf_{j\geq i} e(y, f(x_j))$ for any $y \in Y$.*

Proof. For any $y \in Y$,

$$\begin{aligned}
\widetilde{f}^{\rightarrow}(\varphi)(y) &= \bigvee_{x\in X} \varphi(x) \wedge e_Y(y, f(x)) \\
&\leq \bigvee_{x\in X} sup_{i\in I} inf_{j\geq i} e(x, x_j) \wedge e_Y(y, f(x)) \\
&= \bigvee_{x\in X} sup_{i\in I} inf_{j\geq i} (e(x, x_j) \wedge e_Y(y, f(x))) \\
&\leq \bigvee_{x\in X} sup_{i\in I} inf_{j\geq i} (e_Y(f(x), f(x_j)) \wedge e_Y(y, f(x))) \\
&\leq \bigvee_{x\in X} sup_{i\in I} inf_{j\geq i} e_Y(y, f(x_j))) \\
&= sup_{i\in I} inf_{j\geq i} e_Y(y, f(x_j)).
\end{aligned}$$

Proposition 3.13. *Let L be a completely distributive lattice in which 1 is $\vee-$ irreducible and $\lll$ is multiplicative, $f : X \longrightarrow Y$ be a monotone mapping between continuous fuzzy dcpos (X, e_X) and (Y, e_Y). Then f is fuzzy Scott continuous iff for all nets $(x_i)_{i\in I}$ in X, $f(lim\ x_i) \equiv_{\mathcal{GS}} lim\ f(x_i)$.*

Proof. Necessity. Suppose that f is fuzzy Scott continuous and $x \equiv_{\mathcal{GS}} \lim x_i$. Then there exists $\varphi \in D_L(X)$ such that $e(x, \sqcup\varphi) = 1$ and $\varphi(y) \leq sup_{k\in I} inf_{i\geq k}$ $e(y, x_j)$ for any $y \in X$. By Theorem 2.5 and the assumption, we have $\sqcup\widetilde{f}^{\rightarrow}(\varphi) = f(\sqcup\varphi)$ and $\widetilde{f}^{\rightarrow}(\varphi) \in I_L(Y)$. Then it follows $1 = e(x, \sqcup\varphi) \leq e(f(x), \sqcup\widetilde{f}^{\rightarrow}(\varphi)) = 1$. Furthermore, from Lemma 3.12, we have $\widetilde{f}^{\rightarrow}(\varphi)(y) \leq sup_{i\in I} inf_{j\geq i} e(y, f(x_j))$ for any $y \in Y$. Hence, it follows $f(\lim x_i) \equiv_{\mathcal{GS}} \lim f(x_i)$.

Sufficiency. By Theorem 2.19, we only need to show that f is topological continuous with respect to generalized Scott topology. Suppose that there exists $x_0 \in X$ such that f is not continuous at x_0. Then there exists $V_0 \in \mathscr{N}(f(x_0))$ such that $f^{-1}(V_0) \notin \mathscr{N}(x_0)$. This implies for any $U \in \mathscr{N}(x_0)$, there exists $x_U \in U$, but $f(x) \notin V_0$. Note that the net $\{x_U \in U | U \in \mathscr{N}(x_0)\}$ convergence to x_0, but $f(x_U) \notin \mathscr{N}(f(x_0))$, which implies $f(x_U) \not\rightarrow f(x_0)$. Hence, we have $f(\lim x_U) \not\equiv_{\mathcal{GS}} \lim f(x_U)$ by Theorem 3.11. It is a contradiction.

4. Conclusions

Taking the frame as the structure of truth value, we propose the notation of the generalized $\mathcal{S}$-convergence on fuzzy dcpo. It is shown that for an arbitrary fuzzy dcpo, the generalized $\mathcal{S}$-convergence is topological if and only if the fuzzy dcpo is continuous, and the topology induced by the class of the generalized $\mathcal{S}$-convergence is exactly the generalized Scott topology.

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Solutions and Properties of Some Degenerate Systems of Difference Equations

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ABSTRACT

This paper is devoted to obtain the form of the solution and the qualitative properties of the following systems of a rational difference equations of order two

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n \left(\pm 1 \pm y_n y_{n-1}\right)},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n \left(\pm 1 \pm x_n x_{n-1}\right)},$$

with positive initial conditions $x_{-1},\ x_0,\ y_{-1}$ and y_0 are nonzero real numbers. If we let $u_n = x_n x_{n-1}$ and $v_n = y_n y_{n-1}$, then these systems can be viewed as special cases of the system of the form

$$u_{n+1} = f(v_n),\ v_{n+1} = g(u_n).$$

This system has applications in modeling population growth with age structure or the dynamics of plant-herbivore interaction. Let $w_n = u_{2n}$, we have $w_{n+1} = f(g(w_n)) \equiv h(w_n)$. At a nonzero steady state w^* of the last difference equation, we have

$$|h'^{*})| = |f'(g(w*))g'(w^*)| = 1,$$

indicating that the system is degenerate at this steady state.

Keywords: difference equations, recursive sequences, stability, periodic solution, system of difference equations.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Owing to their rich dynamics, interest and scope in studying the solutions and properties of nonlinear difference equation systems is continuously expanding. In particular, there is a growing need of practical methods that explore and discuss a real life matters described by mathematical models. Such applications we find in environment as biology, genetics and economy [1, 2, 14].

There are some well documented and focused studies deal with some specific nonlinear difference equations system. For example, the periodicity of the positive solutions of the rational difference equations systems

$$x_{n+1} = \frac{1}{y_n},\ \ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},$$

has been obtained by Cinar in [3]. The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n}.$$

has been studied by Kurbanli *et al.* [4]. In [5], Ozban studied the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}},\ y_{n+1} = \frac{by_{n-3}}{x_{n-q}y_{n-q}}.$$

Touafek et al. [6] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Other similar difference equations and nonlinear systems of rational difference equations were investigated see [7]-[14].

In this paper, we deal with the existence and properties of solutions and the periodicity character of the following systems of rational difference equations with order two

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n\left(\pm 1 \pm y_n y_{n-1}\right)},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n\left(\pm 1 \pm x_n x_{n-1}\right)},$$

with nonnegative initial conditions $x_{-1},\ x_0,\ y_{-1}$ and y_0. If we let $u_n = x_n x_{n-1}$ and $v_n = y_n y_{n-1}$, then these systems can be viewed as special cases of the system of the form

$$u_{n+1} = f(v_n),\ v_{n+1} = g(u_n).$$

This system has applications in modeling population growth with age structure [2] or the dynamics of plant-herbivore interaction [14]. Let $w_n = u_{2n}$, we have $w_{n+1} = f(g(w_n)) \equiv h(w_n)$. At a nonzero steady state w^* of the last difference equation, we have

$$|h'^{*})| = |f'(g(w^*))g'(w^*)| = 1,$$

indicating that the system is degenerate at this steady state.

2. SYSTEM $X_{N+1} = \frac{Y_N Y_{N-1}}{X_N\left(1+Y_N Y_{N-1}\right)},\ Y_{N+1} = \frac{X_N X_{N-1}}{Y_N\left(1+X_N X_{N-1}\right)}$

In this section, our main goal is to obtain the solutions of the following second order system of difference equations

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n\left(1+y_n y_{n-1}\right)},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n 1 + x_n x_{n-1})}, \tag{1}$$

where $n = 0, 1, 2, ...,$ and the initial conditions are nonnegative real numbers. Before embarking on our lengthy derivation of the solutions, we would like to present some simple but interesting properties of these solutions.

If we let

$$u_n = x_n x_{n-1},\ v_n = y_n y_{n-1},$$

then

$$u_{n+1} = \frac{v_n}{1+v_n},\ v_{n+1} = \frac{u_n}{1+u_n}$$

and

$$\begin{aligned} u_{n+2} &= \frac{v_{n+1}}{1+v_{n+1}} = \frac{u_n/(1+u_n)}{1+u_n/(1+u_n)} = \frac{u_n}{1+2u_n} = f(u_n), \\ v_{n+2} &= \frac{u_{n+1}}{1+u_{n+1}} = \frac{v_n/(1+v_n)}{1+v_n/(1+v_n)} = \frac{v_n}{1+2v_n} = f(v_n). \end{aligned}$$

From which we see that if $u_0 \geq 0$ $(u_{-1} \geq 0)$, then $u_{2n} \geq 0$ $(u_{2n+1} \geq 0)$ for nonnegative integers n. This system has $(0, 0)$ as the only steady state. Observe that

$$u_{n+2} - u_n = \frac{-2u_n^2}{1+2u_n}.$$

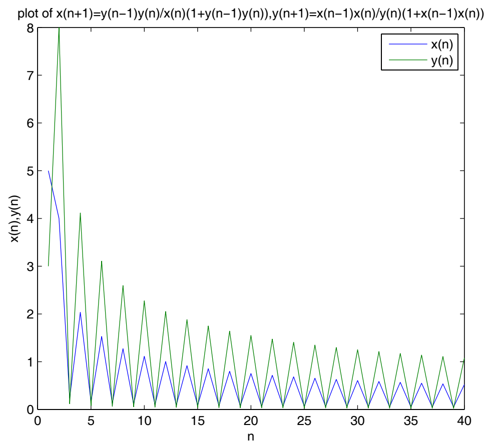

Figure 1. A solution for the difference system (1) with the initial conditions $x_{-1} = 5,\ x_0 = 4,\ y_{-1} = 3$ and $y_0 = 8$.

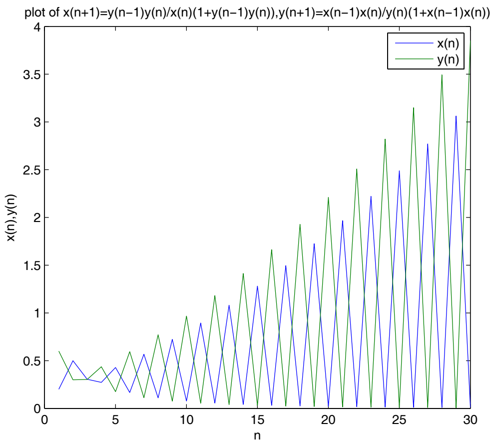

Figure 2. A solution for the difference system (1) with the initial conditions $x_{-1} = 0.2,\ x_0 = 0.5,\ y_{-1} = 0.6$ and $y_0 = 0.3$.

Hence, if $u_0 > 0$ $(u_{-1} > 0)$, then u_{2n} (u_{2n+1}) is a strictly decreasing subsequences and hence must approach the only steady state value 0. Similar argument can be made for v_n. Therefore

$$v_n \to 0 \text{ or } x_n x_{n-1} \to 0 \text{ and } u_n \to 0 \text{ or } y_n y_{n-1} \to 0.$$

In Figures 1 and 2, we present two typical solutions for the difference system (1). Observe that $x_{n+1}x_n \to 0$ and $y_{n+1}y_n \to 0$.

Theorem 2.1. Assume that $\{x_n, y_n\}$ are solutions of system (1). Then for $n = 0, 1, 2, ...$, we see that all solutions of system (1) are given by the following formula

$$x_{2n-1} = \frac{c^n d^n}{a^n b^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(2i)ab)}{(1+(2i+1)cd)}, \quad x_{2n} = \frac{a^{n+1} b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{(1+(2i+1)cd)}{(1+(2i+2)ab)},$$

and

$$y_{2n-1} = \frac{a^n b^n}{c^n d^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(2i)cd)}{(1+(2i+1)ab)}, \quad y_{2n} = \frac{c^{n+1} d^n}{a^n b^n} \prod_{i=0}^{n-1} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)},$$

where $\prod_{i=0}^{-1} A_i \equiv 1$, $x_{-1} = b, x_0 = a, y_{-1} = d$ and $y_0 = c$.

Proof: We prove it by the method of induction, for $n = 0$ the result holds. Assume the theorem is true for $n-1$, that is,

$$\begin{aligned} x_{2n-3} &= \tfrac{c^{n-1}d^{n-1}}{a^{n-1}b^{n-2}} \prod_{i=0}^{n-2} \tfrac{(1+(2i)ab)}{(1+(2i+1)cd)}, \quad x_{2n-2} = \tfrac{a^{n}b^{n-1}}{c^{n-1}d^{n-1}} \prod_{i=0}^{n-2} \tfrac{(1+(2i+1)cd)}{(1+(2i+2)ab)}, \\ y_{2n-3} &= \tfrac{a^{n-1}b^{n-1}}{c^{n-1}d^{n-2}} \prod_{i=0}^{n-2} \tfrac{(1+(2i)cd)}{(1+(2i+1)ab)}, \quad y_{2n-2} = \tfrac{c^{n}d^{n-1}}{a^{n-1}b^{n-1}} \prod_{i=0}^{n-2} \tfrac{(1+(2i+1)ab)}{(1+(2i+2)cd)}, \end{aligned}$$

are true. We will show that the relations given in the above theorem are true.

From Eq.(1) we see that

$$\begin{aligned} x_{2n-1} &= \tfrac{y_{2n-2}y_{2n-3}}{x_{2n-2}(1+y_{2n-2}y_{2n-3})} \\ &= \frac{\frac{a^{n-1}b^{n-1}}{c^{n-1}d^{n-2}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i)cd)}{(1+(2i+1)ab)} \frac{c^{n}d^{n-1}}{a^{n-1}b^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}}{\left(\frac{a^{n}b^{n-1}}{c^{n-1}d^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}\right) \left(1 + \frac{a^{n-1}b^{n-1}}{c^{n-1}d^{n-2}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i)cd)}{(1+(2i+1)ab)} \frac{c^{n}d^{n-1}}{a^{n-1}b^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}\right)} \\ &= \frac{cd \prod\limits_{i=0}^{n-2} \frac{(1+(2i)cd)}{(1+(2i+2)cd)}}{\left(\frac{a^{n}b^{n-1}}{c^{n-1}d^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}\right)\left(1+cd \prod\limits_{i=0}^{n-2} \frac{(1+(2i)cd)}{(1+(2i+2)cd)}\right)} \\ &= \frac{\frac{c^{n}d^{n}}{(1+(2n-2)cd)}}{a^{n}b^{n-1}\left(1+\frac{cd}{(1+(2n-2)cd)}\right)} \prod_{i=0}^{n-2} \frac{(1+(2i+2)ab)}{(1+(2i+1)cd)} \\ &= \frac{c^{n}d^{n}}{a^{n}b^{n-1}(1+(2n-2)cd+cd)} \prod_{i=0}^{n-2} \frac{(1+(2i+2)ab)}{(1+(2i+1)cd)} = \frac{c^{n}d^{n}}{a^{n}b^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(2i)ab)}{(1+(2i+1)cd)}, \end{aligned}$$

$$\begin{aligned} y_{2n-1} &= \tfrac{x_{2n-2}x_{2n-3}}{y_{2n-2}(1+x_{2n-2}x_{2n-3})} \\ &= \frac{\frac{c^{n-1}d^{n-1}}{a^{n-1}b^{n-2}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i)ab)}{(1+(2i+1)cd)} \frac{a^{n}b^{n-1}}{c^{n-1}d^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}}{\left(\frac{c^{n}d^{n-1}}{a^{n-1}b^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}\right) \left(1 + \frac{c^{n-1}d^{n-1}}{a^{n-1}b^{n-2}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i)ab)}{(1+(2i+1)cd)} \frac{a^{n}b^{n-1}}{c^{n-1}d^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}\right)} \\ &= \frac{\frac{ab}{(1+(2n-2)ab)}}{\left(\frac{c^{n}d^{n-1}}{a^{n-1}b^{n-1}} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}\right)\left(1+\frac{ab}{(1+(2n-2)ab)}\right)} \\ &= \frac{a^{n}b^{n} \prod\limits_{i=0}^{n-2} \frac{(1+(2i+2)cd)}{(1+(2i+1)ab)}}{(c^{n}d^{n-1})(1+(2n-2)ab+ab)} = \frac{a^{n}b^{n}}{c^{n}d^{n-1}} \prod_{i=0}^{n-1} \frac{(1+(2i)cd)}{(1+(2i+1)ab)}. \end{aligned}$$

Also, similarly from Eq.(1), we have

$$\begin{aligned}
x_{2n} &= \frac{y_{2n-1}y_{2n-2}}{x_{2n-1}(1+y_{2n-1}y_{2n-2})} \\
&= \frac{\frac{a^nb^n}{c^nd^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)cd)}{(1+(2i+1)ab)}\frac{c^nd^{n-1}}{a^{n-1}b^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}}{\frac{c^nd^n}{a^nb^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)ab)}{(1+(2i+1)cd)}\left(1+\frac{a^nb^n}{c^nd^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)cd)}{(1+(2i+1)ab)}\frac{c^nd^{n-1}}{a^{n-1}b^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}\right)} \\
&= \frac{a^nb^{n-1}\frac{ab}{(1+(2n-1)ab)}}{c^nd^n\left(1+\frac{ab}{(1+(2n-1)ab)}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i)ab)} \\
&= \frac{a^{n+1}b^n}{c^nd^n(1+(2n-1)ab+ab)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i)ab)} = \frac{a^{n+1}b^n}{c^nd^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i+2)ab)},
\end{aligned}$$

$$\begin{aligned}
y_{2n} &= \frac{x_{2n-1}x_{2n-2}}{y_{2n-1}(1+x_{2n-1}x_{2n-2})} \\
&= \frac{\frac{c^nd^n}{a^nb^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)ab)}{(1+(2i+1)cd)}\frac{a^nb^{n-1}}{c^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}}{\frac{a^nb^n}{c^nd^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)cd)}{(1+(2i+1)ab)}\left(1+\frac{c^nd^n}{a^nb^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)ab)}{(1+(2i+1)cd)}\frac{a^nb^{n-1}}{c^{n-1}d^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)ab)}\right)} \\
&= \frac{c^{n+1}d^n\frac{1}{(1+(2n-1)cd)}}{a^nb^n\left(1+\frac{cd}{(1+(2n-1)cd)}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)}{(1+(2i)cd)} = \frac{c^{n+1}d^n}{a^nb^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}.
\end{aligned}$$

The proof is complete.

The following case can be treated similarly

Theorem 2.2. The solutions of the system

$$x_{n+1} = \frac{y_ny_{n-1}}{x_n(1-y_ny_{n-1})},\ y_{n+1} = \frac{x_nx_{n-1}}{y_n(1-x_nx_{n-1})},$$

are given by

$$\begin{aligned}
x_{2n-1} &= \frac{c^nd^n}{a^nb^{n-1}}\prod_{i=0}^{n-1}\frac{(1-(2i)ab)}{(1-(2i+1)cd)},\quad x_{2n} = \frac{a^{n+1}b^n}{c^nd^n}\prod_{i=0}^{n-1}\frac{(1-(2i+1)cd)}{(1-(2i+2)ab)}, \\
y_{2n-1} &= \frac{a^nb^n}{c^nd^{n-1}}\prod_{i=0}^{n-1}\frac{(1-(2i)cd)}{(1-(2i+1)ab)},\quad y_{2n} = \frac{c^{n+1}d^n}{a^nb^n}\prod_{i=0}^{n-1}\frac{(1-(2i+1)ab)}{(1-(2i+2)cd)}.
\end{aligned}$$

Theorem 2.3. Assume that $\{x_n, y_n\}$ are solutions of the system

$$x_{n+1} = \frac{y_ny_{n-1}}{x_n(-1+y_ny_{n-1})},\ y_{n+1} = \frac{x_nx_{n-1}}{y_n(-1-x_nx_{n-1})}.$$

Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{2n-1} &= \frac{c^nd^n}{a^nb^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)ab)}{(-1+(2i+1)cd)},\quad x_{2n} = \frac{a^{n+1}b^n}{c^nd^n}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)cd)}{(1+(2i+2)ab)}, \\
y_{2n-1} &= \frac{a^nb^n}{c^nd^{n-1}}\prod_{i=0}^{n-1}\frac{(-1+(2i)cd)}{(1+(2i+1)ab)},\quad y_{2n} = \frac{c^{n+1}d^n}{a^nb^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)}{(-1+(2i+2)cd)}.
\end{aligned}$$

Theorem 2.4. The system

$$x_{n+1} = \frac{y_ny_{n-1}}{x_n(-1-y_ny_{n-1})},\ y_{n+1} = \frac{x_nx_{n-1}}{y_n(-1+x_nx_{n-1})},$$

has the solutions which given by

$$\begin{aligned} x_{2n-1} &= \tfrac{c^n d^n}{a^n b^{n-1}} \prod_{i=0}^{n-1} \tfrac{(1-(2i)ab)}{(-1-(2i+1)cd)}, \quad x_{2n} = \tfrac{a^{n+1} b^n}{c^n d^n} \prod_{i=0}^{n-1} \tfrac{(-1-(2i+1)cd)}{(1-(2i+2)ab)}, \\ y_{2n-1} &= \tfrac{a^n b^n}{c^n d^{n-1}} \prod_{i=0}^{n-1} \tfrac{(-1-(2i)cd)}{(1-(2i+1)ab)}, \quad y_{2n} = \tfrac{c^{n+1} d^n}{a^n b^n} \prod_{i=0}^{n-1} \tfrac{(1-(2i+1)ab)}{(-1-(2i+2)cd)}. \end{aligned}$$

3. SYSTEM $X_{N+1} = \dfrac{Y_N Y_{N-1}}{X_N\left(1 + Y_N Y_{N-1}\right)}$, $Y_{N+1} = \dfrac{X_N X_{N-1}}{Y_N\left(-1 + X_N X_{N-1}\right)}$

In this section, our main goal is to obtain the solutions of the following second order system of difference equations

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n\left(1 + y_n y_{n-1}\right)}, \; y_{n+1} = \frac{x_n x_{n-1}}{y_n\left(-1 + x_n x_{n-1}\right)}, \tag{2}$$

where $n = 0, 1, 2, ...$, and the initial conditions are nonnegative real numbers with $x_{-1}x_0 \neq 1, \neq \frac{1}{2}$, and $y_{-1}y_0 \neq 1$.

If we let

$$u_n = x_n x_{n-1}, \; v_n = y_n y_{n-1}$$

then

$$u_{n+1} = \frac{v_n}{1 + v_n}, \; v_{n+1} = \frac{u_n}{-1 + u_n},$$

and

$$\begin{aligned} u_{n+2} &= \tfrac{v_{n+1}}{1+v_{n+1}} = \tfrac{u_n/(-1+u_n)}{1+u_n/(-1+u_n)} = \tfrac{u_n}{-1+2u_n} = g(u_n), \\ v_{n+2} &= \tfrac{u_{n+1}}{-1+u_{n+1}} = \tfrac{v_n/(1+v_n)}{-1+v_n/(1+v_n)} = -v_n. \end{aligned}$$

This suggests that the y_n will alternate signs at least every 3 units of time. The system has two steady states (0, 0) and (1, 1). From the fact that $v_{n+2} = -v_n$, we have

$$\frac{y_{n+2}}{y_n} = -\frac{y_{n-1}}{y_{n+1}} = \frac{y_n}{y_{n-2}}$$

Let $r = \frac{y_2}{y_0}$, then we have

$$\frac{y_{2n}}{y_0} = (-1)^{2(n-1)} \frac{y_{2n}}{y_{2n-2}} \frac{y_{2n-2}}{y_{2n-4}} ... \frac{y_2}{y_0} = r^n$$

and hence $y_{2n} = y_0 r^n$. Similarly, we have and $y_{2n+1} = (-1)^n y_{-1} a^n$ where $a = y_1/y_{-1}$. This shows that the values of the highs and lows grow or decay exponentially.

Observe that

$$\frac{dg(0)}{dv_n} = \left.\tfrac{-1+2v_n-2v_n}{(-1+2v_n)^2}\right|_{v_n=0} = -1$$

and

$$\frac{dg(1)}{dv_n} = \left.\tfrac{-1+2v_n-2v_n}{(-1+2v_n)^2}\right|_{v_n=0} = -1.$$

This indicates that both are degenerate steady states. Figure 3 depicts a typical solution of the difference equations system (2).

Theorem 3.1. Let $\{x_n, y_n\}_{n=-1}^{+\infty}$ be solutions of system (2). Then $\{x_n\}_{n=-1}^{+\infty}$ and $\{y_n\}_{n=-1}^{+\infty}$ are given by the following formula for $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{4n-1} &= \tfrac{c^{2n} d^{2n} (-1+2ab)^n}{a^{2n} b^{2n-1} (-1+c^2 d^2)^n}, \quad x_{4n} = \tfrac{a^{2n+1} b^{2n} (-1+c^2 d^2)^n}{c^{2n} d^{2n} (-1+2ab)^n}, \\ x_{4n+1} &= \tfrac{c^{2n+1} d^{2n+1} (-1+2ab)^n}{a^{2n+1} b^{2n} (1+cd)(-1+c^2 d^2)^n}, x_{4n+2} = \tfrac{a^{2n+2} b^{2n+1} (1+cd)(-1+c^2 d^2)^n}{c^{2n+1} d^{2n+1} (-1+2ab)^{n+1}}, \end{aligned}$$

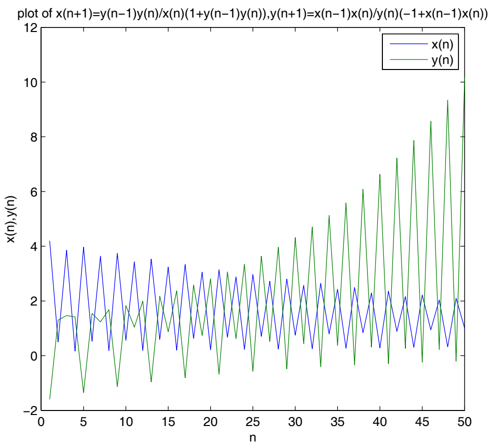

Figure 3. A typical solution of the difference equations system (2). The initial conditions are $x_{-1} = 4.2,\ x_0 = 0.5,\ y_{-1} = -1.6$ and $y_0 = 1.3$. Notice that the y_n alternates signs at least every 3 units of time. and the values of the highs and lows grow or decay exponentially.

and

$$
\begin{aligned}
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(-1+ab)^{2n}},\ y_{4n} = \frac{c^{2n+1}d^{2n}(-1+ab)^{2n}}{a^{2n}b^{2n}},\\
y_{4n+1} &= \frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(-1+ab)^{2n+1}}, y_{4n+2} = -\frac{c^{2n+2}d^{2n+1}(-1+ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}
$$

Proof: For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. that is,

$$
\begin{aligned}
x_{4n-3} &= \frac{c^{2n-1}d^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-2}(1+cd)(-1+c^2d^2)^{n-1}},\ x_{4n-2} = \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{c^{2n-1}d^{2n-1}(-1+2ab)^{n}},\\
y_{4n-3} &= \frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(-1+ab)^{2n-1}}, \quad y_{4n-2} = -\frac{c^{2n}d^{2n-1}(-1+ab)^{2n-1}}{a^{2n-1}b^{2n-1}}.
\end{aligned}
$$

Now it follows from Eq.(2) that

$$
\begin{aligned}
x_{4n-1} &= \frac{y_{4n-2}y_{4n-3}}{x_{4n-2}(1+y_{4n-2}y_{4n-3})}\\
&= \frac{-\frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(-1+ab)^{2n-1}}\frac{c^{2n}d^{2n-1}(-1+ab)^{2n-1}}{a^{2n-1}b^{2n-1}}}{\left(\frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{c^{2n-1}d^{2n-1}(-1+2ab)^{n}}\right)\left(1-\frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(-1+ab)^{2n-1}}\frac{c^{2n}d^{2n-1}(-1+ab)^{2n-1}}{a^{2n-1}b^{2n-1}}\right)}\\
&= \frac{-cdc^{2n-1}d^{2n-1}(-1+2ab)^{n}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}(1-cd)} = \frac{c^{2n}d^{2n}(-1+2ab)^{n}}{a^{2n}b^{2n-1}(-1+c^2d^2)^{n}},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{4n-1} &= \frac{x_{4n-2}x_{4n-3}}{y_{4n-2}(-1+x_{4n-2}x_{4n-3})}\\
&= \frac{\frac{c^{2n-1}d^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-2}(1+cd)(-1+c^2d^2)^{n-1}}\frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{c^{2n-1}d^{2n-1}(-1+2ab)^{n}}}{\left(-\frac{c^{2n}d^{2n-1}(-1+ab)^{2n-1}}{a^{2n-1}b^{2n-1}}\right)\left(-1+\frac{c^{2n-1}d^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-2}(1+cd)(-1+c^2d^2)^{n-1}}\frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{c^{2n-1}d^{2n-1}(-1+2ab)^{n}}\right)}\\
&= \frac{\frac{ab}{(-1+2ab)}a^{2n-1}b^{2n-1}}{-c^{2n}d^{2n-1}(-1+ab)^{2n-1}\left(-1+\frac{ab}{(-1+2ab)}\right)} = \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(-1+ab)^{2n}}.
\end{aligned}
$$

Also, we can prove the other relations. This completes the proof.

We consider the following systems and the proof of the theorems are similar to above theorem and so, left to the reader.

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(1-y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(-1-x_n x_{n-1})} \tag{3}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(-1+y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(1-x_n x_{n-1})} \tag{4}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(-1-y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(1+x_n x_{n-1})} \tag{5}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(1+y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(-1-x_n x_{n-1})} \tag{6}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(1-y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(-1+x_n x_{n-1})} \tag{7}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(-1+y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(1+x_n x_{n-1})} \tag{8}$$
$$x_{n+1} = \frac{y_n y_{n-1}}{x_n(-1-y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1}}{y_n(1-x_n x_{n-1})} \tag{9}$$

We will devote, for example, in the following theorems the form of the solutions of systems (3) and (8).

Theorem 3.2. Let $\{x_n, y_n\}_{n=-1}^{+\infty}$ be solutions of system (3) and $x_{-1}x_0 \neq -1, \neq -\frac{1}{2}$, $y_{-1}y_0 \neq 1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}(-1-2ab)^n}{a^{2n}b^{2n-1}(-1+c^2d^2)^n}, \qquad x_{4n} = \frac{a^{2n+1}b^{2n}(-1+c^2d^2)^n}{c^{2n}d^{2n}(-1-2ab)^n},\\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}(-1-2ab)^n}{a^{2n+1}b^{2n}(1-cd)(-1+c^2d^2)^n}, \quad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(1-cd)(-1+c^2d^2)^n}{c^{2n+1}d^{2n+1}(-1-2ab)^{n+1}},\\
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1+ab)^{2n}}, \qquad y_{4n} = \frac{c^{2n+1}d^{2n}(1+ab)^{2n}}{a^{2n}b^{2n}},\\
y_{4n+1} &= -\frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(1+ab)^{2n+1}}, \quad y_{4n+2} = \frac{c^{2n+2}d^{2n+1}(1+ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}$$

Theorem 3.3. Assume that $\{x_n, y_n\}$ are solutions of system (4) with $x_{-1}x_0 \neq 1, \neq \frac{1}{2}$, and $y_{-1}y_0 \neq 1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}(-1+2ab)^n}{a^{2n}b^{2n-1}(-1+c^2d^2)^n}, \qquad x_{4n} = \frac{a^{2n+1}b^{2n}(-1+c^2d^2)^n}{c^{2n}d^{2n}(-1+2ab)^n},\\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}(-1+2ab)^n}{a^{2n+1}b^{2n}(-1+cd)(-1+c^2d^2)^n}, \quad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(-1+cd)(-1+c^2d^2)^n}{c^{2n+1}d^{2n+1}(-1+2ab)^{n+1}},\\
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(-1+ab)^{2n}}, \qquad y_{4n} = \frac{c^{2n+1}d^{2n}(-1+ab)^{2n}}{a^{2n}b^{2n}},\\
y_{4n+1} &= -\frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(-1+ab)^{2n+1}}, \quad y_{4n+2} = \frac{c^{2n+2}d^{2n+1}(-1+ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}$$

Theorem 3.4. Suppose that $\{x_n, y_n\}$ are solutions of system (5) such that $x_{-1}x_0 \neq -1, \neq -\frac{1}{2}$, and $y_{-1}y_0 \neq 1$.. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}(-1-2ab)^n}{a^{2n}b^{2n-1}(-1+c^2d^2)^n}, \qquad x_{4n} = \frac{a^{2n+1}b^{2n}(-1+c^2d^2)^n}{c^{2n}d^{2n}(-1-2ab)^n},\\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}(-1-2ab)^n}{a^{2n+1}b^{2n}(-1-cd)(-1+c^2d^2)^n}, \quad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(-1-cd)(-1+c^2d^2)^n}{c^{2n+1}d^{2n+1}(-1-2ab)^{n+1}},\\
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1+ab)^{2n}}, \qquad y_{4n} = \frac{c^{2n+1}d^{2n}(1+ab)^{2n}}{a^{2n}b^{2n}},\\
y_{4n+1} &= \frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(1+ab)^{2n+1}}, \quad y_{4n+2} = -\frac{c^{2n+2}d^{2n+1}(1+ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}$$

Theorem 3.5. If $\{x_n, y_n\}$ are solutions of system (6) and $x_{-1}x_0 \neq 1$, and $y_{-1}y_0 \neq -1, \neq -\frac{1}{2}$. Then the solutions are given by

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1+cd)^{2n}}, \qquad x_{4n} = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{c^{2n}d^{2n}},\\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(1+cd)^{2n+1}}, \qquad x_{4n+2} = -\frac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}},\\
y_{4n-1} &= \frac{a^{2n}b^{2n}(-1-2cd)^n}{c^{2n}d^{2n-1}(-1+a^2b^2)^n}, \qquad y_{4n} = \frac{c^{2n+1}d^{2n}(-1+a^2b^2)^n}{a^{2n}b^{2n}(-1-2cd)^n},\\
y_{4n+1} &= \frac{a^{2n+1}b^{2n+1}(-1-2cd)^n}{c^{2n+1}d^{2n}(-1-ab)(-1+a^2b^2)^n}, \quad y_{4n+2} = \frac{c^{2n+2}d^{2n+1}(-1-ab)(-1+a^2b^2)^n}{a^{2n+1}b^{2n+1}(-1-2cd)^{n+1}}.
\end{aligned}$$

Theorem 3.6. The solutions of the system (7) with non zero initial conditions real numbers with $x_{-1}x_0 \neq \pm 1$, and $y_{-1}y_0 \neq 1, \neq \frac{1}{2}$ are given by

$$\begin{aligned}
x_{4n-1} &= \tfrac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(-1+cd)^{2n}}, \qquad x_{4n} = \tfrac{a^{2n+1}b^{2n}(-1+cd)^{2n}}{c^{2n}d^{2n}}, \\
x_{4n+1} &= -\tfrac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(-1+cd)^{2n+1}}, \quad x_{4n+2} = \tfrac{a^{2n+2}b^{2n+1}(-1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\
y_{4n-1} &= \tfrac{a^{2n}b^{2n}(-1+2cd)^{n}}{c^{2n}d^{2n-1}(-1+a^2b^2)^{n}}, \qquad y_{4n} = \tfrac{c^{2n+1}d^{2n}(-1+a^2b^2)^{n}}{a^{2n}b^{2n}(-1+2cd)^{n}}, \\
y_{4n+1} &= \tfrac{a^{2n+1}b^{2n+1}(-1+2cd)^{n}}{c^{2n+1}d^{2n}(-1+ab)(-1+a^2b^2)^{n}}, \quad y_{4n+2} = \tfrac{c^{2n+2}d^{2n+1}(-1+ab)(-1+a^2b^2)^{n}}{a^{2n+1}b^{2n+1}(-1+2cd)^{n+1}}.
\end{aligned}$$

Theorem 3.7. Assume that $\{x_n, y_n\}$ are solutions of system (8) and $x_{-1}x_0 \neq \pm 1$, $y_{-1}y_0 \neq 1, \neq \frac{1}{2}$, then

$$\begin{aligned}
x_{4n-1} &= \tfrac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(-1+cd)^{2n}}, \qquad x_{4n} = \tfrac{a^{2n+1}b^{2n}(-1+cd)^{2n}}{c^{2n}d^{2n}}, \\
x_{4n+1} &= \tfrac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(-1+cd)^{2n+1}}, \quad x_{4n+2} = -\tfrac{a^{2n+2}b^{2n+1}(-1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\
y_{4n-1} &= \tfrac{a^{2n}b^{2n}(-1+2cd)^{n}}{c^{2n}d^{2n-1}(-1+a^2b^2)^{n}}, \qquad y_{4n} = \tfrac{c^{2n+1}d^{2n}(-1+a^2b^2)^{n}}{a^{2n}b^{2n}(-1+2cd)^{n}}, \\
y_{4n+1} &= \tfrac{a^{2n+1}b^{2n+1}(-1+2cd)^{n}}{c^{2n+1}d^{2n}(1+ab)(-1+a^2b^2)^{n}}, \quad y_{4n+2} = \tfrac{c^{2n+2}d^{2n+1}(1+ab)(-1+a^2b^2)^{n}}{a^{2n+1}b^{2n+1}(-1+2cd)^{n+1}}.
\end{aligned}$$

Theorem 3.8. Suppose that $\{x_n, y_n\}$ are solutions of system (9) with $x_{-1}x_0 \neq \pm 1$, and $y_{-1}y_0 \neq -1, \neq -\frac{1}{2}$ then the solutions of system (9) are given by

$$\begin{aligned}
x_{4n-1} &= \tfrac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1+cd)^{2n}}, \quad x_{4n} = \tfrac{a^{2n+1}b^{2n}(1+cd)^{2n}}{c^{2n}d^{2n}}, \\
x_{4n+1} &= -\tfrac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(1+cd)^{2n+1}}, \; x_{4n+2} = \tfrac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\
y_{4n-1} &= \tfrac{a^{2n}b^{2n}(-1-2cd)^{n}}{c^{2n}d^{2n-1}(-1+a^2b^2)^{n}}, \quad y_{4n} = \tfrac{c^{2n+1}d^{2n}(-1+a^2b^2)^{n}}{a^{2n}b^{2n}(-1-2cd)^{n}}, \\
y_{4n+1} &= \tfrac{a^{2n+1}b^{2n+1}(-1-2cd)^{n}}{c^{2n+1}d^{2n}(1-ab)(-1+a^2b^2)^{n}}, \; y_{4n+2} = \tfrac{c^{2n+2}d^{2n+1}(1-ab)(-1+a^2b^2)^{n}}{a^{2n+1}b^{2n+1}(-1-2cd)^{n+1}}.
\end{aligned}$$

4. SYSTEM $X_{N+1} = \dfrac{Y_N Y_{N-1}}{X_N\left(1 + Y_N Y_{N-1}\right)}$, $Y_{N+1} = \dfrac{X_N X_{N-1}}{Y_N\left(1 - X_N X_{N-1}\right)}$

In this section, our main goal is to obtain the solutions of the following second order system of difference equations

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n\left(1 + y_n y_{n-1}\right)}, \; y_{n+1} = \frac{x_n x_{n-1}}{y_n\left(1 - x_n x_{n-1}\right)}, \tag{10}$$

where $n = 0, 1, 2, ...$ and the initial conditions x_{-1}, x_0, y_{-1} *and* y_0 are arbitrary nonzero real numbers with $x_{-1}x_0 \neq 1$, and $y_{-1}y_0 \neq -1$.

From (10), if we take

$$u_n = x_n x_{n-1}, \; v_n = y_n y_{n-1}$$

then

$$u_{n+1} = \frac{v_n}{1 + v_n}, \; v_{n+1} = \frac{u_n}{1 - u_n}$$

and

$$\begin{aligned}
u_{n+2} &= \frac{v_{n+1}}{1 + v_{n+1}} = \frac{u_n/(1-u_n)}{1 + u_n/(1-u_n)} = u_n, \\
v_{n+2} &= \frac{u_{n+1}}{1 - u_{n+1}} = \frac{v_n/(1+v_n)}{1 - v_n/(1+v_n)} = v_n.
\end{aligned}$$

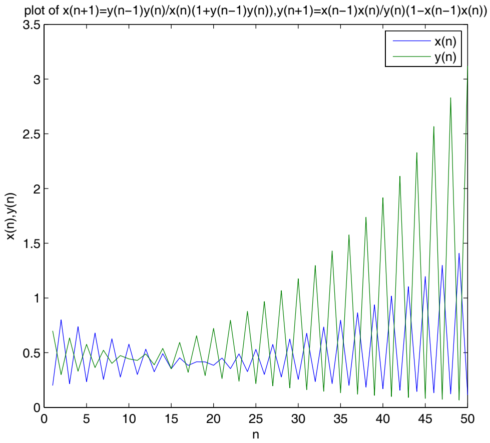

Figure 4. The behavior of a typical solution of the difference system (10). The initial conditions are $x_{-1} = 0.2$, $x_0 = 0.8$, $y_{-1} = 0.7$ and $y_0 = 0.3$. Observe that as indicated by our theoretical result, the peak and trough grow and decay exponentially.

It is easy to see that if initial values are positive and such that $u_0 < 1$, then $u_n < 1$ for all positive integers and hence such a solution stays positive. In addition, from the property $u_{n+2} = u_n$, we see that

$$\frac{x_{n+2}}{x_n} = \frac{x_{n-1}}{x_{n+1}} = \frac{x_n}{x_{n-2}}$$

Let $r = \frac{x_2}{x_0}$, then we have $\frac{x_{2n}}{x_0} = \frac{x_{2n}}{x_{2n-2}}\frac{x_{2n-2}}{x_{2n-4}}...\frac{x_2}{x_0} = r^n$ and hence $x_{2n} = x_0 r^n$. Similarly, we have and $x_{2n-1} = x_{-1}a^n$ where $a = x_1/x_{-1}$. This shows that the values of the highs and lows grow and decay exponentially. Figure **??** shows the behavior of a typical solution of the difference system (10).

Theorem 4.1. If $\{x_n, y_n\}$ are solutions of difference equation system (10), then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1+cd)^{2n}}, \qquad x_{4n} = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{c^{2n}d^{2n}},\\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(1+cd)^{2n+1}}, \qquad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}},\\
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1-ab)^{2n}}, \qquad y_{4n} = \frac{c^{2n+1}d^{2n}(1-ab)^{2n}}{a^{2n}b^{2n}},\\
y_{4n+1} &= \frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(1-ab)^{2n+1}}, \qquad y_{4n+2} = \frac{c^{2n+2}d^{2n+1}(1-ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}$$

Proof: For $n = 0$, the result holds. Now, suppose that $n > 1$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{4n-3} &= \frac{c^{2n-1}d^{2n-1}}{a^{2n-1}b^{2n-2}(1+cd)^{2n-1}}, \ x_{4n-2} = \frac{a^{2n}b^{2n-1}(1+cd)^{2n-1}}{c^{2n-1}d^{2n-1}},\\
y_{4n-3} &= \frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(1-ab)^{2n-1}}, \ y_{4n-2} = \frac{c^{2n}d^{2n-1}(1-ab)^{2n-1}}{a^{2n-1}b^{2n-1}}.
\end{aligned}$$

It follows from Eq.(10) that

$$\begin{aligned}
x_{4n-1} &= \frac{y_{4n-2}y_{4n-3}}{x_{4n-2}(1+y_{4n-2}y_{4n-3})}\\
&= \frac{\frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(1-ab)^{2n-1}}\frac{c^{2n}d^{2n-1}(1-ab)^{2n-1}}{a^{2n-1}b^{2n-1}}}{\left(\frac{a^{2n}b^{2n-1}(1+cd)^{2n-1}}{c^{2n-1}d^{2n-1}}\right)\left(1+\frac{a^{2n-1}b^{2n-1}}{c^{2n-1}d^{2n-2}(1-ab)^{2n-1}}\frac{c^{2n}d^{2n-1}(1-ab)^{2n-1}}{a^{2n-1}b^{2n-1}}\right)}\\
&= \frac{cdc^{2n-1}d^{2n-1}}{a^{2n}b^{2n-1}(1+cd)^{2n-1}(1+cd)} = \frac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1+cd)^{2n}},
\end{aligned}$$

$$\begin{aligned}
y_{4n-1} &= \frac{x_{4n-2}x_{4n-3}}{y_{4n-2}(1-x_{4n-2}x_{4n-3})} \\
&= \frac{\frac{c^{2n-1}d^{2n-1}}{a^{2n-1}b^{2n-2}(1+cd)^{2n-1}}\frac{a^{2n}b^{2n-1}(1+cd)^{2n-1}}{c^{2n-1}d^{2n-1}}}{\left(\frac{c^{2n}d^{2n-1}(1-ab)^{2n-1}}{a^{2n-1}b^{2n-1}}\right)\left(1-\frac{c^{2n-1}d^{2n-1}}{a^{2n-1}b^{2n-2}(1+cd)^{2n-1}}\frac{a^{2n}b^{2n-1}(1+cd)^{2n-1}}{c^{2n-1}d^{2n-1}}\right)} \\
&= \frac{aba^{2n-1}b^{2n-1}}{c^{2n}d^{2n-1}(1-ab)^{2n-1}(1-ab)} = \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1-ab)^{2n}}.
\end{aligned}$$

Also, one can prove other cases. This completes the proof.

The solutions of system (10) are unbounded except in the following case.

Theorem 4.2. System (10) has a periodic solution of period two if and only if $x_{-1}x_0 = 2,\ y_0y_{-1} = -2$, and will be taken the form $\{x_n\} = \{b, a, b, a, ...\}, \{y_n\} = \{d, c, d, c, ...\}$.

Proof: First suppose that there exists a prime period two solution

$$\{x_n\} = \{b, a, b, a, ...\}, \quad \{y_n\} = \{d, c, d, c, ...\},$$

of system (10), we see from the form of the solution of system (10) that

$$\begin{aligned}
b &= \frac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1+cd)^{2n}}, \quad a = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{c^{2n}d^{2n}}, \\
b &= \frac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(1+cd)^{2n+1}}, \quad a = \frac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\
d &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1-ab)^{2n}}, \quad c = \frac{c^{2n+1}d^{2n}(1-ab)^{2n}}{a^{2n}b^{2n}}, \\
d &= \frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(1-ab)^{2n+1}}, \quad c = \frac{c^{2n+2}d^{2n+1}(1-ab)^{2n+1}}{a^{2n+1}b^{2n+1}}.
\end{aligned}$$

Then we get

$$(ab)^{2n} = (cd)^{2n} \quad and \quad 1-ab = -1, \quad 1+cd = -1.$$

Thus

$$ab = 2, \quad cd = -2.$$

Second assume that $ab = 2,\ cd = -2$. Then we see from the form of the solution of system (10) that

$$\begin{aligned}
x_{4n-1} &= b, \ x_{4n} = a, \ x_{4n+1} = b, \ x_{4n+2} = a, \\
y_{4n-1} &= d, \ y_{4n} = c, \ y_{4n+1} = d, \ y_{4n+2} = c.
\end{aligned}$$

Thus we have a periodic solution of period two and the proof is complete.

In a similar fashion, we can obtain the following theorems.

Theorem 4.3. If $\{x_n, y_n\}$ are solutions of the following difference equation system

$$x_{n+1} = \frac{y_ny_{n-1}}{x_n\left(1-y_ny_{n-1}\right)}, \ y_{n+1} = \frac{x_nx_{n-1}}{y_n\left(1+x_nx_{n-1}\right)}, \tag{11}$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-1}x_0 \neq -1,\ y_0y_{-1} \neq 1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{4n-1} &= \frac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(1-cd)^{2n}}, \quad x_{4n} = \frac{a^{2n+1}b^{2n}(1-cd)^{2n}}{c^{2n}d^{2n}}, \\
x_{4n+1} &= \frac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(1-cd)^{2n+1}}, \quad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(1-cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\
y_{4n-1} &= \frac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(1+ab)^{2n}}, \quad y_{4n} = \frac{c^{2n+1}d^{2n}(1+ab)^{2n}}{a^{2n}b^{2n}}, \\
y_{4n+1} &= \frac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(1+ab)^{2n+1}}, \quad y_{4n+2} = \frac{c^{2n+2}d^{2n+1}(1+ab)^{2n+1}}{a^{2n+1}b^{2n+1}},
\end{aligned}$$

and all these solutions are unbounded except if $x_{-1}x_0 = -2,\ y_0y_{-1} = 2$, then the system (11) has a periodic solution of period two in the form $\{x_n\} = \{b, a, b, a, ...\}, \{y_n\} = \{d, c, d, c, ...\}$.

Theorem 4.4. The solutions of the following two systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n \left(-1 \pm y_n y_{n-1}\right)}, \; y_{n+1} = \frac{x_n x_{n-1}}{y_n \left(-1 \pm x_n x_{n-1}\right)}, \tag{12}$$

where the initial conditions are arbitrary nonzero real numbers with $x_{-1}x_0, \; y_0 y_{-1} \neq \pm 1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{4n-1} &= \tfrac{c^{2n}d^{2n}}{a^{2n}b^{2n-1}(-1\pm cd)^{2n}}, \quad x_{4n} = \tfrac{a^{2n+1}b^{2n}(-1\pm cd)^{2n}}{c^{2n}d^{2n}}, \\ x_{4n+1} &= \tfrac{c^{2n+1}d^{2n+1}}{a^{2n+1}b^{2n}(-1\pm cd)^{2n+1}}, \; x_{4n+2} = \tfrac{a^{2n+2}b^{2n+1}(-1\pm cd)^{2n+1}}{c^{2n+1}d^{2n+1}}, \\ y_{4n-1} &= \tfrac{a^{2n}b^{2n}}{c^{2n}d^{2n-1}(-1\pm ab)^{2n}}, \quad y_{4n} = \tfrac{c^{2n+1}d^{2n}(-1\pm ab)^{2n}}{a^{2n}b^{2n}}, \\ y_{4n+1} &= \tfrac{a^{2n+1}b^{2n+1}}{c^{2n+1}d^{2n}(-1\pm ab)^{2n+1}}, \; y_{4n+2} = \tfrac{c^{2n+2}d^{2n+1}(-1\pm ab)^{2n+1}}{a^{2n+1}b^{2n+1}}. \end{aligned}$$

Theorem 4.5. Systems (12) have a periodic solutions of period two if and only if $x_{-1}x_0 = y_0 y_{-1} = \pm 2$, and will be in the form $\{x_n\} = \{b, a, b, a, ...\}, \{y_n\} = \{d, c, d, c, ...\}$.

5. CONCLUSION

This paper discussed the existence of solutions and periodicity of all cases of the systems of difference equations $x_{n+1} = \frac{y_n y_{n-1}}{x_n(\pm 1 \pm y_n y_{n-1})}$, $y_{n+1} = \frac{x_n x_{n-1}}{y_n(\pm 1 \pm x_n x_{n-1})}$. In Section 2, we obtained the form of the solution of the system $x_{n+1} = \frac{y_n y_{n-1}}{x_n(1+y_n y_{n-1})}$, $y_{n+1} = \frac{x_n x_{n-1}}{y_n(1+x_n x_{n-1})}$ and other similar cases. In Section 3, we have got the expressions of the solutions of some cases of the systems especially $x_{n+1} = \frac{y_n y_{n-1}}{x_n(1+y_n y_{n-1})}$, $y_{n+1} = \frac{x_n x_{n-1}}{y_n(-1+x_n x_{n-1})}$. In Section 4, we proved that the solution of the system $x_{n+1} = \frac{y_n y_{n-1}}{x_n(1+y_n y_{n-1})}$, $y_{n+1} = \frac{x_n x_{n-1}}{y_n(1-x_n x_{n-1})}$ unbounded and has a periodic solution of period two under some conditions and we have written the specific solutions of this system, other systems studied. Finally, using Matlab we gave numerical examples of some cases and drew them to support our results.

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Robust Observer Design for Linear Discrete Periodic Systems *

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Abstract

In this paper, an approach to the design of robust observer for linear discrete-time periodic systems is proposed. By utilizing some algebraic techniques, a robustness index is deduced. On the robustness, the needed computation mainly consists in designing parameterized observers for this type of systems and solving an optimization problem. The proposed approach is finally illustrated by a simulation example.

Keywords: Observer design; Linear discrete-time systems; Periodic systems; Robustness.

1 Introduction

In recent years, linear discrete-time periodic systems have attracted considerable attention, since they are regarded as intermediate class of systems connecting LTI systems and time varying ones. Linear periodic systems arise often in a large spectrum of different fields, ranging from economics and management, to biology, control, etc. Thus, this type of systems have been widely researched (see [1]-[3] and references therein). An important aspect of this type of systems is that periodic controllers can be used to deal with problems which time-invariant controllers can not solve (e.g., [4], [5]). In addition, it is turned out that control performance of the closed-loop systems can be improved by using periodic controllers (e.g., [6], [7]).

Lifting techniques, which convert linear discrete periodic systems into LTI systems, are usually used in system analysis and design ([8]). structural properties such as observability, reachability, detectability, and stabilizability can be equivalently analyzed by making reference to the corresponding lifted LTI systems ([9]). Lifting techniques, in particular, have been used to study zeros, robust stabilization, pole assignment, and state and output feedback stabilization for discrete-time linear periodic systems ([10]).

The states of a system can be used to form different control laws to realize different control purposes. But the states of a system are not always measurable in practice. Therefore, the state observers will be used to reconstruct the states of a dynamic system(e.g., [11]-[13]). Observer-based control has been applied to not only linear systems (e.g., [14]-[16]), but also nonlinear systems (e.g., [17], [18]). In linear discrete periodic control area, periodic observer-based residual generators are usually designed to study fault detection problems (e.g., [19]-[21]). But as far as we know, in the existing research results, little attention is given to the robustness of observers for linear discrete periodic systems. This is imprudent especially when there are some pertubations or uncertainties in system matrix data, which can leads to the inaccuracy of observed results.

In this paper, by transforming the state observers design problem into poles assignment problem and utilizing our recent results on poles assignment, numerous parametric explicit observers are provided in the form of recursion. Then an algorithm for parametric observers design is presented. To measure robustness, a performance index is proposed, which reflects the insensitivity of the closed loop system to a certain

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pertubations on system matrices. Degrees of freedom existing in the parametric observers are used to realize robust observer design by solving an optimization problem. A numerical example is employed to illustrate the validity of the algorithms, and the results show that advantage of the robust design approach is very significant.

Notation 1 *Throughout this paper, the notation* $\overline{i,j}$ *denotes the integer set* $\{i, i+1, \ldots, j-1, j\}$, $\|\cdot\|_{\mathrm{F}}$ *represents the Frobenius-norm and* $\kappa_{\mathrm{F}}(A) = \left\|A^{-1}\right\|_{\mathrm{F}} \|A\|_{\mathrm{F}}$ *is the Frobenius-norm conditional number of matrix* A, $\mathrm{O}(\varepsilon)$ *indicates a matrix function of* ε *and all of whose elements have the orders larger than or equal to* ε.

2 Preliminaries

Consider an observable linear discrete periodic system represented by the following state-space model

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, the set of integers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$ and $y(t) \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, $A(t)$, $B(t)$, $C(t)$ are matrices of compatible dimensions with the T-periodic property

$$A(t+T) = A(t), \quad B(t+T) = B(t), C(t+T) = C(t)$$

The lifted LTI system of system (1) has the following description

$$\begin{cases} x^{\mathrm{L}}(t+1) = A^{\mathrm{L}}x^{\mathrm{L}}(t) + B^{\mathrm{L}}u^{\mathrm{L}}(t) \\ y^{\mathrm{L}}(t) = C^{\mathrm{L}}x^{\mathrm{L}}(t) \end{cases} \tag{2}$$

where

$$A^{\mathrm{L}} = A(T-1)\cdots A(0) \tag{3}$$

$$B^{\mathrm{L}} = \left[\begin{array}{cccc} A(T-1)A(T-2)\cdots A(1)B(0) & \cdots & A(T-1)B(T-2) & B(T-1) \end{array}\right]$$

$$C^{\mathrm{L}} = \left[\begin{array}{c} C(0) \\ C(1)A(0) \\ \vdots \\ C(T-1)A(T-2)\cdots A(0) \end{array}\right] \tag{4}$$

$$x^{\mathrm{L}}(t) = x(tT)$$

$$u^{\mathrm{L}}(t) = \left[\begin{array}{cccc} u^{\mathrm{T}}(tT) & u^{\mathrm{T}}(tT+1) & \cdots & u^{\mathrm{T}}(tT+T-1) \end{array}\right]^{\mathrm{T}}$$

It is assumed that the states of system (1) can not be measured by hardware due to some restrictions in practice, but the output $y(t)$ and the input $u(t)$ can be utilized. In this case, we need construct a system which can give an asymptotic estimation of $x(t)$. Such a system can be taken the form as

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(C(t)\hat{x} - y(t)), \quad \hat{x}(0) = \hat{x}_0 \tag{5}$$

where $\hat{x} \in \mathbb{R}^n$ and $L(t) \in \mathbb{R}^{n\times m}$, $t \in \mathbb{Z}$ are real matrices of periodic T.

Combining system (1) with (5) gives the following closed loop system

$$\hat{x}(t+1) = (A(t) + L(t)C(t))\hat{x}(t) - L(t)C(t)x(t) + B(t)u(t), \quad \hat{x}(0) = \hat{x}_0 \tag{6}$$

This system is also a T-periodic linear system and its monodromy matrix is

$$\Psi_{\mathrm{c}} = A_{\mathrm{c}}(T-1)A_{\mathrm{c}}(T-2)\cdots A_{\mathrm{c}}(0)$$

where

$$A_{\mathrm{c}}(i) = A(i) + L(i)C(i), \ i \in \overline{0, T-1}$$

Here, we give an existence condition for a full order state observer (5) which is simple and its proof is omitted.

Proposition 1 *For an observable system (1), there exist matrices $L(t)$, $t \in \overline{0, T-1}$ such that system (5) becomes a full order state observer of system (1) if and only if all the eigenvalues of the monodromy matrix Ψ_c of system (6) lie in the open unit disk.*

Let $\Gamma = \{s_i, s_i \in \mathbb{C}, i \in \overline{1,n}\}$ be the set of the desired poles of the closed-loop system (6), which is symmetric with respect to the real axis. Let $F \in \mathbb{R}^{n\times n}$ be a given real matrix satisfying $\lambda(F) = \Gamma$. Then, clearly, $\lambda(\Psi_c) = \Gamma$ if and only if there exists a nonsingular matrix V such that

$$\Psi_c V = VF \tag{7}$$

Then the problem of observer design for system (1) can be reduced as following:

Problem 1 *Given a completely observable discrete-time linear periodic system (1) and matrix $F \in \mathbb{R}^{n\times n}$, find matrices $L(t) \in \mathbb{R}^{n\times m}$, $t \in \overline{0, T-1}$, such that (7) is satisfied for some nonsingular matrix $V \in \mathbb{R}^{n\times n}$.*

When there exist parameter perturbations in system matrices $A(t)$ and $C(t)$, the closed-loop system matrices will deviate the nominal matrix $A_c(t)$. Without loss of generality, we assumed that the closed-loop periodic system matrices are perturbed as follows:

$$A(t) + L(t)C(t) \mapsto A(t) + \Delta_{a,t}(\varepsilon) + L(t)(C(t) + \Delta_{c,t}(\varepsilon)),\ t \in \overline{0, T-1}$$

where $\Delta_{at}(\varepsilon) \in \mathbb{R}^{n\times n}, \Delta_{ct}(\varepsilon) \in \mathbb{R}^{m\times n}$, $t \in \overline{0, T-1}$, are matrix functions of ε satisfying

$$\lim_{\varepsilon\to 0^+} \frac{\Delta_{a,t}(\varepsilon)}{\varepsilon} = \Delta_{a,t},\ \ \lim_{\varepsilon\to 0^+} \frac{\Delta_{c,t}(\varepsilon)}{\varepsilon} = \Delta_{c,t}$$

in which $\Delta_{a,t} \in \mathbb{R}^{n\times n}$, $\Delta_{c,t} \in \mathbb{R}^{m\times n}$, $t \in \overline{0, T-1}$, are constant matrices. Then the monodromy matrix of perturbed closed-loop system is

$$\Psi_c(\varepsilon) = (A_c(T-1) + \Delta_{a,T-1}(\varepsilon) + L(T-1)\Delta_{c,T-1}(\varepsilon)) \cdots (A_c(0) + \Delta_{c,0}(\varepsilon) + L(0)\Delta_{c,0}(\varepsilon))$$

Since nondefective matrices possess better robustness than defective ones (see [23]), the robust observer design problem for system (1) can be described as follows.

Problem 2 *Given a completely observable discrete-time linear periodic system (1) and matrix $F \in \mathbb{R}^{n\times n}$, find real matrices $L(t) \in \mathbb{R}^{n\times m}$, $t \in \overline{0, T-1}$, such that the following conditions are met.*

1. *Matrix Ψ_c is nondefective, and (7) is satisfied for some nonsingular matrix $V \in \mathbb{R}^{n\times n}$;*
2. *The eigenvalues of matrix $\Psi_c(\varepsilon)$ at $\varepsilon = 0$, $i \in \overline{1,l}$ are as insensitive as possible to small variations in ε.*

At the end of this section, we introduce the following right coprime principle for polynomial matrix pair (see, for example, [22]).

Definition 1 *A pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ are said to be right coprime if*

$$\operatorname{rank} \begin{bmatrix} N(\lambda) \\ D(\lambda) \end{bmatrix} = r \text{ for any } \lambda \in \mathbb{C}$$

and a pair of polynomial matrices $H(s) \in \mathbb{R}^{m\times n}[s]$ and $L(s) \in \mathbb{R}^{m\times m}[s]$ are said to be left coprime if

$$\operatorname{rank} \begin{bmatrix} H(\lambda) & L(\lambda) \end{bmatrix} = m \text{ for any } \lambda \in \mathbb{C}$$

3 Main results

Let A^{LT} and C^{LT} denote the lifted system matrices corresponding to periodic matrix pair $(A^{\mathrm{T}}(\cdot), C^{\mathrm{T}}(\cdot))$. Introducing the following polynomial matrix factorization:

$$(zI - A^{\mathrm{LT}})^{-1}C^{\mathrm{LT}} = N(z)D^{-1}(z) \tag{8}$$

where $N(z) \in \mathbb{R}^{n\times Tm}$, $D(z) \in \mathbb{R}^{Tm\times Tm}$ are right coprime matrix polynomials in z. If we denote

$$D(z) = [d_{ij}(z)]_{Tm\times Tm}\,,\; N(z) = [n_{ij}(z)]_{n\times Tm}$$

and $\omega = \max\{\omega_1, \omega_2\}$, where

$$\omega_1 = \max_{i,\, j\in\overline{1,Tm}} \{\deg(d_{ij}(z))\}\,,\; \omega_2 = \max_{i\in\overline{1,n},\, j=\overline{1,Tm}} \{\deg(n_{ij}(z)\}$$

then $N(z)$ and $D(z)$ can be rewritten as

$$\begin{cases} N(z) = \sum\limits_{i=0}^{\omega} N_i z^i, & N_i \in \mathbb{C}^{n\times Tm} \\ D(z) = \sum\limits_{i=0}^{\omega} D_i z^i, & D_i \in \mathbb{C}^{Tm\times Tm} \end{cases} \tag{9}$$

Denote

$$\begin{cases} V(Z) = N_0 Z + N_1 ZF + \cdots + N_\omega ZF^\omega \\ W(Z) = D_0 Z + D_1 ZF + \cdots + D_\omega ZF^\omega \end{cases} \tag{10}$$

and

$$\mathcal{Z} = \left\{ Z \,\middle|\, \det\left(\sum_{i=0}^{\omega} N_i ZF^i\right) \neq 0 \right\} \tag{11}$$

where $Z \in \mathbb{R}^{Tm\times n}$ is an arbitrary parameter matrix.

Let

$$X(Z) = W(Z)V^{-1}(Z) \triangleq \begin{bmatrix} X_0^{\mathrm{T}} & X_1^{\mathrm{T}} & \cdots & X_{T-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},\; Z \in \mathcal{Z} \tag{12}$$

Theorem 1 *Let periodic matrix pairs $(A(\cdot), C(\cdot))$ are system matrices of system (1), $V(Z)$ and $W(Z)$ are given by (10), X_i, $i \in \overline{0, T-1}$ are given by (12). Then the whole set of solutions to Problem 1 are characterized as*

$$\mathcal{L} = \left\{ \begin{pmatrix} L(0) \\ L(1) \\ \vdots \\ L(T-1) \end{pmatrix} \,\middle|\, \begin{cases} X(Z) = W(Z)V^{-1}(Z)\,, \quad Z \in \mathcal{Z} \\ L(0) = [X_1]^{\mathrm{T}},\; \det(A_{\mathrm{c}}(0)) \neq 0 \\ L(t) = \left[X_{t+1} \prod\limits_{j=0}^{i-1} A_{\mathrm{c}}^{-1}(j) \right]^{\mathrm{T}},\; \det(A_{\mathrm{c}}(t)) \neq 0,\; t \in \overline{1, T-1} \end{cases} \right\} \tag{13}$$

Proof. Since matrix Ψ_{c} has the same eigenvalues with matrix $\Psi_{\mathrm{c}}^{\mathrm{T}}$, problem 1 can be converted into find matrices $L^{\mathrm{T}}(t)$, such that

$$\Psi_{\mathrm{c}}^{\mathrm{T}} = \left(A^{\mathrm{T}}(0) + C^{\mathrm{T}}(0)L^{\mathrm{T}}(0)\right)\cdots\left(A^{\mathrm{T}}(T-1) + C^{\mathrm{T}}(T-1)L^{\mathrm{T}}(T-1)\right)$$

has the desired eigenvalues. Utilizing theorem 1 of literature [23], we can conclude that the solution to Problem 1 has the form shown in (13). ■

Based upon Theorem 1, an algorithm for solving Problem 1 follows.

Algorithm 1 *(Parametric Observers Design)*

1. *Assign the desired poles set $\{s_i, s_i \in \mathbb{C}, i \in \overline{1,n}\}$ for the united system (6);*
2. *Compute A^{LT}, C^{LT} according to (3) and (4);*

3. *Solve the right coprime polynomial matrices $N(z)$ and $D(z)$ satisfying factorizations (8) to obtain matrices N_i, D_i, $i \in \overline{0,\omega}$;*
4. *By virtue of formula (10), compute $V(Z), W(Z)$;*
5. *Compute $L(t)$, $t \in \overline{0,T-1}$ according to (13).*

Since above parametric observers design algorithm can present numerous explicit solutions, it is convenient to exploit free parameter Z to achieve some other system performances by imposing additional conditions on the gains $L(t), t \in \overline{0,T-1}$ and matrix V. An important aspect in observer design problem is that the designed observer should be insensitive to the changes in the matrix data. Thus the robust observer design problem for linear periodic systems consists of choosing $L(t) \in \mathbb{R}^{n\times m}$, $t \in \overline{0,T-1}$ so that Ψ_{c} has the prescribed set of eigenvalues, and so that these eigenvalues are as insensitive to perturbations in the closed-loop system as possible. The question remaining is how to choose cost function, i.e., the index characterizing insensitivity of the poles to small changes in the system data. For this purpose, we formulate the following theorem.

Theorem 2 *Let $\Psi_{\mathrm{c}} = A_{\mathrm{c}}(T-1)A_{\mathrm{c}}(T-2)\cdots A_{\mathrm{c}}(0) \in \mathbb{R}^{n\times n}$ be diagonalizable and $V \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $\Psi_{\mathrm{c}} = V^{-1}\Lambda V \in \mathbb{R}^{n\times n}$, where $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the Jordan canonical form of matrix Ψ_{c}. Assume that for real scalar $\varepsilon > 0$, $t \in \overline{0,T-1}$,*

$$\lim_{\varepsilon\to 0^+}\frac{\Delta_{a,t}(\varepsilon)}{\varepsilon} = \Delta_{a,t}, \quad \lim_{\varepsilon\to 0^+}\frac{\Delta_{c,t}(\varepsilon)}{\varepsilon} = \Delta_{c,t} \tag{14}$$

where $\Delta_{a,t} \in \mathbb{R}^{n\times n}$, $\Delta_{c,t} \in \mathbb{R}^{m\times n}$, $t \in \overline{0,T-1}$ are constant matrices. Then for any eigenvalue λ of matrix

$$\Psi_{\mathrm{c}}(\varepsilon) = (A_{\mathrm{c}}(T-1) + \Delta_{a,T-1}(\varepsilon) + L(T-1)\Delta_{c,T-1}(\varepsilon))\cdots(A_{\mathrm{c}}(0) + \Delta_{c,0}(\varepsilon) + L(0)\Delta_{c,0}(\varepsilon))$$

the following relation holds:

$$\min_j\{|\lambda_j - \lambda|\} \le \varepsilon n\kappa_{\mathrm{F}}(V)\left(\sum_{t=0}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1}\right)\left(1+\sum_{t=0}^{T-1}\|L(t)\|_{\mathrm{F}}\right)\max_t\left\{\|\Delta_{a,t}\|_{\mathrm{F}}, \|\Delta_{c,t}\|_{\mathrm{F}}\right\} + \mathrm{O}(\varepsilon^2) \tag{15}$$

where $\kappa_{\mathrm{F}}(V) \triangleq \left\|V^{-1}\right\|_{\mathrm{F}}\|V\|_{\mathrm{F}}$ is the Frobenius-norm conditional number of matrix V.

Proof. Since λ is an eigenvalue of matrix $\Psi_{\mathrm{c}}(\varepsilon)$, we have

$$\begin{aligned} 0 &= \det\left(V^{-1}(\Psi_{\mathrm{c}}(\varepsilon) - \lambda I)V\right) \\ &= \det\left(V^{-1}\left(\Psi_{\mathrm{c}} - \lambda I + \Pi\right)V\right) \\ &= \det\left(\Lambda - \lambda I + V^{-1}\Pi V\right) \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Pi &= (\Delta_{a,T-1}(\varepsilon) + L(T-1)\Delta_{c,T-1}(\varepsilon))\prod_{t=T-2}^{0}A_{\mathrm{c}}(t) \\ &\quad + A_{\mathrm{c}}(T-1)\left(\Delta_{a,T-2}(\varepsilon) + L(T-2)\Delta_{c,T-2}(\varepsilon)\right)\prod_{t=T-3}^{0}A_{\mathrm{c}}(t) \\ &\quad + \cdots + \left(\prod_{t=T-1}^{1}A_{\mathrm{c}}(t)\right)(\Delta_{a,0}(\varepsilon) + L(0)\Delta_{c,0}(\varepsilon)) + \mathrm{O}(\varepsilon^2) \end{aligned}$$

If matrix $\Lambda - \lambda I$ is singular, it is obvious that there exists j satisfying $\lambda = \lambda_j$, thus relation (15) holds automatically. If the matrix $\Lambda - \lambda I$ is nonsingular, then it follows from (16) that

$$0 = \det\left((\Lambda - \lambda I)\left(I + (\Lambda - \lambda I)^{-1}V^{-1}\Pi V\right)\right)$$

which implies that the matrix $I+(\Lambda-\lambda I)^{-1}V^{-1}\Pi V$ must be singular. Consequently, we get

$$\left\|(\Lambda-\lambda I)^{-1}V^{-1}\Pi V\right\|_{\mathrm{F}}\geq 1$$

from which we deduce

$$\begin{aligned} 1 &\leq \left\|(\Lambda-\lambda I)^{-1}V^{-1}\Pi V\right\|_{\mathrm{F}} \\ &\leq \left\|(\Lambda-\lambda I)^{-1}\right\|_{\mathrm{F}}\left\|V^{-1}\Pi V\right\|_{\mathrm{F}} \\ &\leq n\max_{j}\left\{|\lambda_j-\lambda|^{-1}\right\}\kappa_{\mathrm{F}}(V)\|\Pi\|_{\mathrm{F}} \end{aligned}$$

or equivalently,

$$\min_{j}\{|\lambda_j-\lambda|\}\leq n\kappa_{\mathrm{F}}(V)\|\Pi\|_{\mathrm{F}} \tag{17}$$

Note that

$$\begin{aligned} \|\Pi\|_{\mathrm{F}} &= \Bigg\|(\Delta_{a,T-1}(\varepsilon)+L(T-1)\Delta_{c,T-1}(\varepsilon))\prod_{t=T-2}^{0}A_{\mathrm{c}}(t) \\ &\quad +A_{\mathrm{c}}(T-1)\left(\Delta_{a,T-2}(\varepsilon)+L(T-2)\Delta_{c,T-2}(\varepsilon)\right)\prod_{t=T-3}^{0}A_{\mathrm{c}}(t) \\ &\quad +\cdots+\left(\prod_{t=T-1}^{1}A_{\mathrm{c}}(t)\right)(\Delta_{a,0}(\varepsilon)+L(0)\Delta_{c,0}(\varepsilon))\Bigg\|_{\mathrm{F}} \\ &\leq \varepsilon\left(\prod_{t=T-1,\ t\neq T-1}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}+\prod_{t=T-1,\ t\neq T-2}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}\right. \\ &\quad \left.+\cdots+\prod_{t=T-1,\ t\neq 0}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}\right)\max_{t}\left\{\|\Delta_{a,t}\|_{\mathrm{F}}+\|L(t)\|_{\mathrm{F}}\|\Delta_{c,t}\|_{\mathrm{F}}\right\}+\mathrm{O}(\varepsilon^2) \end{aligned}$$

On the other hand, in terms of the inequality

$$\prod_{i=1}^{n}a_i\leq\frac{1}{n}\sum_{i=1}^{n}a_i^n,\quad a_i\geq 0$$

the following series of inequalities hold:

$$\begin{aligned} \prod_{t=T-1,\ t\neq T-1}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}} &\leq \frac{1}{T-1}\sum_{t=0,\ t\neq T-1}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \\ \prod_{t=T-1,\ t\neq T-2}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}} &\leq \frac{1}{T-1}\sum_{t=0,\ t\neq T-2}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \\ &\ \ \vdots \\ \prod_{t=T-1,\ t\neq 0}^{0}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}} &\leq \frac{1}{T-1}\sum_{t=0,\ t\neq 0}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \end{aligned}$$

Therefore, we can get

$$\begin{aligned} \|\Pi\|_{\mathrm{F}} &\leq \frac{\varepsilon}{T-1}\left(\sum_{t=0,\ t\neq T-1}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1}+\sum_{t=0,\ t\neq T-2}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1}+\cdots+\right. \\ &\quad \left.\sum_{t=0,\ t\neq 0}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1}\right)\max_{t}\left\{\|\Delta_{a,t}\|_{\mathrm{F}}+\|L(t)\|_{\mathrm{F}}\|\Delta_{c,t}\|_{\mathrm{F}}\right\} \\ &= \varepsilon\left(\sum_{t=0}^{T-1}\|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1}\right)\max_{t}\left\{\|\Delta_{a,t}\|_{\mathrm{F}}+\|L(t)\|_{\mathrm{F}}\|\Delta_{c,t}\|_{\mathrm{F}}\right\}+\mathrm{O}(\varepsilon^2) \end{aligned}$$

Combining inequality (17), we have

$$\begin{aligned}\min_j \{|\lambda_j - \lambda|\} &\leq \varepsilon n \kappa_{\mathrm{F}}(V) \left(\sum_{t=0}^{T-1} \|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \right) \max_t \left\{ \|\Delta_{a,t}\|_{\mathrm{F}} + \|L(t)\|_{\mathrm{F}} \|\Delta_{c,t}\|_{\mathrm{F}} \right\} + \mathrm{O}(\varepsilon^2) \\ &\leq \varepsilon n \kappa_{\mathrm{F}}(V) \left(\sum_{t=0}^{T-1} \|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \right) \left(1 + \sum_{t=0}^{T-1} \|L(t)\|_{\mathrm{F}} \right) \max_t \left\{ \|\Delta_{a,t}\|_{\mathrm{F}}, \|\Delta_{c,t}\|_{\mathrm{F}} \right\} + \mathrm{O}(\varepsilon^2)\end{aligned}$$

Thus, the proof is accomplished. ∎

In view of theorem 2, the sensitivity of the observers for system (1) with respect to the perturbations on system matrices can be measured by the following index:

$$J(Z) \triangleq \kappa_{\mathrm{F}}(V) \sum_{t=0}^{T-1} \|A_{\mathrm{c}}(t)\|_{\mathrm{F}}^{T-1} \left(1 + \sum_{t=0}^{T-1} \|L(t)\|_{\mathrm{F}} \right) \tag{18}$$

So far, the robust observer design problem can be converted into a static optimization problem. Here, we present an algorithm for problem 2.

Algorithm 2 *(Robust Observer Design)*

1. *Select the desired poles of the closed-loop system (6) such that they all lie in the open unit circle;*
2. *Compute* A^{LT}, C^{LT} *according to (3) and (4). Solve the coprime factorization (8) to compute matrices* $N(z), D(z)$*, and further obtain matrices* $N_i, D_i, i \in \overline{0, \omega}$
3. *Construct general expressions for matrices* V *and* $L(t)$, $t \in \overline{0, T-1}$ *according to (10) and (13);*
4. *Solving optimization problem*
$$\mathrm{Minimize} J(Z)$$
by using gradient based searching method. The optimal decision matrix is denoted by Z_{opt} *;*
5. *Compute matrices* V_{opt} *and* W_{opt} *according to (10) by using optimal decision matrix* Z_{opt}*;*
6. *Calculate matrices* $L_{\mathrm{opt}}(t)$, $t \in \overline{0, T-1}$ *by formulae (13).*

4 A numerical example

Consider linear discrete periodic system (1) with parameters as follows:

$$\begin{aligned} A_0 &= \begin{bmatrix} -4.5 & -1 \\ 2.5 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \\ B_0 &= B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C_0 &= \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$

This is an oscillation system and it is easily verified that this system is completely observable. Therefore, we can design an observer represented by system (5) for it. Without loss of generality, the poles of closed-loop system (6) can be taken as -0.5 and 0.5.

By right coprime factorization to the lifted system matrix pair, we can obtain

$$N(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{2}{3}s - \frac{1}{3} & -\frac{2}{3} \\ 2s - 2 & -\frac{28}{3}s - \frac{4}{3} & \frac{16}{3} \end{bmatrix}$$

Randomly choosing a parameter matrix

$$Z = \begin{bmatrix} -1 & 0.5 \\ 2 & 3 \\ -3 & 1 \end{bmatrix}$$

we get a group of periodic observer gains as follows:

$$L_{\mathrm{rand}}(0) = \begin{bmatrix} 2.1481 \\ -1.2037 \end{bmatrix}, L_{\mathrm{rand}}(1) = \begin{bmatrix} -15.5000 \\ -5.0000 \end{bmatrix}, L_{\mathrm{rand}}(2) = \begin{bmatrix} -1.8333 \\ -4.0833 \end{bmatrix}$$

Applying algorithm 2 gives the following robust observer gains

$$L_{\mathrm{robu}}(0) = \begin{bmatrix} 1.7880 \\ -1.0259 \end{bmatrix}, L_{\mathrm{robu}}(1) = \begin{bmatrix} -0.6496 \\ -0.0063 \end{bmatrix}, L_{\mathrm{robu}}(2) = 10^{-4} \times \begin{bmatrix} 0.4673 \\ 0.2320 \end{bmatrix}$$

For convenience, denote

$$L_{\mathrm{rand}} = (L_{\mathrm{rand}}(0),\ L_{\mathrm{rand}}(1),\ L_{\mathrm{rand}}(2)), \quad L_{\mathrm{robu}} = (L_{\mathrm{robu}}(0),\ L_{\mathrm{robu}}(1),\ L_{\mathrm{robu}}(2))$$

When the reference input signal is taken to be $v(t) = 0.1 * \sin(t + \pi/2)$ and the initial states of system (1) and observer (5) are respectively taken to be $x_0 = [-1, 1]^{\mathrm{T}}$, $\hat{x}_0 = [0, 0]^{\mathrm{T}}$, we depict the states histories of system (1) and its observing system (5) respectively by L_{rand} and L_{robu} in figure. 1. Here, the solid line denotes the trajectory of $x(t)$ and the dotted line denotes the trajectory of $\hat{x}(t)$. In this figure, the two observers can both track the nominal system very well.

Let the closed-loop system matrices be perturbed as follows:

$$A(t) + L(t)C(t) \mapsto A(t) + \mu\Delta_{at} + L(t)(C(t) + \mu\Delta_{ct}),\ t \in \overline{0,2}$$

where $\Delta_{at} \in \mathbb{R}^{2\times 2}$, $\Delta_{ct} \in \mathbb{R}^{1\times 2}, t \in \overline{0,2}$ are random perturbations normalized such that $\|\Delta_{at}\|_{\mathrm{F}} = 1, \|\Delta_{ct}\|_{\mathrm{F}} = 1,\ t \in \overline{0,2}$ and $\mu > 0$ is a parameter controlling the level of perturbations. Let $\mu = 0.01$, we depict the response histories of $x(t)$ and $\hat{x}(t)$ with observer gains L_{rand} and L_{robu} in figure. 2. It is obvious that the observer with L_{rand} can not track the system state $x(t)$ even the perturbation level is reduced to $\mu = 0.01$. To measure robustness of the designed observer with gains L_{robu}, we continuously increase the pertubation controlling level until $\mu = 0.25$ and depict the results in figure. 3. From simulation results, we can see the designed robust observer has strong anti-interference. In addition, we notice that L_{robu} has a very small norm compared with L_{rand}. This means that the robust observer can possess less energy consumption, since small gains lead to small control signals.

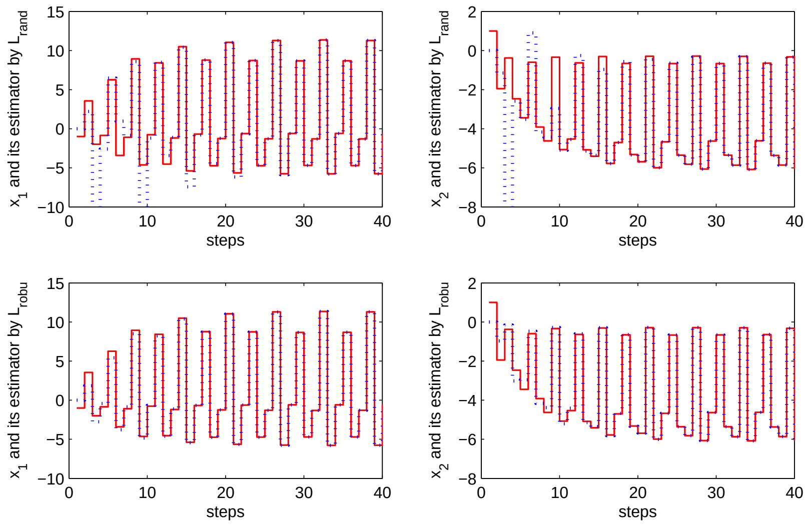

Figure 1: $x(t)$ and $\hat{x}(t)$ respectively by L_{rand} and L_{robu} for the nominal system

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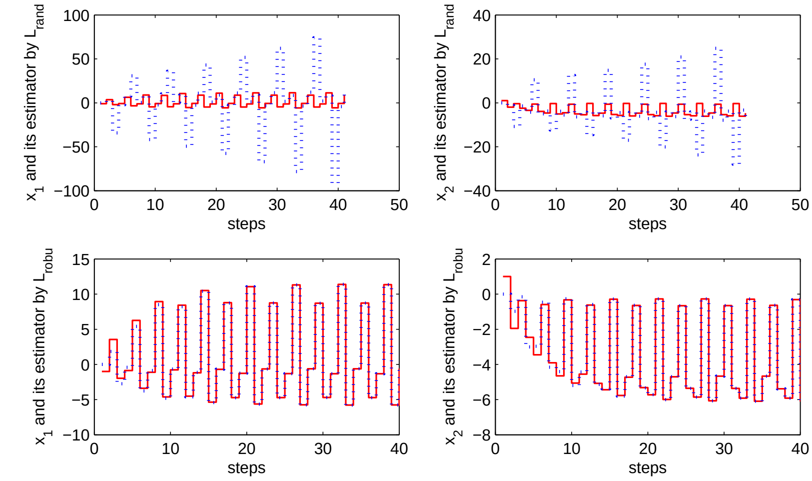

Figure 2: $x(t)$ and $\hat{x}(t)$ respectively by L_{rand} and L_{robu} for the perturbed system with $\mu = 0.01$

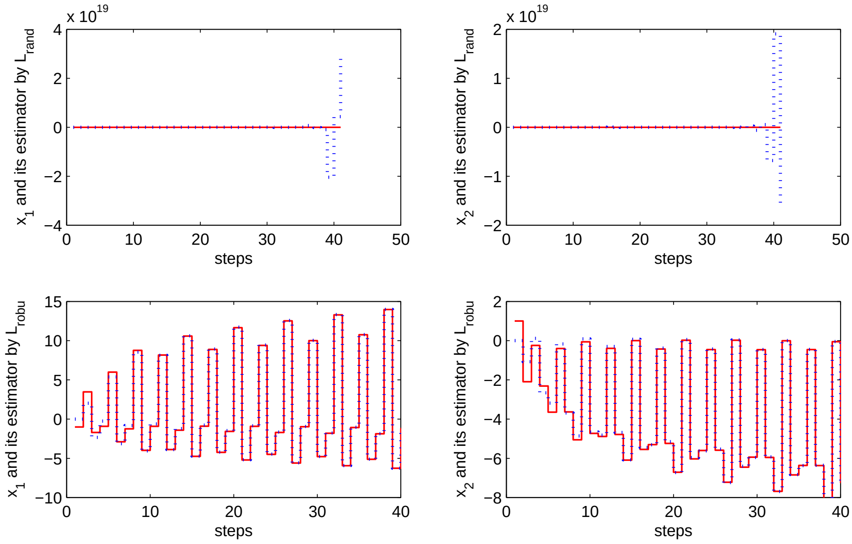

Figure 3: $x(t)$ and $\hat{x}(t)$ respectively by L_{rand} and L_{robu} for the perturbed system with $\mu = 0.25$

5 Conclusion

Based on periodic parametric observers design approach and a deduced robust performance index, the robust observer design problem for linear discrete-time periodic systems is studied by solving an optimization problem. In this case, it is shown that the robust performance of an observer can be related with the norm of the monodromy matrix of the closed-loop system. The provided robust observer design algorithm has been tested on a numerical example, giving satisfactory results.

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SOME FAMILIES OF TWO-SIDED GENERATING FUNCTIONS FOR CERTAIN CLASS OF $r-$VARIABLE POLYNOMIALS

CEM KAANOGLU

Abstract. In 1972, H.M. Srivastava considered certain class of one variable polynomials, the Srivastava polynomials and introduced a research for these polynomials [22]. In the following period, the various classes of Srivastava polynomials have considered in the papers [1], [13], [15], [16], [17], [19], [20]. In [14], Kaanoglu and Özarslan have introduced certain class of $r-$variable polynomials which is multivariable extension of Srivastava polynomials and investigated two-sided generating relations between these $r-$variable polynomials and certain class of $r-1-$variable polynomials. In this paper, we obtain a number of new two-sided linear generating functions for these polynomials by applying certain hypergeometric transformation. Furthermore, various generating relations are presented for $r-$variable Lagrange polynomials.

1. Introduction

It was H.M. Srivastava [22] who introduced the so-called Srivastava polynomials,

$$S_n^N(z)=\sum_{k=0}^{\left[\frac{n}{N}\right]}\frac{(-n)_{Nk}}{k!}A_{n,k}z^k \qquad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\,;N\in\mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a\in\mathbb{R}$, and $(\lambda)_\nu$, $(\lambda)_0\equiv 1$ denotes the Pochhammer symbol defined by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

by means of familiar Gamma functions.

In 2002, González et al. [13] extended the Srivastava polynomials $S_n^N(z)$ as follows:

$$S_{n,m}^N(z)=\sum_{k=0}^{\left[\frac{n}{N}\right]}\frac{(-n)_{Nk}}{k!}A_{n+m,k}z^k \qquad (m,n\in\mathbb{N}_0;N\in\mathbb{N}),$$

and studied their properties extensively. Because of the fact that $S_{n,0}^N(z)=S_n^N(z)$, we call the family by "extended Srivastava polynomials".

Key words and phrases. Generating functions, Srivastava polynomials, Lagrange polynomials, Hypergeometric transformation.

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In [1], the following family of bivariate polynomials has been introduced

$$S_n^{m,N}(x,y)=\sum_{k=0}^{\left[\frac{n}{N}\right]}A_{m+n,k}\frac{x^{n-Nk}}{(n-Nk)!}\,\frac{y^k}{k!}\qquad (n,m\in\mathbb{N}_0,N\in\mathbb{N}),$$

and it has shown that the polynomials $S_n^{m,N}(x,y)$ includes many well known polynomials such as Lagrange-Hermite polynomials, Lagrange polynomials and Hermite-Kampé de Feriét polynomials.

In [23], Srivastava et al. have introduced the three-variable polynomials

(1.2)
$$S_n^{m,M,N}(x,y,z)=\sum_{k=0}^{\left[\frac{n}{N}\right]}\sum_{l=0}^{\left[\frac{k}{M}\right]}A_{m+n,k,l}\frac{x^l}{l!}\frac{y^{k-Ml}}{(k-Ml)!}\frac{z^{n-Nk}}{(n-Nk)!},\ (m,n\in\mathbb{N}_0;M,N\in\mathbb{N}),$$

where $\{A_{m,n,k}\}$ be a triple sequence of complex numbers.

Suitable choices of $\{A_{m,n,k}\}$ in equation (1.2) gives three variable version of well-known polynomials. In [23], a number of two-sided linear generating functions between three-variable polynomials $S_n^{m,M,N}(x,y,z)$ and a family of two-variable polynomials

$$P_n^{m,M,N}(x,y)=\sum_{l=0}^{\left[\frac{n}{M}\right]}A_{m+Nn,n,l}\frac{x^{n-Ml}}{(n-Ml)!}\frac{y^l}{l!},\quad (m,n\epsilon\mathbb{N}_0;\ M,N\epsilon\mathbb{N}) \tag{1.3}$$

have been investigated. Furthermore, in [15] some new two-sided generating relations have been proved by applying a certain class of hypergeometric transformations.

In [14], C. Kaanoğlu and M.A. Özarslan introduced $r-$variable Srivastava polynomials

(1.4)
$$S_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)$$
$$=\sum_{k_{r-1}=0}^{\left[\frac{n}{N_{r-1}}\right]}\sum_{k_{r-2}=0}^{\left[\frac{k_{r-1}}{N_{r-2}}\right]}...\sum_{k_2=0}^{\left[\frac{k_3}{N_2}\right]}\sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]}A_{m+n,k_{r-1},k_1,k_2,...,k_{r-2}}\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-N_1k_1}}{(k_2-N_1k_1)!}\cdots\frac{x_r^{n-N_{r-1}k_{r-1}}}{(n-N_{r-1}k_{r-1})!}$$
$$(m,n\in\mathbb{N}_0;N_1,N_2,...,N_{r-1}\in\mathbb{N}).$$

and obtained a number of two-sided linear generating functions between these polynomials and $r-1-$ variable polynomials

(1.5)
$$P_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_{r-1})$$
$$=\sum_{k_{r-2}=0}^{\left[\frac{n}{N_{r-2}}\right]}\sum_{k_{r-3}=0}^{\left[\frac{k_{r-2}}{N_{r-3}}\right]}...\sum_{k_2=0}^{\left[\frac{k_3}{N_2}\right]}\sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]}A_{m+N_{r-1}n,\ n,k_1,k_2,k_3,...,k_{r-2}}$$
$$\cdot\frac{{x_1}^{k_1}}{k_1!}\frac{{x_2}^{(k_2-N_1k_1)}}{(k_2-N_1k_1)!}\cdots\frac{{x_{r-1}}^{(n-N_{r-2}k_{r-2})}}{(n-N_{r-2}k_{r-2})!},$$
$$(m,n\in\mathbb{N}_0;N_1,,N_2,...,N_{r-1}\in\mathbb{N}).$$

Note that appropriate choices of the sequence $\{A_{n,k_{r-1},k_1,k_2,...,k_{r-2}}\}$ in (1.4) give the $r-$variable versions of the well known polynomials.

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The aim of this paper is to obtain new two-sided linear generating functions between r-variable polynomials $S_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)$ and $(r-1)$-variable polynomials $P_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_{r-1})$ by applying hypergeometric transformations [21]

$$\tag{1.6} {}_2F_1(\alpha,\alpha+\frac{1}{2};\gamma;z^2)=(1+z)^{-2\alpha}\ {}_2F_1(2\alpha,\gamma-\frac{1}{2};2\gamma-1;\frac{2z}{1+z})$$
$$(|\arg(1+z)|\leq\pi-\varepsilon\ (0\prec\varepsilon\prec\pi);2\gamma-1\notin\mathbb{Z}_0^-),$$

and

$$\tag{1.7} {}_2F_1(\alpha,\alpha+\frac{1}{2};\gamma;z) = (1-z)^{-\alpha}\ {}_2F_1(2\alpha,2\gamma-2\alpha-1;\gamma;\frac{\sqrt{1-z}-1}{2\sqrt{1-z}})$$
$$(|\arg(1-z)| \leq \pi-\varepsilon\ (0<\varepsilon<\pi);\gamma\notin\mathbb{Z}_0^-).$$

In [5], the given hypergeometric transformations have been used to investigate families of double series identities involving various hypergeometric functions in one and two variables .

2. Main Results

The main object of this section is to derive a number of two-sided linear generating relations between $r-$variable polynomials and $r-1-$ variable polynomials which are defined by (1.4) and (1.5), respectively.

We start by recalling an infinite series identities which were obtained by Srivastava et al. [23].

Lemma 2.1. *(see [23 ,Lemma1]) Let $N_1,N_2,...,N_{r-1}\in\mathbb{N}$, $r=\{2,3,...\}$. Then*

$$\tag{2.1} \sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\infty}...\sum_{n_1=0}^{\infty}A(n_1,n_2,...,n_r)$$
$$=\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]}...\sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]}A(n_1,n_2-N_1n_1,...,n_r-N_{r-1}n_{r-1})$$

and

$$\tag{2.2} \sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]}...\sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]}A(n_1,n_2,...,n_r)$$
$$=\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\infty}...\sum_{n_1=0}^{\infty}A(n_1,n_2+N_1n_1,n_3+N_2N_1n_1+N_2n_2,...,$$
$$n_r+\prod_{j=1}^{r-1}N_jn_1+\prod_{j=2}^{r-1}N_jn_2+...+N_{r-1}n_{r-1})$$

where $\{A(n_1,n_2,...,n_r)\}$ is a bounded $r-$tuple sequence of real or complex numbers.

The main result of this paper is given by the following theorem.

Theorem 2.2. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \tag{2.3}$$

$$= \sum_{n,m,k_{r-1},...,k_1=0}^{\infty} \Omega(n+m+F(r-1))\ A_{n+m+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,...,k_{r-2}+F(r-3)}$$

$$\cdot\frac{(x_1 w^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2 w^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}\cdots\frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1}N_j)})^{k_{r-1}}}{k_{r-1}!}\frac{(x_r w+2t)^n}{n!}\frac{(v)_m}{(2v)_m}\frac{(-4t)^m}{m!}$$

provided that each member of the series identity (2.3) exists.

Proof. Let the left hand side of (2.3) denote by $\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)$. Then using the definition of $S_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)$ on the left hand side of (2.3), we have

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)\\
=&\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n\\
=&\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m}\sum_{k_{r-1}=0}^{\left[\frac{n}{N_{r-1}}\right]}\sum_{k_{r-2}=0}^{\left[\frac{k_{r-1}}{N_{r-2}}\right]}\cdots\sum_{k_2=0}^{\left[\frac{k_3}{N_2}\right]}\sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]} A_{2m+n,k_{r-1},k_1,k_2,...,k_{r-2}}\\
&\cdot\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-N_1k_1}}{(k_2-N_1k_1)!}\cdots\frac{x_r^{n-N_{r-1}k_{r-1}}}{(n-N_{r-1}k_{r-1})!}\frac{t^{2m}}{m!}w^n.
\end{aligned}$$

Let define

$$F(r-1) := \prod_{j=1}^{r-1}N_jk_1 + \prod_{j=2}^{r-1}N_jk_2 + ... + N_{r-1}k_{r-1}.$$

By applying Lemma (2.1), we find

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)\\
=&\sum_{m,n=0}^{\infty}\sum_{k_{r-1},...,k_1=0}^{\infty}\frac{\Omega(2m+n+F(r-1))}{(v+\frac{1}{2})_m}\\
&\cdot A_{2m+n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}\\
&\cdot\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots\frac{x_r^n}{n!}\frac{t^{2m}}{m!}w^{(n+\prod_{j=1}^{r-1}N_jk_1+\prod_{j=2}^{r-1}N_jk_2+...+N_{r-1}k_{r-1})}
\end{aligned}$$

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$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)\\
= &\sum_{m,n=0}^{\infty}\sum_{k_{r-1},...,k_1=0}^{\infty}\frac{\Omega(2m+n+F(r-1))}{(v+\frac{1}{2})_m}\\
&\cdot A_{2m+n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}\\
&\cdot\frac{(x_1w^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2w^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}\cdots\frac{(x_rw)^n}{n!}\frac{t^{2m}}{m!}.
\end{aligned}$$

Now taking $n\to n-2m$ $(0\le m\le[\frac{n}{2}];m,n\epsilon\mathbb{N}_0)$, we get

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)\\
= &\sum_{n,k_{r-1},...,k_1=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{\Omega(n+F(r-1))}{(v+\frac{1}{2})_m}\\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}\\
&\cdot\frac{(x_1w^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2w^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}\cdots\frac{(x_rw)^{n-2m}}{(n-2m)!}\frac{t^{2m}}{m!}.
\end{aligned}$$

Using the following elementary identity:

$$(n-2m)!=\frac{n!}{2^{2m}(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m},(0\le m\le[\frac{n}{2}];m,n\epsilon\mathbb{N}_0),$$

we obtain

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w)\\
&=\sum_{n,k_{r-1},...,k_1=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m}{(v+\frac{1}{2})_m}\frac{((\frac{2t}{x_rw})^2)^m}{m!}\Omega(n+F(r-1))\\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}\\
&\cdot\frac{(x_1w^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2w^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}\cdots\frac{(x_rw)^n}{n!}
\end{aligned}$$

$$\begin{aligned}
(2.4)\qquad = &\sum_{n,k_{r-1},...,k_1=0}^{\infty}\Omega(n+F(r-1))\\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}\\
&\cdot\frac{(x_1w^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2w^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}\cdots\frac{(x_rw)^n}{n!}\\
&\cdot{}_2F_1(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};v+\frac{1}{2};(\frac{2t}{x_rw})^2)\ .
\end{aligned}$$

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If we consider the hypergeometric transformation (1.6) in the last member of (2.4), we get

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w) \\
&= \sum_{n,k_{r-1},...,k_1=0}^{\infty} \Omega(n+F(r-1))\,{}_2F_1(-n,v;2v;\frac{\frac{4t}{x_r w}}{1+\frac{2t}{x_r w}}) \\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)} \\
&\cdot \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} \frac{(x_2 w^{(\prod_{j=2}^{r-1} N_j)})^{k_2}}{k_2!} ... \frac{(x_r w)^n}{n!} \; (1+\frac{2t}{x_r w})^n
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n,k_{r-1},...,k_1=0}^{\infty} \sum_{m=0}^{n} \Omega(n+F(r-1)) \\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)} \\
&\cdot \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} ... \frac{(x_r w)^n}{n!} \; (1+\frac{2t}{x_r w})^n \frac{(-n)_m (v)_m}{(2v)_m} \frac{(\frac{4t}{x_r w+2t})^m}{m!} \\
&= \sum_{n,k_{r-1},...,k_1=0}^{\infty} \sum_{m=0}^{n} \Omega(n+F(r-1)) \\
&\cdot A_{n+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)} \\
&\cdot \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} ... \frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)})^{k_{r-1}}}{k_{r-1}!} \frac{(x_r w+2t)^{n-m}}{(n-m)!} \frac{(v)_m}{(2v)_m} \frac{(-4t)^m}{m!}.
\end{aligned}$$

Setting $n \to n+m$

$$\begin{aligned}
&\Psi_{v,m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r,t,w) \\
&= \sum_{n,m,k_{r-1},...,k_1=0}^{\infty} \Omega(n+m+F(r-1)) \\
&\cdot A_{n+m+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)} \\
&\cdot \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} ... \frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)})^{k_{r-1}}}{k_{r-1}!} \frac{(x_r w+2t)^{n}}{n!} \frac{(v)_m}{(2v)_m} \frac{(-4t)^m}{m!}.
\end{aligned}$$

Whence the result. □

Setting $w = \frac{-2t}{x_r}$ in Theorem 2.2 we get the next corollary.

Corollary 2.3. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

(2.5)

$$\sum_{m,n=0}^{\infty}\frac{\Omega(2m+n)}{(v+\frac{1}{2})_m}S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}(\frac{-2t}{x_r})^n$$
$$=\sum_{m,k_{r-1},...,k_1=0}^{\infty}\Omega(m+F(r-1))\ A_{m+F(r-1),k_{r-1}+F(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+F(r-3)}$$
$$\cdot\frac{(x_1(\frac{-2t}{x_r})^{(\prod_{j=1}^{r-1}N_j)})^{k_1}}{k_1!}\frac{(x_2(\frac{-2t}{x_r})^{(\prod_{j=2}^{r-1}N_j)})^{k_2}}{k_2!}...\frac{(x_{r-1}(\frac{-2t}{x_r})^{(\prod_{j=r-1}^{r-1}N_j)})^{k_{r-1}}}{k_{r-1}!}\frac{(v)_m}{(2v)_m}\frac{(-4t)^m}{m!}$$

provided that each member of the series identity (2.5) exists.

Theorem 2.4. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers and $S_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)$ be defined by (1.4). Suppose also that $(r-1)-$variable polynomials $P_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_{r-1})$ be defined by (1.5). Then the family of two sided linear generating relations holds true between these polynomials:*

$$\sum_{m,n=0}^{\infty}\frac{\Omega(2m+n)}{(v+\frac{1}{2})_m}S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \tag{2.6}$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_{r-1}=0}^{\infty}\Omega(n+m+k_{r-1}N_{r-1})$$
$$\cdot P_{k_{r-1}}^{m+n,N_1,N_2,...,N_{r-1}}(x_1w^{(\prod_{j=1}^{r-1}N_j)},x_2w^{(\prod_{j=2}^{r-1}N_j)},...,x_{r-1}w^{(\prod_{j=r-1}^{r-1}N_j)})$$
$$\cdot\frac{(x_rw+2t)^n}{n!}\frac{(v)_m}{(2v)_m}\frac{(-4t)^m}{m!}$$

provided that each member of the assertion (2.6) exists.

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Proof. Setting $k_2 - N_1k_1, k_3 - N_2k_2, ..., k_{r-1} - k_{r-2}N_{r-2}$ in Theorem 2.2 we get

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \\
= &\sum_{n,m=0}^{\infty} \sum_{k_{r-1}=0}^{\infty} \Omega(n+m+k_{r-1}N_{r-1}) \\
&\cdot \sum_{k_{r-2}=0}^{\left[\frac{k_{r-1}}{N_{r-2}}\right]} \sum_{k_{r-3}=0}^{\left[\frac{k_{r-2}}{N_{r-3}}\right]} \cdots \sum_{k_2=0}^{\left[\frac{k_3}{N_2}\right]} \sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]} A_{n+m+k_{r-1}N_{r-1},k_{r-1},k_1,k_2,k_3,...,k_{r-2}} \\
&\cdot \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} \frac{(x_2 w^{(\prod_{j=2}^{r-1} N_j)})^{k_2-N_1k_1}}{(k_2-N_1k_1)!} \cdots \frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)})^{k_{r-1}-k_{r-2}N_{r-2}}}{(k_{r-1}-k_{r-2}N_{r-2})!} \\
&\cdot \frac{(x_r w+2t)^n}{n!} \frac{(v)_m}{(2v)_m} \frac{(-4t)^m}{m!}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \\
= &\sum_{n,m=0}^{\infty} \sum_{k_{r-1}=0}^{\infty} \Omega(n+m+k_{r-1}N_{r-1}) \\
&\cdot P_{k_{r-1}}^{m+n,N_1,N_2,...,N_{r-1}}(x_1 w^{(\prod_{j=1}^{r-1} N_j)}, x_2 w^{(\prod_{j=2}^{r-1} N_j)}, ..., x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)}) \\
&\cdot \frac{(x_r w+2t)^n}{n!} \frac{(v)_m}{(2v)_m} \frac{(-4t)^m}{m!}.
\end{aligned}$$

□

Corollary 2.5. *Setting $t = 0$ in Theorem 2.4, we get the following two-sided generating relation:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Omega(n) S_n^{0,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)w^n \\
= &\sum_{n,m=0}^{\infty} \Omega(m+nN_{r-1}) P_n^{m,N_1,N_2,...,N_{r-1}}(x_1 w^{(\prod_{j=1}^{r-1} N_j)}, ..., x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)}) \frac{(x_r w)^m}{m!}
\end{aligned}$$

which was earlier derived by [14] *and $r = 2$ case was presented by Srivastava et al* [23].

Using the similar technique in the proof of Theorem 2.2 and considering hypergeometric transformation (1.7), one can obtain the following theorem.

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Theorem 2.6. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\begin{aligned}(2.7)\quad &\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v)_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \\ &= \sum_{n,m,k_{r-1},...,k_1=0}^{\infty} \Omega(n+m+\digamma(r-1)) \\ &\cdot A_{n+m+\digamma(r-1),k_{r-1}+\digamma(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+\digamma(r-3)} \\ &\cdot \frac{(2v+n+m-1)_m}{(v)_m} \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} \frac{(x_2 w^{(\prod_{j=2}^{r-1} N_j)})^{k_2}}{k_2!} \cdots \frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)})^{k_{r-1}}}{k_{r-1}!} \\ &\cdot \frac{\left(\sqrt{x_r^2w^2-4t^2}\right)^n}{n!} \frac{\left(\frac{1}{2}(x_r w-\sqrt{x_r^2w^2-4t^2})\right)^m}{m!}, \\ &v \notin \mathbb{Z}_0^-\end{aligned}$$

provided that each member of the series identity (2.7) exists.

Setting $t=\frac{x_r w}{2}$ in Theorem 2.6 we get the following corollary.

Corollary 2.7. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\begin{aligned}(2.8)\qquad &\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v)_m} S_n^{2m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)\frac{(\frac{x_r w}{2})^{2m}}{m!}w^n \\ &= \sum_{m,k_{r-1},...,k_1=0}^{\infty} \Omega(m+\digamma(r-1)) A_{m+\digamma(r-1),k_{r-1}+\digamma(r-2),k_1,k_2+N_1k_1,k_3+N_2k_2+N_2N_1k_1,...,k_{r-2}+\digamma(r-3)} \\ &\cdot \frac{(2v+m-1)_m}{(v)_m} \frac{(x_1 w^{(\prod_{j=1}^{r-1} N_j)})^{k_1}}{k_1!} \frac{(x_2 w^{(\prod_{j=2}^{r-1} N_j)})^{k_2}}{k_2!} \cdots \frac{(x_{r-1} w^{(\prod_{j=r-1}^{r-1} N_j)})^{k_{r-1}}}{k_{r-1}!} \frac{\left(\frac{1}{2}x_r w\right)^m}{m!}, \\ &v \notin \mathbb{Z}_0^-\end{aligned}$$

provided that each member of the series identity (2.8) exists.

Next, by Theorem 2.6 we can state the following theorem.

Theorem 2.8. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers and $S_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_r)$ be defined by (1.4). Suppose also that $(r-1)-$variable polynomials $P_n^{m,N_1,N_2,...,N_{r-1}}(x_1,x_2,...,x_{r-1})$ are given by (1.5). Then the family*

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of two sided linear generating relations holds true between these polynomials:

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v)_m} S_n^{2m,N_1,N_2,\ldots,N_{r-1}}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}w^n \tag{2.9}$$

$$= \sum_{n,m,k_{r-1}=0}^{\infty} \Omega(n+m+k_{r-1}N_{r-1})P_{k_{r-1}}^{m+n,N_1,N_2,\ldots,N_{r-1}}(x_1 w^{(\prod_{j=1}^{r-1} N_j)},...,x_{r-1}w^{(\prod_{j=r-1}^{r-1} N_j)})$$

$$\cdot\frac{(2v+n+m-1)_m}{(v)_m}\frac{\left(\sqrt{x_r^2w^2-4t^2}\right)^n}{n!}\frac{\left(\frac{1}{2}(x_rw-\sqrt{x_r^2w^2-4t^2})\right)^m}{m!},\ v\notin\mathbb{Z}_0^-$$

provided that each member of the series identity (2.9) exists.

3. Some Applications of The Main Results

The polynomials $S_n^{m,N_1,N_2,\ldots,N_{r-1}}(x_1,x_2,...,x_r)$ involve the well known polynomials . For instance, if the take

$$N_1 = N_2 = ... = N_{r-1} = 1 \tag{3.1}$$

and

$$A_{m,n,k_2,k_3,\ldots,k_{r-2},k_{r-1}} = (\alpha_1)_{k_2}(\alpha_2)_{k_3-k_2}...(\alpha_{r-2})_{k_{r-1}-k_{r-2}}(\alpha_{r-1})_{n-k_{r-1}}(\alpha_r)_{m-n} \tag{3.2}$$

in equation (1.4), we have

$$\begin{aligned} A_{m+n,k_{r-1},k_1,k_2,\ldots,k_{r-2}} &= (\alpha_1)_{k_1}(\alpha_2)_{k_2-k_1}...(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{m+n-k_{r-1}} \\ &= (\alpha_1)_{k_1}(\alpha_2)_{k_2-k_1}...(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_m(\alpha_r+m)_{n-k_{r-1}} \end{aligned}$$

and therefore

$$S_n^{m,1,1,\ldots,1}(x_1,x_2,...,x_r) = (\alpha_r)_m g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r+m)}(x_1,x_2,...,x_r), \tag{3.3}$$

where $g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,...,x_r)$ are the Lagrange polynomials in several variables, which are known as Chan-Chyan Srivastava polynomials, defined through the generating function [4]

$$\begin{aligned} \prod_{j=1}^{r}(1-x_jt)^{-\alpha_j} &= \sum_{n=0}^{\infty} g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,...,x_r)t^n \\ (\alpha_j &\in \mathbb{C}\ (j=1,...,r); |t| < \min\left\{|x_1|^{-1},...,|x_r|^{-1}\right\}). \end{aligned}$$

It is obvious that the explicit expression of the $g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,...,x_r)$ in the form

$$g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,...,x_r) \tag{3.4}$$

$$= \sum_{k_{r-1}=0}^{n} ... \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(\alpha_1)_{k_1}(\alpha_2)_{k_2-k_1}...(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{n-k_{r-1}}$$

$$\cdot\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-k_1}}{(k_2-k_1)!}...\frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!}\frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!}.$$

Many authors have studied some properties of these polynomials. For example, the bilateral generating functions for these polynomials and miscellaneous properties are given in Liu et al. [18]. In [12], the orthogonality properties and various integral representations for these polynomials are given (see also [2], [3], [6], [7], [10]). Furthermore, these polynomials are used in approximation theory. In [8], they investigated some approximation properties of positive linear operators constructed by these polynomials (see also [9], [11]).

Furthermore, under the assumptions (3.1) and (3.2) we have

$$P_n^{m,1,1,...,1}(x_1,x_2,...,x_{r-1}) = (\alpha_r)_m g_n^{(\alpha_1,\alpha_2,...,\alpha_{r-1})}(x_1,x_2,...,x_{r-1}). \tag{3.5}$$

Hence, upon setting $N_1 = N_2 = ... = N_{r-1} = 1$ and considering (3.2) in Corollary 2.3, we get

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(v+\frac{1}{2})_m} (\alpha_r)_{2m} g_n^{(\alpha_1,\alpha_2,...,\alpha_{r-1},\alpha_r+2m)}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}(\frac{-2t}{x_r})^n \\ &= \sum_{m,k_{r-1},...,k_1=0}^{\infty} \Omega(m+k_1+k_2+...+k_{r-1})\ (\alpha_1)_{k_1}(\alpha_2)_{k_2}...(\alpha_{r-1})_{k_{r-1}}(\alpha_r)_m \\ &\qquad \cdot \frac{(x_1(\frac{-2t}{x_r}))^{k_1}}{k_1!}\frac{(x_2(\frac{-2t}{x_r}))^{k_2}}{k_2!}...\frac{(x_{r-1}(\frac{-2t}{x_r}))^{k_{r-1}}}{k_{r-1}!}\frac{(v)_m}{(2v)_m}\frac{(-4t)^m}{m!}. \end{aligned} \tag{3.6}$$

Choosing $\Omega(n) = 1$ in (3.6), we can state the following generating relation for $g_n^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,x_2,...,x_r)$,

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{(\alpha_r)_{2m}}{(v+\frac{1}{2})_m} g_n^{(\alpha_1,\alpha_2,...,\alpha_{r-1},\alpha_r+2m)}(x_1,x_2,...,x_r)\frac{t^{2m}}{m!}(\frac{-2t}{x_r})^n \\ &= {}_2F_1(\alpha_r,v;2v;-4t)(1+\frac{2tx_1}{x_r})^{-\alpha_1}(1+\frac{2tx_2}{x_r})^{-\alpha_2}...(1+\frac{2tx_{r-1}}{x_r})^{-\alpha_{r-1}} \\ &\qquad \left(\left|\frac{2tx_1}{x_r}\right|<1, \left|\frac{2tx_2}{x_r}\right|<1, ..., \left|\frac{2tx_{r-1}}{x_r}\right|<1\right), \end{aligned} \tag{3.7}$$

which $r = 3$ case was earlier derived by [15].

Also we note that, for every suitable choice of the function $\Omega(n)$ used in equation (3.6) , we derive various families of linear generating functions for the polynomials in $r-$variables defined by (3.4).

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Relationship between subsethood measure and entropy of interval-valued intuitionistic fuzzy sets

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Abstract

The main purpose of this paper is to establish a unified formulation of subsethood, entropy and cardinality of interval-valued intuitionistic fuzzy sets. The notion of average possible cardinality is presented and its connection to least and biggest cardinalities is established. Moreover, the entropy-subsethood theorem of interval-valued intuitionistic fuzzy sets, which generalizes the works of Kosko [Inform. Sci. 40 (1986) 165] of fuzzy sets and Vlachos and Sergiadis [Fuzzy Sets Syst. 158 (2007) 1384] of interval-valued fuzzy sets, is algebraically proved.

1 Introduction

Since Zadeh [23] introduced fuzzy sets, many scholars introduced various notions of higher-order fuzzy sets. Among them, interval-valued fuzzy sets (IVFSs) [24, 18] provides us with a flexible mathematical framework to deal with imperfect and/or imprecise information. In [9], Deng called IVFSs as grey sets. Atanassov [1, 2] introduced the concept of the intuitionstic fuzzy set (IFS), as a generalization of a fuzzy set. Atanassov and Gargov [4] showed that IFSs and IVFSs are equipollent generalizations of fuzzy sets. Bustince and Burillo [6] proved that the concept of vague sets identifies with the one of IFSs. As a generalization of an IFS, Atanassov and Gargov [4] introduced the notion of interval-valued intuitionstic fuzzy set (IVIFS), which is characterized by the membership and nonmembership functions whose values are intervals rather

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than exact numbers. Some researchers have investigated IVIFS theory and its relevant topics. Atanassov [3] gave some relations and operations over IVIFSs, and studied their basic properties. Bustince and Burillo [7] defined the concepts of correlation and correlation coefficient of IVIFSs, and obtained two decomposition theorems of the correlation of IVIFSs in terms of the correlation of IVFSs, the entropy and the correlation of IFSs. Hong [11] generalized the results of Bustince and Burillo into a general probability space. Hung and Wu [12] and Xu [22] proposed other approaches to deriving the correlation coefficients of IVIFSs. Park et al. [17] proposed three methods for measuring distances between IVIFSs and showed that these reduce to the Burillo and Bustince's distances [5] and Grzegorzewski's distances [10] of IVFSs, respectively. Zhang et al. [25] proposed the entropy and cross-entropy of IVIFSs, and discussed the relationships among some information measures of IVIFSs.

In this paper, we establish a unified framework between the concepts of subsethood, entropy and cardinality of IVIFSs. Then we review the axioms of subsethood of IVIFSs and propose an alternative axiomatic skeleton, in order for subsethood to reduce to entropy. Based on the axioms, we give an interval-valued intuitionistic version of the entropy-subsethood theorem and derive new measures of subsethood and entropy of IVIFSs. Furthermore, the concepts of cardinality and average possible cardinality of IVIFSs is presented. We carry out an algebraic and geometrical analysis, which demonstrates a connection between the above-mentioned cardinality and the least and biggest cardinalities. Finally, based on the average possible cardinality, we extend the fuzzy entropy theorem in the interval-valued intuitionistic fuzzy setting and provide connections between it and ones of IVFSs and fuzzy sets.

2 Basic notions

Throughout this paper, X denotes the universe set, and IVIFS(X), IVFS(X), IFS(X) and FS(X) stand for the set of all IVIFSs, IVFSs, IFSs and fuzzy sets on X, respectively. The operation "c" is the complements of IVIFS, IVFS, IFS or fuzzy set on X, respectively.

Let $I = [0, 1]$ and $[I]$ be the set of all closed subintervals of the interval $[0, 1]$. Then, by Zadeh's extension principle [23], we can popularize these operations such as $\vee$, $\wedge$ and c to $[I]$ and thus $([I], \vee, \wedge, c)$ is a complete lattice with a minimal element $\bar{0} = [0, 0]$ and a maximal element $\bar{1} = [1, 1]$. Furthermore, let $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$, then we have $\bar{a} = \bar{b} \Longleftrightarrow a^- = b^-, a^+ = b^+$, $\bar{a} \leq \bar{b} \Longleftrightarrow a^- \leq b^-, a^+ \leq b^+$, and $\bar{a} < \bar{b} \Longleftrightarrow \bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$.

Definition 2.1 [24, 18] An interval-valued fuzzy set (IVFS) A on X is defined as

$$A = \{(x, M_A(x)) : x \in X\}, \tag{1}$$

where the function $M_A : X \to [I]$ determines the degree of membership of an element x in A.

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For each $A \in \text{IVFS}(X)$, let $M_A(x) = [M_A^-(x), A_A^+(x)]$, where $M_A^-(x) \leq M_A^+(x)$ for any $x \in X$. Then fuzzy sets $M_A^- : X \to I$ and $M_A^+ : X \to I$ are called a lower fuzzy set of A and a upper fuzzy set of A, respectively. For simplicity, we denote $A = [M_A^-, M_A^+]$.

Definition 2.2 [1, 2] An intuitionistic fuzzy set (IFS) A on X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{2}$$

where the functions $\mu_A, \nu_A : X \to I$ determine the degree of membership and the degree of non-membership of $x \in X$ in A, respectively, and satisfy $\mu_A(x) + \nu_A(x) \leq 1$ for any $x \in X$. For simplicity, we denote $A = \langle \mu_A, \nu_A\rangle$.

For an IFS A on X, we call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ the intuitionistic index of an element $x \in X$ in A, which denotes the hesitancy degree of x to A. The Atanassov's operator [2] $D_\alpha : \text{IFS}(X) \to \text{FS}(X)$ is defined by

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle : x \in X\},\ \alpha \in [0,1], \tag{3}$$

for each IFS A.

Definition 2.3 [4] An interval-valued intuitionistic fuzzy set (IVIFS) A on X is defined as

$$A = \{(x, M_A(x), N_A(x)) : x \in X\}, \tag{4}$$

where $M_A : X \to [I]$ and $N_A : X \to [I]$ denote, respectively, membership function and nonmembership function of A and satisfy $0 \leq M_A^+(x) + N_A^+(x) \leq 1$ for any $x \in X$.

By $\mathcal{I} = \{(x, [1,1], [0,0]) : x \in X\}$ and $\mathcal{O} = \{(x, [0,0], [1,1]) : x \in X\}$, we denote the greatest and the smallest IVIFSs, respectively, and denote frequently $\mathcal{I} = ([1,1], [0,0])$ and $\mathcal{O} = ([0,0], [1,1])$ for simplicity. For $A \in \text{IVIFS}(X)$, let $A = ([M_A^-, M_A^+], [N_A^-, N_A^+])$. Then $A_L = \langle M_A^-, N_A^-\rangle$ and $A_U = \langle M_A^+, N_A^+\rangle$ is called lower IFS and upper IFS of A, respectively. Thus, under these notions, we can give another representation of an IVIFS A as $A = (A_L, A_U)$. This representation gives us to describe the pseudo average possible cardinality of an IVIFS in the Euclidean plane, as it will be demonstrated in Section 3. An IVIFS A defined on a single-element universe $X = \{x\}$, is represented by two points, whose coordinates correspond to the membership and nonmembership degrees, as shown in Fig. 1.

Definition 2.4 Let $A, B \in \text{IVIFS}(X)$. Then

(1) $A \cup B = \{(x, [\max(M_A^-(x), M_B^-(x)), \max(M_A^+(x), M_B^+(x))], [\min(N_A^-(x), N_B^-(x)), \min(N_A^+(x), N_B^+(x))]) : x \in X\}$.

(2) $A \cap B = \{(x, [\min(M_A^-(x), M_B^-(x)), \min(M_A^+(x), M_B^+(x))], [\max(N_A^-(x), N_B^-(x)), \max(N_A^+(x), N_B^+(x))]) : x \in X\}$.

(3) A is subset of B, denoted by $A \subset B$, iff $M_A(x) \leq M_B(x)$ and $N_A(x) \geq M_B(x)$, for all $x \in X$.

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(4) A refines B (i.e., A is less fuzzy than B), denoted as $A \leq B$, iff $M_A(x) \geq M_B(x)$ and $N_A(x) \leq N_B(x)$ for $M_B(x) \geq N_B(x)$, or $M_A(x) \leq M_B(x)$ and $N_A(x) \geq N_B(x)$ for $M_B(x) \leq N_B(x)$.

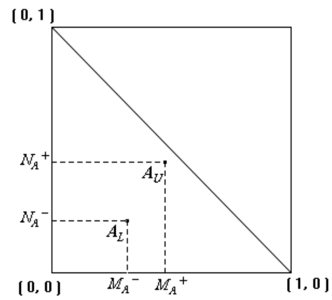

Figure 1: Geometrical interpretation of an IVIFS $A = (A_L, A_U)$ in $X = \{x\}$.

3 Cardinality of IVIFSs

Szmit and Kacprzyk [20] defined the concept of cardinality of IFSs. Vlachos and Sergiadis [21] provided an interpretation of cardinality under a geometrical framework and present the concept of average possible cardinality of IFSs. We extend these concepts in the interval-valued intuitionistic fuzzy setting.

Definition 3.1 For a set $A \in$ IVIFS(X) the following two cardinalities are defined:

- the least cardinality or min-sigma-count, which is given by

$$\min \sum Count(A) = \sum_{x_i \in X} \frac{M_A^+(x_i) + M_A^-(x_i)}{2} \tag{5}$$

- the biggest cardinality or max-sigma-count defined as

$$\max \sum Count(A) = \sum_{x_i \in X} \frac{2 - (N_A^+(x_i) + N_A^-(x_i))}{2}. \tag{6}$$

The cardinality of the IVIFS A is defined as the interval

$$card(A) = \left[\min \sum Count(A), \max \sum Count(A)\right]. \tag{7}$$

For the smallest IVIFS $\mathcal{O}$, according to Vlachos and Sergiadis's definition of cardinality for IFSs, we call the magnitude $\mathcal{M}(A)$ of the vector $\overrightarrow{\mathcal{O}A}$, using Hamming distance, the *average possible cardinality* of the IVIFS A. The

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characterization of *average possible cardinality* will be justified by the following analysis. The Hamming distance between two IVIFSs A and B is given in [17]

$$d_1'(A,B) = \frac{1}{4}\sum_{i=1}^{n}\big(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ +|N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|\big). \tag{8}$$

Definition 3.2 For $A \in \text{IVIFS}(X)$, the average possible cardinality $\mathcal{M}(A)$ is defined as

$$\begin{aligned}\mathcal{M}(A) &= d_1'(\mathcal{O}, A) \\ &= \frac{1}{4}\sum_{x_i\in X}\big(2 + M_A^-(x_i) - N_A^-(x_i) + M_A^+(x_i) - N_A^+(x_i)\big).\end{aligned} \tag{9}$$

From (9), taking into account (7), it follows that $\mathcal{M}(A)$ is the midpoint of the interval $[\min\sum Count(A), \max\sum Count(A)]$. So, (9) encompasses the notions of least, biggest and average possible cardinalities.

In particular, if an IVIFS A becomes an IFS, then $\mathcal{M}(A)$ reduces to average possible cardinality of IFS. To derive connection between the cardinality of IVIFSs and that of IFSs, for $A \in \text{IVIFS}(X)$, we consider the lower IFS $A_L = \langle M_A^-, N_A^-\rangle$ and the upper IFS $A_U = \langle M_A^+, N_A^+\rangle$ of A. From (9) and Definition 18 of [21], we obtain

$$\begin{aligned}\mathcal{M}(A) &= \frac{1}{2}\sum_{x_i\in X}\left(\frac{M_A^-(x_i) + M_A^+(x_i)}{2} + \frac{2 - N_A^-(x_i) - N_A^+(x_i)}{2}\right) \\ &= \frac{1}{2}\sum_{x_i\in X}\left(M_A^-(x_i) + \frac{\pi_{A_L}(x_i)}{2} + M_A^+(x_i) + \frac{\pi_{A_U}(x_i)}{2}\right) \\ &= \frac{\mathcal{M}(A_L) + \mathcal{M}(A_U)}{2}.\end{aligned} \tag{10}$$

Thus, the average possible cardinality of IVIFS A is half of the sum of the average possible cardinalities of A_L and A_U. Moreover, from Proposition 20 of [21], we obtain the following result

$$\mathcal{M}(A) = M(D_{0.5}(A_L)) + M(D_{0.5}(A_U)), \tag{11}$$

where $M(D_{0.5}(A_L))$ and $M(D_{0.5}(A_U))$ are cardinalities of fuzzy sets $D_{0.5}(A_L)$ and $D_{0.5}(A_U)$, respectively.

Vlachos and Sergiadis [21] provided a geometrical interpretation of the average possible cardinality of IFSs. In order to consist with Vlachos and Sergiadis's work, we modify the Hamming distance of IVIFSs as follows

$$d_1''(A,B) = \frac{1}{2}\sum_{i=1}^{n}\big(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ +|N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|\big). \tag{12}$$

Then, using (12), a modified definition of the average possible cardinality is obtained as

$$\begin{aligned}\mathcal{M}'(A) &= d_1''(\mathcal{O}, A)\\ &= \frac{1}{2}\sum_{x_i \in X}\left(2 + M_A^-(x_i) - N_A^-(x_i) + M_A^+(x_i) - N_A^+(x_i)\right), \qquad (13)\end{aligned}$$

which will be called *pseudo-average possible cardinality*, since $\mathcal{M}'(A)$ does not coincide with the midpoint of the interval $[\min \sum Count(A), \max \sum Count(A)]$.

From (13), taking into account (10), we obtain the following

$$\mathcal{M}'(A) = \frac{\mathcal{M}'(A_L) + \mathcal{M}'(A_U)}{2}. \qquad (14)$$

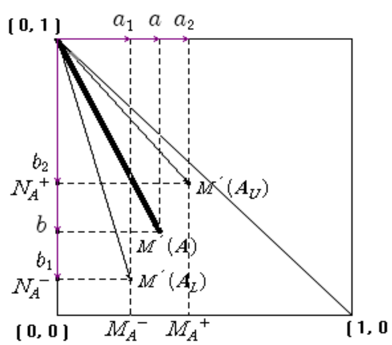

Figure 2: Representation of the pseudo-average possible cardinality $\mathcal{M}'(A)$ in the unit square and its geometrical connection with least and biggest cardinalities.

Fig. 2 depicts the notion of pseudo-average possible cardinality $\mathcal{M}'(A)$ in case of an IVIFS A in $X = \{x\}$. One can observe that the vectors $\overrightarrow{\mathcal{O}A_L}$ and $\overrightarrow{\mathcal{O}A_U}$ can be projected onto the membership and nonmembership axes, deriving the vectors a_1, a_2, b_1 and b_2, respectively, as illustrated in Fig. 2. The magnitudes of projections are

$$|a_1| = \mu_{A_L}(x) = M_A^-(x), \quad |a_2| = \mu_{A_U}(x) = M_A^+(x), \qquad (15)$$

and

$$\begin{aligned}|b_1| &= 1 - \nu_{A_L}(x) = \mu_{A_L}(x) + \pi_{A_L}(x) = M_A^-(x) + \pi_{A_L}(x),\\ |b_2| &= 1 - \nu_{A_L}(x) = \mu_{A_L}(x) + \pi_{A_L}(x) = M_A^+(x) + \pi_{A_U}(x). \qquad (16)\end{aligned}$$

Moreover, from (15) and (16), we deduce that the magnitudes of the projections on the membership axis equal the min-sigma-counts of A_L and A_U, respectively,

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in the case of an one element universe $X = \{x\}$, while the magnitudes of the projections on the nonmembership axis coincide with the max-sigma-counts, respectively. Thus, from (14), the magnitude $|a|$ of the projection on the membership axis equals the min-sigma-count of A, while the magnitude $|b|$ of the projection on the nonmembership axis coincides with the max-sigma-count.

4 Subsethood of IVIFSs

We extend the works of Liu and Xiong [16] and Vlachos and Sergiadis [21], in order to establish a connection between subsethood, entropy and cardinality in interval-valued intuitionistic fuzzy setting.

Definition 4.1 A real function $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ is called a subsethood measure of IVIFSs if S satisfies the following properties:
(S1) $S(A,B) = 1$ iff $A \subset B$;
(S2) If $A^c \subset A$, then $S(A, A^c) = 0$ iff $A = \mathcal{I}$;
(S3) If $B \subset A_1 \subset A_2$, then $S(A_1, B) \geq S(A_2, B)$, and if $B_1 \subset B_2$, then $S(A, B_1) \leq S(A, B_2)$.

Theorem 4.2 *For two* IVIFS*s A and B on $X = \{x_1, x_2, \ldots, x_n\}$,*

$$\begin{aligned} & S_{\mathrm{IVIFS}}(A,B) \\ & \quad = 1 - \Bigg(\frac{\sum_{x_i \in X} \big(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\} \big)}{\sum_{x_i \in X} \big(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) \big)} \\ & \qquad + \frac{\sum_{x_i \in X} \big(\max\{0, N_B^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\} \big)}{\sum_{x_i \in X} \big(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) \big)} \Bigg) \end{aligned} \tag{17}$$

is a subsethood measure of IVIFS*s.*

Proof (S1) Let $A \subset B$. Then $M_A^-(x_i) \leq M_B^-(x_i)$, $M_A^+(x_i) \leq M_B^+(x_i)$, $N_A^-(x_i) \geq N_B^-(x_i)$ and $N_A^+(x_i) \geq N_B^+(x_i)$ for all $x_i \in X$. Thus, we have $\max\{0, M_A^-(x_i) - M_B^-(x_i)\} = 0$, $\max\{0, M_A^+(x_i) - M_B^+(x_i)\} = 0$, $\max\{0, N_B^-(x_i) - N_A^-(x_i)\} = 0$ and $\max\{0, N_B^+(x_i) - N_A^+(x_i)\} = 0$ for all $x_i \in X$. Therefore, we obtain $S_{\mathrm{IVIFS}}(A,B) = 1$. Suppose that $S_{\mathrm{IVIFS}}(A,B) = 1$. Then,

$$\begin{aligned} \sum_{x_i \in X} & \big(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\} \\ & + \max\{0, N_B^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\} \big) = 0. \end{aligned} \tag{18}$$

Since every term of the sum is non-negative, we deduce $\max\{0, M_A^-(x_i) - M_B^-(x_i)\} = 0$, $\max\{0, M_A^+(x_i) - M_B^+(x_i)\} = 0$, $\max\{0, N_B^-(x_i) - N_A^-(x_i)\} = 0$ and $\max\{0, N_B^+(x_i) - N_A^+(x_i)\} = 0$ for all $x_i \in X$, which implies that $M_A^-(x_i) \leq M_B^-(x_i)$, $M_A^+(x_i) \leq M_B^+(x_i)$, $N_A^-(x_i) \geq N_B^-(x_i)$ and $N_A^+(x_i) \geq N_B^+(x_i)$. Hence $A \subset B$.

(S2) From (17) we obtain that

$$S_{\mathrm{IVIFS}}(A, A^c) = 1 - \frac{2\sum_{x_i\in X}(\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\}}{\sum_{x_i\in X}(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))}. \tag{19}$$

Assume that $A = \mathcal{I} \supset A^c$. Evaluating (19) for $A = \mathcal{I}$, we deduce that $S_{\mathrm{IVIFS}}(A, A^c) = 0$. Let us now consider that $S_{\mathrm{IVIFS}}(A, A^c) = 0$ and $A^c \subset A$. Then, (17) yields

$$\sum_{x_i\in X}\big[(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) \\ -2(\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\})\big] = 0. \tag{20}$$

Since $A^c \subset A$, we obtain that $N_A^-(x_i) \le M_A^-(x_i)$ and $N_A^+(x_i) \le M_A^+(x_i)$ for all x_i. Hence $\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\} = (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))$ and from (20) we derive that $\sum_{x_i\in X}(2 - (M_A^-(x_i) - N_A^-(x_i)) - (M_A^+(x_i) - N_A^+(x_i))) = 0$. However, $2 - (M_A^-(x_i) - N_A^-(x_i)) - (M_A^+(x_i) - N_A^+(x_i)) \ge 0$ for all $x_i \in X$. Thus, every summand should equal zero, that is $2 - (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) = 0$ for all $x_i \in X$. Therefore, $A = \mathcal{I}$.

(S3) Let $A_1, A_2, B \in \mathrm{IVIFS}(X)$ such that $B \subset A_1 \subset A_2$. Since $B \subset A_1$, we obtain

$$S_{\mathrm{IVIFS}}(A_1, B) = \frac{\sum_{x_i\in X}(2 + (M_B^-(x_i) - N_B^-(x_i)) + (M_B^+(x_i) - N_B^+(x_i)))}{\sum_{x_i\in X}(2 + (M_{A_1}^-(x_i) - N_{A_1}^-(x_i)) + (M_{A_1}^+(x_i) - N_{A_1}^+(x_i)))}. \tag{21}$$

Similarly, we get

$$S_{\mathrm{IVIFS}}(A_2, B) = \frac{\sum_{x_i\in X}(2 + (M_B^-(x_i) - N_B^-(x_i)) + (M_B^+(x_i) - N_B^+(x_i)))}{\sum_{x_i\in X}(2 + (M_{A_2}^-(x_i) - N_{A_2}^-(x_i)) + (M_{A_2}^+(x_i) - N_{A_2}^+(x_i)))}. \tag{22}$$

Since $A_1 \subset A_2$, we obtain that $M_{A_2}^-(x_i) - N_{A_2}^-(x_i) \ge M_{A_1}^-(x_i) - N_{A_1}^-(x_i)$ and $M_{A_2}^+(x_i) - N_{A_2}^+(x_i) \ge M_{A_1}^+(x_i) - N_{A_1}^+(x_i)$ for all $x_i \in X$. Hence, $\sum_{x_i\in X}(2 + (M_{A_1}^-(x_i) - N_{A_1}^-(x_i)) + (M_{A_1}^+(x_i) - N_{A_1}^+(x_i))) \le \sum_{x_i\in X}(2 + (M_{A_2}^-(x_i) - N_{A_2}^-(x_i)) + (M_{A_2}^+(x_i) - N_{A_2}^+(x_i)))$. Thus, from (21) and (22), we get $S_{\mathrm{IVIFS}}(A_1, B) \ge S_{\mathrm{IVIFS}}(A_2, B)$.

Now assume that $B_1 \subset B_2$. Then $M_A^-(x_i) - M_{B_1}^-(x_i) \ge M_A^-(x_i) - M_{B_2}^-(x_i)$, $M_A^+(x_i) - M_{B_1}^+(x_i) \ge M_A^+(x_i) - M_{B_2}^+(x_i)$, $N_{B_1}^-(x_i) - N_A^-(x_i) \ge N_{B_2}^-(x_i) - N_A^-(x_i)$ and $N_{B_1}^+(x_i) - N_A^+(x_i) \ge N_{B_2}^+(x_i) - N_A^+(x_i)$ for all $x_i \in X$. Due to the

monotonicity of the max operator, it follows that

$$\begin{aligned}\sum_{x_i\in X}\big(&\max\{0,M_A^-(x_i)-M_{B_1}^-(x_i)\}+\max\{0,M_A^+(x_i)-M_{B_1}^+(x_i)\}\\&+\max\{0,N_{B_1}^-(x_i)-N_A^-(x_i)\}+\max\{0,N_{B_1}^+(x_i)-N_A^+(x_i)\}\big)\\\geq\sum_{x_i\in X}(&\max\{0,M_A^-(x_i)-M_{B_2}^-(x_i)\}+\max\{0,M_A^+(x_i)-M_{B_2}^+(x_i)\}\\&+\max\{0,N_{B_2}^-(x_i)-N_A^-(x_i)\}+\max\{0,N_{B_2}^+(x_i)-N_A^+(x_i)\}).\quad(23)\end{aligned}$$

Therefore, $S_{\text{IVIFS}}(A,B_1)\leq S_{\text{IVIFS}}(A,B_2)$.

Remark 4.3 Note that if $A=\mathcal{O}$, (17) is undefined, due to the fact that $\mathcal{M}(\mathcal{O})=0$. However, since $\mathcal{O}$ is a proper subset of any IVIFS B, from Definition 4.1, we must have that $S_{\text{IVIFS}}(\mathcal{O},B)=1$.

5 Entropy of IVIFSs

In this section, similar to the works of Szmidt and Kacprzyk [20] and Zeng and Li [26], we extend De Luca and Termini axioms [8] for fuzzy entropy to introduce the entropy concept of IVIFSs.

Definition 5.1 A real function $E:\text{IVIFS}(X)\to[0,1]$ is called an entropy of IVIFSs if E satisfies the following properties:
(E1) $E(A)=0$ iff A is a crisp set;
(E2) $E(A)=1$ iff $M_A(x)=N_A(x)$ for all $x\in X$;
(E3) $E(A)\leq E(B)$ if A refines B;
(E4) $E(A)=E(A^c)$.

Generalizing the works of Kosko [13] and Vlachos and Sergiadis [21], we state the entropy-subsethood theorem for IVIFSs, based on the axiomatic skeleton (S1)-(S3).

Theorem 5.2 *Suppose S is a subsethood measure of* IVIFS*s on X and $A\in$* IVIFS(X)*, then*

$$E(A)=S(A\cup A^c,A\cap A^c)\tag{24}$$

is an entropy measure of A.

Proof (E1) Let A be a crisp set. Then $A\cup A^c=\mathcal{I}$ and $A\cap A^c=\mathcal{O}$. Since $A\cap A^c=(A\cup A^c)^c$, we have $A\cup A^c=\mathcal{I}\supset(A\cup A^c)^c$ and thus, from (S2), we obtain $E(A)=0$. Suppose that $E(A)=0$; that is $S(A\cup A^c,A\cap A^c)=0$, which can be written as $(A\cup A^c,(A\cup A^c)^c)=0$. Then, since $A\cup A^c\supset A\cap A^c=(A\cup A^c)^c$, by (S2) we obtain $A\cup A^c=\mathcal{I}$. Hence A is a crisp set.

(E2) Let us consider that $M_A(x_i)=N_A(x_i)$ for all $x_i\in X$. Then, $A\cup A^c=A\cap A^c=A=A^c$ and thus from (S1), we obtain $E(A)=1$. Let us assume that

$E(A) = 1$; that is $S(A \cup A^c, A \cap A^c) = 1$. Then, from (S1), we deduce that $A \cup A^c \subset A \cap A^c$. However, for any IVIFS A, it hold that $A \cup A^c \supset A \cap A^c$. Hence, $A \cup A^c = A \cap A^c$, which implies $M_A(x_i) = N_A(x_i)$ for all $x_i \in X$.

(E3) Suppose that A refines B. Then, since $A \cap A^c \subset B \cap B^c \subset B \cup B^c \subset A \cup A^c$, from (S3), we derive that $E(A) = S(A \cup A^c, A \cap A^c) \leq S(B \cup B^c, B \cap B^c) = E(B)$. Hence $E(A) \leq E(B)$.

(E4) It is evident that $E(A^c) = S(A^c \cup A, A^c \cap A) = E(A)$.

Remark 5.3 Theorem 5.2 describes an interesting relationship between the entropy and subsethood measure of IVIFSs. It states that the entropy (24) expresses the degree to which the $A \cup A^c$ is a subset of its own subset $A \cap A^c$. Evaluating (24) for the subsethood measure (17), yields a new entropy of IVIFSs given by

$$\begin{aligned} &E_{\text{IVIFS}}(A) \\ &= \frac{\sum_{x_i \in X} \big(2 - \max\{M_A^-(x_i), N_A^-(x_i)\} + \min\{M_A^-(x_i), N_A^-(x_i)\}}{\sum_{x_i \in X} \big(2 + \max\{M_A^-(x_i), N_A^-(x_i)\} - \min\{M_A^-(x_i), N_A^-(x_i)\}} \\ &\qquad \frac{- \max\{M_A^+(x_i), N_A^+(x_i)\} + \min\{M_A^+(x_i), N_A^+(x_i)\}\big)}{+ \max\{M_A^+(x_i), N_A^+(x_i)\} - \min\{M_A^+(x_i), N_A^+(x_i)\}\big)}. \end{aligned} \tag{25}$$

Now, we state a relationship between the entropy and average possible cardinality of IVIFSs, which generalize the works of Kosko [13, 14, 15] and Vlachos and Sergiadis [21].

Theorem 5.4 *Suppose $\mathcal{M}$ is an average possible cardinality of* IVIFS*s on X and $A \in$ IVIFS(X), then*

$$E(A) = \frac{\mathcal{M}(A \cap A^c)}{\mathcal{M}(A \cup A^c)} \tag{26}$$

is an entropy measure of A.

Proof For $A \in$ IVIFS(X) and its complement A^c, it hold that

$$\begin{aligned} A \cup A^c = \{(x_i, &[\max\{M_A^-(x_i), N_A^-(x_i)\}, \max\{M_A^+(x_i), N_A^+(x_i)\}], \\ &[\min\{N_A^-(x_i), M_A^-(x_i)\}, \min\{N_A^+(x_i), M_A^+(x_i)\}]) : x_i \in X\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} A \cap A^c = \{(x_i, &[\min\{M_A^-(x_i), N_A^-(x_i)\}, \min\{M_A^+(x_i), N_A^+(x_i)\}], \\ &[\max\{N_A^-(x_i), M_A^-(x_i)\}, \max\{N_A^+(x_i), M_A^+(x_i)\}]) : x_i \in X\}. \end{aligned} \tag{28}$$

From Definition 3.2, we obtain that

$$\begin{aligned} &\mathcal{M}(A \cup A^c) \\ &= \frac{1}{4} \sum_{x_i \in X} \big(2 + \max\{M_A^-(x_i), N_A^-(x_i)\} - \min\{M_A^-(x_i), N_A^-(x_i)\} \\ &\qquad + \max\{M_A^+(x_i), N_A^+(x_i)\} - \min\{M_A^+(x_i), N_A^+(x_i)\}\big) \end{aligned} \tag{29}$$

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and

$$\begin{aligned}&\mathcal{M}(A\cap A^c)\\&\quad=\frac{1}{4}\sum_{x_i\in X}\big(2+\min\{M_A^-(x_i),N_A^-(x_i)\}-\max\{M_A^-(x_i),N_A^-(x_i)\}\\&\qquad\qquad+\min\{M_A^+(x_i),N_A^+(x_i)\}-\max\{M_A^+(x_i),N_A^+(x_i)\}\big).\end{aligned}\tag{30}$$

Substituting (29) and (30) into (25) yields (26). This completes the proof.

6 Modified entropy-subsethood theorem

A desirable property that an entropy measure should possess, is described as

$$E(A)=\sum_{i=1}^{n}E(A_i),\tag{31}$$

which states that the sum of the entropy of separate of a set is equal to the entropy of the set. It is evident that the entropy $E_{\rm IVIFS}$ (25) does not satisfy the condition (31), that is

$$E_{\rm IVIFS}(A)\neq\sum_{i=1}^{n}E_{\rm IVIFS}(A_i).\tag{32}$$

In order to overcome the above-mentioned drawback, we can consider the analysis of Section 4 to be carried out element-wisely, instead of considering the entire set. Thus, we modify the subsethood mesure (17) as follows

$$\begin{aligned}&S'_{\rm IVIFS}(A,B)\\&\quad=1-\frac{1}{n}\sum_{x_i\in X}\left(\frac{\big(\max\{0,M_A^-(x_i)-M_B^-(x_i)\}+\max\{0,M_A^+(x_i)-M_B^+(x_i)\}\big)}{\big(2+(M_A^-(x_i)-N_A^-(x_i))+(M_A^+(x_i)-N_A^+(x_i))\big)}\right.\\&\qquad\left.+\frac{\big(\max\{0,N_B^-(x_i)-N_A^-(x_i)\}+\max\{0,N_B^+(x_i)-N_A^+(x_i)\}\big)}{\big(2+(M_A^-(x_i)-N_A^-(x_i))+(M_A^+(x_i)-N_A^+(x_i))\big)}\right)\end{aligned}\tag{33}$$

It is easy to verify that $S'_{\rm IVIFS}$ also satisfies the conditions (S1)-(S3). Thus, from the entropy-subsethood theorem (24), the following modified entropy measure $E'_{\rm IVIFS}$ is derived

$$\begin{aligned}&E'_{\rm IVIFS}(A)\\&\quad=\frac{1}{n}\sum_{x_i\in X}\left(\frac{2-\max\{M_A^-(x_i),N_A^-(x_i)\}+\min\{M_A^-(x_i),N_A^-(x_i)\}}{2+\max\{M_A^-(x_i),N_A^-(x_i)\}-\min\{M_A^-(x_i),N_A^-(x_i)\}}\right.\\&\qquad\left.\frac{-\max\{M_A^+(x_i),N_A^+(x_i)\}+\min\{M_A^+(x_i),N_A^+(x_i)\}}{+\max\{M_A^+(x_i),N_A^+(x_i)\}-\min\{M_A^+(x_i),N_A^+(x_i)\}}\right),\end{aligned}\tag{34}$$

which can be re-written as

$$E'_{\mathrm{IVIFS}}(A) = \frac{1}{n}\sum_{i=1}^{n}\frac{\mathcal{M}(A_i \cap A_i^c)}{\mathcal{M}(A_i \cup A_i^c)}. \tag{35}$$

From (34) and (35), we obtain $E'_{\mathrm{IVIFS}}(A) = \sum_{i=1}^{n} E'_{\mathrm{IVIFS}}(A_i)$. Note that, in (35), the average possible cardinality $\mathcal{M}$ is calculated over the single-element universe $X = \{x_i\}$, where the set A_i is defined. Furthermore, if IVIFS A become either an IVFS with $M_A^+ = (M_A^-)^c$ or a fuzzy set, then (35) reduces the modified entropy measure E'_{IVFS}, proposed by Vlachos and Sergiadis [21], of IVFSs and the entropy measure E_{SJ}, proposed by Shang and Jiang [19], of fuzzy sets, respectively.

7 Conclusions

In this paper, a framework for susethood, entropy and cardinality for IVIFSs under conditions (S1)-(S3) and (E1)-(E4) was established. We introduced an axiomatic skeleton for subsethood, and proposed new subsethood and entropy measures in the interval-valued intuitionistic fuzzy setting, and presented the notion of average possible cardinality and derived both algebraically and geometrically its properties. Connections with the cardinalities of fuzzy sets and IVFSs were established. Moreover, we stated and proved interval-valued intuitionistic fuzzy versions of the entropy and entropy-subsethood theorems, which generalize the works of Kosko for fuzzy sets and Vlachos and Sergiadis for IVFSs.

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On preconditioned GAOR methods for weighted linear least squares problems

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Abstract

In this paper, we consider the preconditioned generalized accelerated overrelaxation (GAOR) methods for solving weighted linear least squares problems. New type of preconditioners are proposed and the convergence rates of the new preconditioned GAOR methods are studied. Comparison results show the effectiveness of the proposed preconditioners in this paper. A numerical example is given to confirm our theoretical results.

Keywords: Preconditioner; GAOR method; Preconditioned GAOR method; Weighted linear least squares problem

1 Introduction

In this paper, we consider the weighted linear least squares problem

$$\min_{x\in\mathbb{R}^n}(Ax-b)^TW^{-1}(Ax-b),$$

where $A\in\mathbb{R}^{n\times n}$, $b\in\mathbb{R}^n$ and $W\in\mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. This problem has many scientific applications, a typical source is a parameter estimation in mathematical modelling, see [9, 10] for details. In order to solve it, one has to solve a linear system of the form

$$Hy=f,\quad y,f\in\mathbb{R}^n, \tag{1.1}$$

where

$$H=\begin{pmatrix} I_p-B & U \\ L & I_q-C \end{pmatrix}$$

is a non-singular matrix with $B=(b_{ij})\in\mathbb{R}^{p\times p}$, $C=(c_{ij})\in\mathbb{R}^{q\times q}$, $L=(l_{ij})\in\mathbb{R}^{q\times p}$, $U=(u_{ij})\in\mathbb{R}^{p\times q}$, and $p+q=n$. Here and elsewhere in the paper, I_k denotes the identity matrix with dimension k.

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Many iterative methods for solving the linear system (1.1) have been studied by many authors [5, 6]. Such iterative methods require the inverses of $I_p - B$ and $I_q - C$, which is a main drawback of these methods. To avoid this problem, Yuan and Jin [10] proposed the generalized accelerated overrelaxation (GAOR) method for solving the linear system (1.1), and the convergence of the GAOR method have been studied in [3, 4, 10]. The GAOR method for solving the linear system (1.1) is defined by

$$y_{k+1} = T_{\gamma\omega} y_k + \omega g, \quad k = 0, 1, 2, \cdots, \tag{1.2}$$

where

$$\begin{aligned} T_{\gamma\omega} &= \begin{pmatrix} I_p & 0 \\ \gamma L & I_q \end{pmatrix}^{-1} \left[(1-\omega) I_n + (\omega - \gamma) \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix} \right] \\ &= \begin{pmatrix} (1-\omega) I_p + \omega B & -\omega U \\ \omega(\gamma - 1) L - \omega\gamma L B & (1-\omega) I_q + \omega C + \omega\gamma L U \end{pmatrix} \end{aligned} \tag{1.3}$$

is the iteration matrix,

$$g = \begin{pmatrix} I_p & 0 \\ -\gamma L & I_q \end{pmatrix} f,$$

and ω and γ are real parameters with $\omega \neq 0$.

The spectral radius of the iteration matrix $T_{\gamma\omega}$ is decisive for the convergence, and the smaller it is, the faster the method converges. For improving the convergent rate of the corresponding iterative method, preconditioning techniques are used [2]. Especially, we consider the following equivalent left preconditioned linear system of (1.1)

$$PHy = Pf, \tag{1.4}$$

where $P \in \mathbb{R}^{n \times n}$, called the left preconditioner, is nonsingular.

If we express PH as

$$PH = \begin{pmatrix} I_p - \widehat{B} & \widehat{U} \\ \widehat{L} & I_q - \widehat{C} \end{pmatrix},$$

then the GAOR method for solving the preconditioned linear system (1.4), which is called *the preconditioned GAOR method* for solving the linear system (1.1), is defined by

$$y_{k+1} = \widehat{T}_{\gamma\omega} y_k + \omega \widehat{g}, \quad k = 0, 1, 2, \cdots, \tag{1.5}$$

where

$$\widehat{T}_{\gamma\omega} = \begin{pmatrix} (1-\omega) I_p + \omega \widehat{B} & -\omega \widehat{U} \\ \omega(\gamma - 1) \widehat{L} - \omega\gamma \widehat{L} \widehat{B} & (1-\omega) I_q + \omega \widehat{C} + \omega\gamma \widehat{L} \widehat{U} \end{pmatrix}$$

and

$$\widehat{g} = \begin{pmatrix} I_p & 0 \\ -\gamma \widehat{L} & I_q \end{pmatrix} Pf.$$

Recently, some left preconditioners P have been proposed for GAOR methods for solving the linear system (1.1), see [12, 11, 7]. Yun in [11] have modified the preconditioners in [12]. Based on the ideals of Yun [11], in this paper, we propose a new type of preconditioners, which are the modifications of the preconditioners in [7], and study the convergence rates of the new preconditioned GAOR methods for solving the linear system (1.1).

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The remainder of the paper is organized as follows. Next section is the preliminaries. In Section 3, a new type of preconditioners are introduced and the corresponding preconditioned GAOR methods are proposed. In Section 4, we compare the convergence rates of the new preconditioned GAOR methods with that of the original GAOR method, those of the preconditioned GAOR methods proposed by Wang *et al.* [7]. Comparison results show that the convergence rates of the new preconditioned GAOR methods are better than those of the preconditioned GAOR methods proposed by Wang *et al.* [7], that of the GAOR method whenever these methods are convergent. In Section 5, a numerical example is provided in order to confirm the theoretical results. Finally, some conclusions are drawn in Section 6.

2 Preliminaries

For $A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geq B(A > B)$ if $a_{ij} \geq b_{ij}(a_{ij} > b_{ij})$ holds for all $i, j = 1, 2\cdots, n$. We say that A is nonnegative (positive) if $A \geq 0(A > 0)$, and $A - B \geq 0$ if and only if $A \geq B$. These definitions carry immediately over to vectors by identifying them with $n \times 1$ matrices. $\rho(*)$ denotes the spectral radius of a square matrix. A is called irreducible if the directed graph of A is strongly connected [6].

Some useful results which we refer to later are provided below.

Lemma 2.1. [6] *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative and irreducible matrix. Then*

(a). *A has a positive eigenvalue equal to $\rho(A)$;*

(b). *A has an eigenvector $x > 0$ corresponding to $\rho(A)$.*

Lemma 2.2. [1] *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then*

(a). *If $\alpha x \leq Ax$ for a vector $x \geq 0$ and $x \neq 0$, then $\alpha \leq \rho(A)$.*

(b). *If $Ax \leq \beta x$ for a vector $x > 0$, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\alpha < \rho(A) < \beta$ and $x > 0$.*

3 Preconditioned GAOR methods

In this section, we will propose a new type of preconditioners for GAOR methods for solving the linear system (1.1). Let us recall the preconditioners, proposed by Wang *et al.* [7], are of the form

$$P_i^{(1)} = \begin{pmatrix} I_p + S_i & 0 \\ 0 & I_q \end{pmatrix}, \quad i = 1, 2,$$

where for $\alpha_s > 0$ and $\beta_s > 0$ $(s = 2, 3, \cdots, p)$,

$$S_1 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & \alpha_p b_{1p} \\ \beta_2 b_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_p b_{p1} & 0 & \cdots & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & 0 & 0 \\ \beta_2 b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \alpha_p b_{p-1,p} \\ 0 & 0 & \cdots & \beta_p b_{p,p-1} & 0 \end{pmatrix}.$$

Applying the GAOR method to the peconditioned linear system (1.4) with the preconditioners $P_i^{(1)}$, Wang *et al.* [7] studied the preconditioned GAOR methods

$$y_{k+1} = \widehat{T}^{(1)}_{\gamma\omega i} y_k + \omega \widehat{g}^{(1)}, \quad k = 0, 1, 2, \cdots, \ i = 1, 2, \tag{3.1}$$

where $\widehat{T}^{(1)}_{\gamma\omega i}$ are the iteration matrices, $\widehat{g}^{(1)}$ are the corresponding known vectors.

Based on the ideals of Yun [11], in this paper, we propose new preconditioners $P_i^{(2)}$ of the form

$$P_i^{(2)} = \begin{pmatrix} I_p + S_i & 0 \\ 0 & I_q + V_i \end{pmatrix}, \quad i = 1, 2,$$

where S_i are defined as above, and for $\delta_t > 0$ and $\tau_t > 0$ $(t = 2, 3, \cdots, q)$,

$$V_1 = \begin{pmatrix} 0 & \delta_2 c_{12} & \cdots & \delta_q c_{1q} \\ \tau_2 c_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \tau_q c_{q1} & 0 & \cdots & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & \delta_2 c_{12} & \cdots & 0 & 0 \\ \tau_2 c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \delta_q c_{q-1,q} \\ 0 & 0 & \cdots & \tau_q c_{q,q-1} & 0 \end{pmatrix}.$$

Then the preconditioned matrices $P_i^{(2)} H$ can be expressed as

$$P_i^{(2)} H = \begin{pmatrix} I_p - \widehat{B}_i & \widehat{U}_i \\ \widehat{L}_i & I_q - \widehat{C}_i \end{pmatrix}, \quad i = 1, 2,$$

where

$$\begin{aligned}\widehat{B}_i &= B - S_i(I_p - B),\\ \widehat{U}_i &= (I_p + S_i)U,\\ \widehat{L}_i &= (I_q + V_i)L,\\ \widehat{C}_i &= C - V_i(I_q - C).\end{aligned}$$

Note that $\widehat{B}_i$ and $\widehat{C}_i$ can be written as follows:

$$\widehat{B}_1 = B - S_1(I_p - B)$$
$$= \begin{pmatrix} b_{11} + \alpha_2 b_{12} b_{21} + \cdots + \alpha_p b_{1p} b_{p1} & \cdots & b_{1p} + \alpha_2 b_{12} b_{2p} + \cdots + \alpha_p b_{1p}(1 - b_{pp}) \\ b_{21} - \beta_2 b_{21}(1 - b_{11}) & \cdots & b_{2p} + \beta_2 b_{21} b_{1p} \\ \vdots & \ddots & \vdots \\ b_{p1} - \beta_p b_{p1}(1 - b_{11}) & \cdots & b_{pp} + \beta_p b_{p1} b_{1p} \end{pmatrix},$$
$$\widehat{B}_2 = B - S_2(I_p - B)$$
$$= \begin{pmatrix} b_{11} + \alpha_2 b_{21} b_{12} & \cdots & b_{1p} + \alpha_2 b_{2p} b_{12} \\ b_{21} + \alpha_3 b_{23} b_{31} - \beta_2 b_{21}(1 - b_{11}) & \cdots & b_{2p} + \beta_2 b_{1p} b_{21} + \alpha_3 b_{3p} b_{23} \\ \vdots & \ddots & \vdots \\ b_{p-1,1} + \beta_{p-1} b_{p-1,p-2} b_{p-2,1} + \alpha_p b_{p-1,p} b_{p1} & \cdots & b_{p-1,p} + \beta_{p-1} b_{p-1,p-2} b_{p-2,p} + \alpha_p b_{p-1,p-2} b_{p-2,p} \\ b_{p1} + \beta_p b_{p,p-1} b_{p-1,1} & \cdots & b_{pp} + \beta_p b_{p,p-1} b_{p-1,p} \end{pmatrix},$$

$$\widehat{C}_1 = V - V_1(I_q - C)$$
$$= \begin{pmatrix} c_{11} + \delta_2 c_{12} c_{21} + \cdots + \delta_q c_{1q} c_{q1} & \cdots & c_{1q} + \delta_2 c_{12} c_{2q} + \cdots + \delta_q c_{1q}(1 - c_{qq}) \\ c_{21} - \tau_2 c_{21}(1 - c_{11}) & \cdots & c_{2q} + \tau_2 c_{21} c_{1q} \\ \vdots & \ddots & \vdots \\ c_{q1} - \tau_q c_{q1}(1 - c_{11}) & \cdots & c_{qq} + \tau_q c_{q1} c_{1q} \end{pmatrix},$$
$$\widehat{C}_2 = C - V_2(I_q - C)$$
$$= \begin{pmatrix} c_{11} + \delta_2 c_{21} c_{12} & \cdots & c_{1q} + \delta_2 c_{2q} c_{12} \\ c_{21} + \delta_3 c_{23} c_{31} - \tau_2 c_{21}(1 - c_{11}) & \cdots & c_{2q} + \tau_2 c_{1q} c_{21} + \delta_3 c_{3q} c_{23} \\ \vdots & \ddots & \vdots \\ c_{q-1,1} + \tau_{q-1} c_{q-1,q-2} c_{q-2,1} + \delta_q c_{q-1,q} c_{q1} & \cdots & c_{q-1,q} + \tau_{q-1} c_{q-1,q-2} c_{q-2,q} + \delta_q c_{q-1,q-2} c_{q-2,q} \\ c_{q1} + \tau_q c_{q,q-1} c_{q-1,1} & \cdots & c_{qq} + \tau_q c_{q,q-1} c_{q-1,q} \end{pmatrix}.$$

Now, applying the GAOR method to the peconditioned linear system (1.4) with the preconditioners $P_i^{(2)}$, we have the preconditioned GAOR methods

$$y_{k+1} = \widehat{T}^{(2)}_{\gamma\omega i} y_k + \omega \widehat{g}^{(2)}, \quad k = 0, 1, 2, \cdots, \tag{3.2}$$

where for $i = 1, 2$

$$\widehat{T}^{(2)}_{\gamma\omega i} = \begin{pmatrix} (1-\omega)I_p + \omega\widehat{B}_i & -\omega\widehat{U}_i \\ \omega(\gamma-1)\widehat{L}_i - \omega\gamma\widehat{L}_i\widehat{B}_i & (1-\omega)I_q + \omega\widehat{C}_i + \omega\gamma\widehat{L}_i\widehat{U}_i \end{pmatrix} \tag{3.3}$$

are the iteration matrices and
$$\widehat{g}^{(2)} = \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_i & I_q \end{pmatrix} P_i^{(2)} f.$$

4 Comparison results

In this section, some comparison theorems are established. We first compare the convergence rates of the preconditioned GAOR methods defined by Equation (3.2) with that of the GAOR method defined by Equation (1.2).

Theorem 4.1. *Let $T_{\gamma\omega}$ and $\widehat{T}^{(2)}_{\gamma\omega 1}$ be the iteration matrices defined by Equations (1.3) and (3.3), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \gamma < 1$, $b_{i,1} > 0$, $b_{1,i} > 0$ for some $i \in \{2, 3, \cdots, p\}$, $c_{i,1} > 0$, $c_{1,i} > 0$ for some $i \in \{2, 3, \cdots, q\}$. If the parameters α_s, β_s, $(s \in \{2, \cdots, p\})$ and δ_t, τ_t $(t \in \{2, \cdots, q\})$ satisfies*

(i) when $0 \le b_{11} < 1$ and $0 \le c_{11} < 1$,
$$\begin{cases} 0 < \alpha_s < \frac{b_{1,2}+\alpha_2 b_{12} b_{2,s}+\cdots+\alpha_{s-1} b_{1,s-1} b_{s-1,s}+\alpha_{s+1} b_{1,s+1} b_{s+1,s}+\cdots+\alpha_p b_{1,p} b_{p,s}}{b_{1,s}(1-b_{s,s})}, \\ 0 < \beta_s < \frac{1}{1-b_{11}}, \\ 0 < \delta_t < \frac{c_{1,2}+\delta_2 c_{12} c_{2,t}+\cdots+\delta_{t-1} c_{1,t-1} c_{t-1,t}+\delta_{t+1} c_{1,t+1} c_{t+1,t}+\cdots+\delta_q c_{1,q} c_{q,t}}{c_{1,t}(1-c_{t,t})}, \\ 0 < \tau_t < \frac{1}{1-c_{11}} \end{cases}$$

or

(ii) when $b_{11} > 1$ and $c_{11} > 1$,
$$\begin{cases} \alpha_s > 0, \ \beta_s > 0, \\ \delta_t > 0, \ \tau_t > 0, \end{cases}$$

then either
$$\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) < \rho(T_{\gamma\omega}) < 1$$
or
$$\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) > \rho(T_{\gamma\omega}) > 1.$$

Proof. The iteration matrix $T_{\gamma\omega}$ in (1.3) can be rewritten as
$$T_{\gamma\omega} = \begin{pmatrix} (1-\omega)I_p + \omega B & -\omega U \\ -\omega(1-\gamma)L & (1-\omega)I_q + \omega C \end{pmatrix} + \omega\gamma \begin{pmatrix} 0 & 0 \\ -LB & LU \end{pmatrix}.$$
Thus, we known that $T_{\gamma\omega}$ is nonnegative as $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$ and $0 \le \gamma < 1$. Since H is irreducible, it is easy to see that $T_{\gamma\omega}$ is irreducible.

Similarly, it can be proved that $\widehat{T}^{(2)}_{\gamma\omega 1}$ is nonnegative and irreducible under the conditions of the theorem.

By Lemma 2.1, there is a positive vector x such that
$$T_{\gamma\omega} x = \lambda x, \tag{4.1}$$
where $\lambda = \rho(T_{\gamma\omega})$. Clearly, $\lambda \ne 1$, for otherwise the matrix H is singular. Therefore, one gets that $\lambda < 1$ or $\lambda > 1$. Note that
$$\omega H x = \begin{pmatrix} I_p & 0 \\ \gamma L & I_q \end{pmatrix} (I_n - T_{\gamma\omega}) x = (1-\lambda) \begin{pmatrix} I_p & 0 \\ \gamma L & I_q \end{pmatrix} x \tag{4.2}$$

holds.

From (4.1) and (4.2), one can obtain that

$$
\begin{aligned}
&\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \\
= \;& \begin{pmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{pmatrix}^{-1} \left[(1-\omega)I_n + (\omega-\gamma)\begin{pmatrix} 0 & 0 \\ -\widehat{L}_1 & 0 \end{pmatrix} + \omega\begin{pmatrix} \widehat{B}_1 & -\widehat{U}_1 \\ 0 & \widehat{C}_1 \end{pmatrix}\right] x - \lambda x \\
= \;& \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} -\omega(I_p - B) & -\omega U \\ (-\omega+\gamma-\lambda\gamma)L & -\omega(I_q - C) \end{pmatrix} x \\
= \;& \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\left[-\omega H + \begin{pmatrix} 0 & 0 \\ (1-\lambda)\gamma L & 0 \end{pmatrix}\right] x \\
= \;& \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} (\lambda-1)I_p & 0 \\ 0 & (\lambda-1)I_q \end{pmatrix} x \\
= \;& (\lambda-1)\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} x.
\end{aligned}
$$

By assumptions, $\widehat{L}_1 = (I_q + V_1)L \le 0$, S_1 and V_1 are non-negative matrices which are nonzero, so we have

$$
\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} x \ge 0 \quad\text{and}\quad \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} x \ne 0.
$$

If $\lambda < 1$, then $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \le 0$ and $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ne 0$, Lemma 2.2 implies that $\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) < \rho(T_{\omega\gamma}) < 1$. If $\lambda > 1$, then $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ge 0$ and $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ne 0$, Lemma 2.2 yields $\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) > \rho(T_{\omega\gamma}) > 1$. $\square$

Similarly, we can obtain the following comparison theorem between $\rho(\widehat{T}^{(2)}_{\gamma\omega 2})$ with $\rho(T_{\gamma\omega})$. The only difference is that some assumptions are changed so that $S_2 \ne 0, V_2 \ne 0$ and $\widehat{T}^{(2)}_{\gamma\omega 2}$ is irreducible.

Theorem 4.2. *Let $T_{\gamma\omega}$ and $\widehat{T}^{(2)}_{\gamma\omega 2}$ be the iteration matrices defined by Equations (1.3) and (3.3), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \gamma < 1$, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{2, 3, \cdots, p\}$, $c_{i,i+1} > 0$, $c_{i+1,i} > 0$ for some $i \in \{2, 3, \cdots, q\}$. If the parameters α_s, β_s $(s \in \{2, \cdots, p\})$ and δ_t, τ_t $(t \in \{2, \cdots, q\})$ satisfies*

(i) when $0 \le b_{s,s} < 1$ and $0 \le c_{t,t} < 1$,

$$
\begin{cases}
0 < \alpha_s < \frac{b_{s-1,s-2}b_{s-2,s}+b_{s-1,s}(1-b_{s-2,s-2})}{b_{s-1,s-2}[(1-b_{s,s})(1-b_{s-2,s-2})-b_{s,s-2}b_{s-2,s}]} \ \textit{for } s \in \{3, \cdots, p\}, \\
\qquad 0 < \alpha_2 < \frac{1}{1-b_{22}}, \\
0 < \beta_s < \frac{b_{s,s-1}(1-b_{s+1,s+1})+b_{s-1,s}b_{s+1,s-1}}{b_{s,s-1}[(1-b_{s-1,s-1})(1-b_{s+1,s+1})-b_{s-1,s+1}b_{s+1,s-1}]} \ \textit{for } s \in \{2, \cdots, p-1\}, \\
\qquad 0 < \beta_p < \frac{1}{1-b_{p,p}}, \\
0 < \delta_t < \frac{c_{t-1,t-2}c_{t-2,t}+c_{t-1,t}(1-c_{t-2,t-2})}{c_{t-1,t-2}[(1-c_{t,t})(1-c_{t-2,t-2})-c_{t,t-2}c_{t-2,t}]} \ \textit{for } t \in \{3, \cdots, q\}, \\
\qquad 0 < \delta_2 < \frac{1}{1-c_{22}}, \\
0 < \tau_t < \frac{c_{t,t-1}(1-c_{t+1,t+1})+c_{t-1,t}c_{t+1,t-1}}{c_{t,t-1}[(1-c_{t-1,t-1})(1-c_{t+1,t+1})-c_{t-1,t+1}c_{t+1,t-1}]} \ \textit{for } t \in \{2, \cdots, q-1\}, \\
\qquad 0 < \tau_q < \frac{1}{1-c_{q,q}}
\end{cases}
$$

or

(ii) when $b_{s,s} > 1$ *and* $c_{t,t} > 1$,

$$\begin{cases} \alpha_s > 0,\ \beta_s > 0,\ s \in \{2, \cdots, p\}, \\ \delta_t > 0,\ \tau_t > 0,\ t \in \{2, \cdots, q\}, \end{cases}$$

then either

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 2}) < \rho(T_{\gamma\omega}) < 1$$

or

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 2}) > \rho(T_{\gamma\omega}) > 1.$$

In Theorem 4.2, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{2, 3, \cdots, p\}$, $c_{i,i+1} > 0$, $c_{i+1,i} > 0$ for some $i \in \{2, 3, \cdots, q\}$ imply that $S_2 \neq 0$ and $V_2 \neq 0$. The conditions which were α_s, β_s $(s \in \{2, \cdots, p\})$ and δ_t, τ_t $(t \in \{2, \cdots, q\})$ satisfied ensure that $\widehat{T}^{(2)}_{\gamma\omega 2}$ is irreducible.

We next compare the convergence rates of the preconditioned GAOR methods defined by Equation (3.2) with those of the preconditioned GAOR methods defined by Equation (3.1).

Theorem 4.3. *Let* $T^{(1)}_{\gamma\omega 1}$ *and* $\widehat{T}^{(2)}_{\gamma\omega 1}$ *be the iteration matrices of preconditioned GAOR methods (3.1) and (3.2), respectively. Assume that the matrix* H *in Equation (1.1) is irreducible with* $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \gamma < 1$, $b_{i,1} > 0$, $b_{1,i} > 0$ *for some* $i \in \{2, 3, \cdots, p\}$, $c_{i,1} > 0$, $c_{1,i} > 0$ *for some* $i \in \{2, 3, \cdots, q\}$. *If the parameters* α_s, β_s, $(s \in \{2, \cdots, p\})$ *and* δ_t, τ_t $(t \in \{2, \cdots, q\})$ *satisfies*

(i) when $0 \le b_{11} < 1$ *and* $0 \le c_{11} < 1$,

$$\begin{cases} 0 < \alpha_s < \frac{b_{1,2}+\alpha_2 b_{12} b_{2,s}+\cdots+\alpha_{s-1} b_{1,s-1} b_{s-1,s}+\alpha_{s+1} b_{1,s+1} b_{s+1,s}+\cdots+\alpha_p b_{1,p} b_{p,s}}{b_{1,s}(1-b_{s,s})}, \\ 0 < \beta_s < \frac{1}{1-b_{11}}, \\ 0 < \delta_t < \frac{c_{1,2}+\delta_2 c_{12} c_{2,t}+\cdots+\delta_{t-1} c_{1,t-1} c_{t-1,t}+\delta_{t+1} c_{1,t+1} c_{t+1,t}+\cdots+\delta_q c_{1,q} c_{q,t}}{c_{1,t}(1-c_{t,t})}, \\ 0 < \tau_t < \frac{1}{1-c_{11}} \end{cases}$$

or

(ii) when $b_{11} > 1$ *and* $c_{11} > 1$,

$$\begin{cases} \alpha_s > 0,\ \beta_s > 0, \\ \delta_t > 0,\ \tau_t > 0, \end{cases}$$

then either

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) < \rho(T^{(1)}_{\gamma\omega 1}) < 1$$

or

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) > \rho(T^{(1)}_{\gamma\omega 1}) > 1.$$

Proof. By assumptions, it is easy to show that $T^{(1)}_{\gamma\omega 1}$ and $\widehat{T}^{(2)}_{\gamma\omega 1}$ are nonnegative and irreducible matrices. By Lemma 2.1, there is a positive vector x such that

$$T^{(1)}_{\gamma\omega 1} x = \lambda x, \tag{4.3}$$

where $\lambda = \rho(T^{(1)}_{\gamma\omega 1})$. Clearly, $\lambda \neq 1$, for otherwise the matrix H is singular. Therefore, one gets that $\lambda < 1$ or $\lambda > 1$. Note that

$$\omega P^{(1)}_1 H x = \begin{pmatrix} I_p & 0 \\ \gamma L & I_q \end{pmatrix} (I_n - \widehat{T}^{(1)}_{\gamma\omega 1}) x = (1-\lambda) \begin{pmatrix} I_p & 0 \\ \gamma L & I_q \end{pmatrix} x \tag{4.4}$$

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holds.

In a manner similar to that done for Theorem 1, from (4.3) and (4.4), one can obtain that

$$\begin{aligned}
&\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \\
= &\begin{pmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{pmatrix}^{-1} \left[(1-\omega)I_n + (\omega-\gamma)\begin{pmatrix} 0 & 0 \\ -\widehat{L}_1 & 0 \end{pmatrix} + \omega\begin{pmatrix} \widehat{B}_1 & -\widehat{U}_1 \\ 0 & \widehat{C}_1 \end{pmatrix}\right] x - \lambda x \\
= &\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix}\left[-\omega P_1^{(1)}H + \begin{pmatrix} 0 & 0 \\ (1-\lambda)\gamma L & 0 \end{pmatrix}\right] x \\
= &\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} (\lambda-1)I_p & 0 \\ 0 & (\lambda-1)I_q \end{pmatrix} x \\
= &(\lambda-1)\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} x.
\end{aligned}$$

As $\widehat{L}_1 \le 0$ and $V_1 \ge 0$ are nonzero, we have

$$\begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} x \ge 0 \ \text{ and } \ \begin{pmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} x \ne 0.$$

If $\lambda < 1$, then $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \le 0$ and $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ne 0$, Lemma 2.2 implies that $\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) < \rho(T^{(1)}_{\omega\gamma 1}) < 1$. If $\lambda > 1$, then $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ge 0$ and $\widehat{T}^{(2)}_{\gamma\omega 1}x - \lambda x \ne 0$, Lemma 2.2 yields $\rho(\widehat{T}^{(2)}_{\gamma\omega 1}) > \rho(T^{(1)}_{\omega\gamma 1}) > 1$. □

Similarly, we can obtain the following comparison theorem between $\rho(\widehat{T}^{(2)}_{\gamma\omega 2})$ with $\rho(T^{(1)}_{\omega\gamma 2})$.

Theorem 4.4. *Let $T^{(1)}_{\omega\gamma 2}$ and $\widehat{T}^{(2)}_{\gamma\omega 2}$ be the iteration matrices of preconditioned GAOR methods (3.1) and (3.2), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \gamma < 1$, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{2,3,\cdots,p\}$, $c_{i,i+1} > 0$, $c_{i+1,i} > 0$ for some $i \in \{2,3,\cdots,q\}$. If the parameters α_s, β_s $(s \in \{2,\cdots,p\})$ and δ_t, τ_t $(t \in \{2,\cdots,q\})$ satisfies*

(i) when $0 \le b_{s,s} < 1$ and $0 \le c_{t,t} < 1$,

$$\begin{cases}
0 < \alpha_s < \frac{b_{s-1,s-2}b_{s-2,s}+b_{s-1,s}(1-b_{s-2,s-2})}{b_{s-1,s-2}[(1-b_{s,s})(1-b_{s-2,s-2})-b_{s,s-2}b_{s-2,s}]} \ \textit{for } s \in \{3,\cdots,p\}, \\
\qquad 0 < \alpha_2 < \frac{1}{1-b_{22}}, \\
0 < \beta_s < \frac{b_{s,s-1}(1-b_{s+1,s+1})+b_{s-1,s}b_{s+1,s-1}}{b_{s,s-1}[(1-b_{s-1,s-1})(1-b_{s+1,s+1})-b_{s-1,s+1}b_{s+1,s-1}]} \ \textit{for } s \in \{2,\cdots,p-1\}, \\
\qquad 0 < \beta_p < \frac{1}{1-b_{p,p}}, \\
0 < \delta_t < \frac{c_{t-1,t-2}c_{t-2,t}+c_{t-1,t}(1-c_{t-2,t-2})}{c_{t-1,t-2}[(1-c_{t,t})(1-c_{t-2,t-2})-c_{t,t-2}c_{t-2,t}]} \ \textit{for } t \in \{3,\cdots,q\}, \\
\qquad 0 < \delta_2 < \frac{1}{1-c_{22}}, \\
0 < \tau_t < \frac{c_{t,t-1}(1-c_{t+1,t+1})+c_{t-1,t}c_{t+1,t-1}}{c_{t,t-1}[(1-c_{t-1,t-1})(1-c_{t+1,t+1})-c_{t-1,t+1}c_{t+1,t-1}]} \ \textit{for } t \in \{2,\cdots,q-1\}, \\
\qquad 0 < \tau_q < \frac{1}{1-c_{q,q}}
\end{cases}$$

or

(ii) when $b_{s,s} > 1$ and $c_{t,t} > 1$,

$$\begin{cases}
\alpha_s > 0, \ \beta_s > 0, \ s \in \{2,\cdots,p\}, \\
\delta_t > 0, \ \tau_t > 0, \ t \in \{2,\cdots,q\},
\end{cases}$$

then either

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 2}) < \rho(T^{(1)}_{\omega\gamma 2}) < 1$$

or

$$\rho(\widehat{T}^{(2)}_{\gamma\omega 2}) > \rho(T^{(1)}_{\omega\gamma 2}) > 1.$$

From Theorem 4.1–4.2, we can see that the preconditioned GAOR methods (3.2) are better than the GAOR method (1.2) whenever the GAOR method is convergent. And from Theorem 4.3–4.4, it is clearly that the preconditioners $P_i^{(2)}$ are better than the preconditioners $P_i^{(1)}$ [7] for $i = 1, 2$ whenever these methods are convergent.

5 Numerical example

In this section, an example with numerical experiments is given to illustrate the theoretical results provided in the present paper.

Example 5.1. *This example is introduced in [12], also studied in [7, 11]. The coefficient matrix H in Equation (1.1) is given by*

$$H = \begin{pmatrix} I_p - B & U \\ L & I_q - C \end{pmatrix},$$

where $B = (b_{ij}) \in \mathbb{R}^{p\times p}$, $C = (c_{ij}) \in \mathbb{R}^{q\times q}$, $L = (l_{ij}) \in \mathbb{R}^{q\times p}$, and $U = (u_{ij}) \in \mathbb{R}^{p\times q}$ with

$$\begin{aligned}
b_{ii} &= \frac{1}{10(i+1)}, \quad 1 \le i \le p,\\
b_{ij} &= \frac{1}{30} - \frac{1}{30j+i}, \quad 1 \le i < j \le p,\\
b_{ij} &= \frac{1}{30} - \frac{1}{30(i-j+1)+i}, \quad 1 \le j < i \le p,\\
c_{ii} &= \frac{1}{10(p+i+1)}, \quad 1 \le i \le q,\\
c_{ij} &= \frac{1}{30} - \frac{1}{30(p+j)+p+i}, \quad 1 \le i < j \le q,\\
c_{ij} &= \frac{1}{30} - \frac{1}{30(i-j+1)+p+i}, \quad 1 \le j < i \le q,\\
l_{ij} &= \frac{1}{30(p+i-j+1)+p+i} - \frac{1}{30}, \quad 1 \le i \le q,\ 1 \le j \le p,\\
u_{ij} &= \frac{1}{30(p+j)+i} - \frac{1}{30}, \quad 1 \le i \le p,\ 1 \le j \le q.
\end{aligned}$$

Table 5 displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters ω, γ, p and q. The randomly chosen parameters α_s, β_s, δ_t and τ_t satisfy the conditions of Theorem 4.1–4.4. In which $\rho = \rho(T_{\omega,\gamma})$ and $\rho_i^j = \rho(T^{(j)}_{\omega\gamma i})$ for $i = 1, 2$ and $j = 1, 2$.

Table 1: Spectral radii of GAOR and preconditioned GAOR iteration matrices

n	p	ω	γ	ρ	ρ_1^1	ρ_2^1	ρ_1^2	ρ_2^2
10	5	0.9	0.8	0.2830	0.2823	0.2524	0.2813	0.2513
20	5	0.8	0.6	0.6259	0.6258	0.6162	0.6247	0.6151
20	10	0.8	0.6	0.6146	0.6142	0.5847	0.6135	0.5838
30	8	0.9	0.6	0.8843	0.8842	0.8796	0.8839	0.8793
40	16	0.9	0.5	1.2220	1.2222	1.2397	1.2228	1.2403

From Table 5, it can be seen that $\rho_i^2 < \rho_i^1 < \rho < 1$ and $\rho_i^2 > \rho_i^1 > \rho > 1$ for $i = 1, 2$. These numerical results are in accordance with the theoretical results given in Section 4.

Moreover, we find that the preconditioned GAOR methods (3.2) need fewer iteration numbers than the preconditioned GAOR methods (3.1) and the GAOR method (1.2) when the iterative methods are started from the same vector and terminated rule. Therefore, from the above numerical example and the theoretical analysis, we see that the effectiveness of the preconditioners constructed in this paper is obvious.

6 Conclusions

In this paper, a new type of preconditioners for the GAOR method are proposed, the convergence rates of the new preconditioned GAOR methods for solving generalized least squares problems are studied. Comparison theorems in Section 4 as well as numerical results in Section 5 show that the convergence rates of the new preconditioned GAOR methods are better than those of the preconditioned GAOR methods proposed by Wang et al. [7] whenever these methods are convergent.

Like all-parameter based iterative methods, how to choose the optimal iteration parameters ω and γ, the optimal parameters α_s, β_s, δ_t and τ_t is a very difficult task. This aspect needed further in-depth study.

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MIXING PROPERTIES IN THE OPERATOR ALGEBRA USING HILBERT-SCHMIDT OPERATORS

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Abstract. In the present paper, we discuss the relation between the mixing property of any operator T on a Hilbert space and the mixing property of the corresponding left multiplication operator induced by T in the strong operator topology. Besides, we further prove that the Hypercyclicity Criterion with respect to some syndetic sequence for any weighted backward shift T on ℓ^2 (the Hilbert space of square summable sequences) is equivalent to the mixing property of the corresponding left multiplication operators induced by T in the $\|.\|_2$ topology.

1. Introduction

Suppose that $\mathbb{N}$ is the set of all positive integral numbers and H is a separable infinite dimensional complex Hilbert space and T is a bounded linear operator on H. We say that T is hypercyclic if there is an $f \in H$ such that the orbit $\{T^n f\}_{n\geq 0}$ is dense in H. In such a case, f is called a hypercyclic vector for T. A vector $f \in X$ is called supercyclic for T if its projective orbit, $\{\lambda T^n x; n \geq 0, \lambda \in \mathbb{C}\}$ is dense in H. Besides, for every pair U, V of nonempty open subsets of H, there is a non-negative integer N, such that $T^n(U) \cap U \neq \emptyset$, for all $n \geq N$, then we call T mixing. Roughly speaking, the iterates of any open set become well spread throughout the space.

Recently, there have been an increasing interest in studying the mixing operators. G. Costakis and M. Sambarinoa [7] proved that linear operator $T : X \to X$ satisfying a special case of the Hypercyclicity Criterion is topologically mixing. Besides, A. Bonilla and P. Mianaour [2] provided sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous cosine function and proved that every separable infinite dimensional complex Banach space admits a topologically mixing uniformly continuous cosine family. In 2012, in [4], the authors showed that every separable infinite-dimensional Fréchet space supports an arbitrarily large finite and commuting disjoint mixing collection of operators. When this space is a Banach space, it supports an arbitrarily large finite disjoint mixing collection of C_0-semigroups. For several works, see, e.g., [3, 5, 8].

Many results for supercyclicity and hypercyclicity have been given. In [10], bilateral weighted backward shifts on l^2 spaces are also discussed and hypercyclic and supercyclic properties are characterized, respectively. Some necessary and sufficient conditions for Hypercyclicity Criterion were discussed, see, e.g., [15, 16, 18]. For discussion of hypercyclicity of composition operators, see, e.g., [1, 6, 9, 11, 12, 13, 14, 17] and the references therein.

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The operator algebra $B(H)$ consists of all bounded linear operators from H to H. Recall that $\{e_i\}$ is a basis for a separable Hilbert space H and $A \in B(H)$, then $\|A\|_2 = [\sum\limits_{i=1}^{\infty} \|Ae_i\|^2]^{\frac{1}{2}}$ which is independent of the basis chosen and hence is well defined. If $\|A\|_2 < \infty$, then A is called a Hilbert-Schmidt operator. The set of Hilbert-Schmidt operators on H is denoted by $B_2(H)$. For a separable infinite dimensional complex Hilbert space H, the operator algebra $B(H)$ has many topologies, but we use only three, namely the operator norm topology, the strong operator topology and $\|.\|_2$-topology. And we use the convention that when a topology term is used for $B(H)$, it always refers to the operator norm topology, otherwise we add the prefix "*SOT*" in front of the term with reference to the strong operator topology and add the prefix $\|.\|_2$ in front of the term with reference to $\|.\|_2$-topology.

We note that the operator algebra $B(H)$ and $B_2(H)$ respectively with the strong operator topology and $\|.\|_2$-topology are separable. Suppose that $\{e_i\}$ is a basis for a separable Hilbert space H and S is a dense subset in H. Then $S(H)$ will denote the set of all finite rank operators E such that there exists $N \in \mathbb{N}$ satisfying $E(e_n) = 0$ for $n \geq N$ and $E(e_n) \in S$ for $n < N$.

For any operator $B(H)$, the left multiplication $L_T : B(H) \to B(H)$ is defined by $L_T F = TF$ for all F in $B(H)$. B. Yousefi and H. Rezaei [20] proved that the Supercyclicity Criterion for any operator T on a Hilbert space is equivalent to the supercyclicty of the left multiplication operator induced by T in the strong operator topology and Supercyclicity can occur on the operator algebra $B(H)$ with strong operator topology and some equivalent conditions for Supercyclicity Criterion were given in [19].

2. Mixing properties of left multiplication operators on the operator algebra

In this section, we will discuss mixing properties of left multiplication operators on $B(H)$ with the strong operator topology and on $B_2(H)$ with the $\|.\|_2$-topology. Besides, some necessary conditions for Hypercyclicity Criterion with respect to a syndetic sequence will be given. In the following, note that for vectors g, h in H, the operator $g \bigotimes h$ denotes a rank one operator and is defined by $(g \otimes h)(f) = \langle f, g\rangle g$.

Definition 2.1. *For $T \in B(H)$ and any sets $A, B \subset H$, the return set from A to B is defined as*

$$N_T(A, B) = N(A, B) = \{n \in N_0 : T^n(A) \cap B \neq \emptyset\}.$$

Definition 2.2. *For $T \in B(H)$, let $L_T : B(H) \to B(H)$ be a continuous linear mapping. We say that the operator L_T is SOT-mixing if for any two nonempty SOT-open set U and V, there exists an integer $N \geq 1$ such that $L_T^n(U) \cap V \neq \emptyset$, for all $n \geq N$. Similarly, we say that $L_T : B_2(H) \to B_2(H)$ is $\|.\|_2$-mixing if for any two nonempty $\|.\|_2$-open set U and V, there exists an integer $N \geq 1$ such that $L_T^n(U) \cap V \neq \emptyset$, for all $n \geq N$.*

Definition 2.3. *(Hypercyclicity Criterion). Let T be a bounded linear operator on a separable Hilbert space H. Suppose that there are $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ strictly increasing, dense subsets $X, Y \subset H$ and mappings $S_{n_k} : Y \to H$ so that*
(i) $T^{n_k} \to 0$ pointwise on X,
(ii) $S_{n_k} \to 0$ pointwise on Y and
(iii) $T^{n_k} S_{n_k} \to id$ pointwise on Y. Then T is hypercyclic.

An increasing sequence of positive integers $\{n_k\}$ is syndetic if

$$\sup_k \{n_{k+1} - n_k\} < \infty.$$

We say that T satisfies the Hypercyclicity Criterion for a syndetic sequence if the sequence $\{n_k\}$ is syndetic in the above criterion . Notice that a large class of hypercyclic operators satisfies the Hypercyclicity Criterion for a syndetic sequence, for instance: λB where $|\lambda| > 1$ and B is the backward shift on $\ell^2 = \ell^2(\mathbb{N})$ (the Hilbert space of square summable sequences).

Lemma 2.4. [12, Proposition 2.40]. *Let $T \in B(H)$. Then $T \oplus T$ is mixing on $H \oplus H$ if and only if T is mixing on H.*

Proposition 2.5. *Let $T \in B(H)$. Then the following are equivalent:*
i) $\overset{\infty}{\underset{n=1}{\oplus}} T$ is mixing on $\overset{\infty}{\underset{n=1}{\oplus}} H$.
ii) $T \oplus T$ is mixing on $H \oplus H$.
iii) T is mixing on H.

Proof. $(i) \Rightarrow (ii)$. It is clear.

$(ii) \Leftrightarrow (iii)$. By Lemma 2.4, $T \oplus T$ is mixing on $H \oplus H$ if and only if T is mixing on H.

$(iii) \Rightarrow (i)$. Now suppose that T is mixing on H. Let any $U, V \subset \left(\overset{\infty}{\underset{n=1}{\oplus}} H\right)$ be nonempty open sets. Then there are $\varepsilon > 0$, $m \geq 1$ and points $x := (x_1, ..., x_m, 0, 0, ...) \in U$ and $y := (y_1, ..., y_m, 0, 0, ...) \in V$ such that the open balls of radius ε around these points belong to U and V, respectively. Since T is mixing, there is some $N \geq 1$ such that, for each $1 \leq k \leq m$ and $n \geq N$, there are $x_k^n \in H$ such that $\left\|x_k^{(n)} - x_k\right\| < \varepsilon$ and $\left\|T^{n_k} x_k^{(n)} - y_k\right\| < \varepsilon$. Then, for all $n \geq N$, $x^{(n)} := \left(x_1^{(n)}, ..., x_m^{(n)}, 0, 0, ...\right) \in U$ and $\left(\overset{\infty}{\underset{k=1}{\oplus}} T\right)^n x^{(n)} \in V$, which implies that $\overset{\infty}{\underset{n=1}{\oplus}} T$ is mixing. $\square$

Lemma 2.6. [7, Theorem 1.1]. *Let $T \in B(H)$ and T satisfy the Hypercyclicity Criterion for a syndetic sequence. Then T is topologically mixing.*

Theorem 2.7. *Suppose that T satisfies the Hypercyclicity Criterion for the syndetic sequence $(n_k)_k$. Then $\overset{\infty}{\underset{n=1}{\oplus}} T$ is mixing on $\overset{\infty}{\underset{n=1}{\oplus}} H$.*

Proof. Let T be a bounded linear operator on a separable Hilbert space H. Suppose that there are $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ strictly increasing, dense subsets $X, Y \subset H$ and mappings $S_{n_k} : Y \to H$ so that

(i) $T^{n_k} \to 0$ pointwise on X,
(ii) $S_{n_k} \to 0$ pointwise on Y and
(iii) $T^{n_k} S_{n_k} \to id$ pointwise on Y.

Now let χ_0 be the set of all sequence $(x_n)_n \in \overset{\infty}{\underset{i=1}{\oplus}} X$ such that $x_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly, let χ_1 be the set of all sequence $(y_n)_n \in \overset{\infty}{\underset{i=1}{\oplus}} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Consider $\mathbb{S}_k = \overset{\infty}{\underset{i=1}{\oplus}} S_{n_k}$ acting on χ_1. Then χ_0 and χ_1 are dense in $\overset{\infty}{\underset{i=1}{\oplus}} H$ and clearly the hypotheses of the Hypercyclicity Criterion with respect to the syndetic sequence $(n_k)_k$ are satisfied. By lemma 2.6, $\overset{\infty}{\underset{i=1}{\oplus}} T$ is mixing on $\overset{\infty}{\underset{i=1}{\oplus}} H$. $\square$

Lemma 2.8. [19, Proposition 2.3]. *For any operator $T \in B(H)$, the left multiplication operator $L_T : B_2(H) \to B_2(H)$ is unitary equivalent to the operator*

$$\overset{\infty}{\underset{n=1}{\oplus}} T : \overset{\infty}{\underset{n=1}{\oplus}} H \to \overset{\infty}{\underset{n=1}{\oplus}} H.$$

Lemma 2.9. [12, Proposition 2.37]. *The operator T is mixing if and only if, for any nonempty open set $U \subset H$ and any 0- neighbourhood W, the return sets $N(U,W)$ and $N(W,U)$ are cofinite.*

Theorem 2.10. *If an operator T satisfies the the Hypercyclicity Criterion for some syndetic sequence $(n_k)_k$, then L_T is mixing on $B(H)$ with the strong operator topology.*

For the above proof, we will prove the following proposition first.

Proposition 2.11. *For the left multiplication operator $L_T : B(H) \to B(H)$, the following are equivalent:*

(i) the operator L_T is mixing on $B(H)$ with the strong operator topology.

(ii) the operator L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology.

Proof. Note that $B_2(H)$ is a SOT-dense subset of $B(H)$ and $L_T : B_2(H) \to B_2(H)$ is well defined. So (ii) implies (i).

It suffices to prove that (i) implies (ii). To see that, suppose that any non-empty V_0, V_1 are $\|\|_2$-open subsets in $B_2(H)$ and let $S(H)$ be the set that was defined as before. Choose $A_j \in V_j \cap S(H)$, $j = 0,1$ such that for some $N_0 \in \mathbb{N}, A_0e_i = A_1e_i$ for $i > N_0$. Now let E be a finite rank operator that is defined by $E = \sum\limits_{n=1}^{N_0} e_n \otimes e_n$. Then $A_jE = A_j$, $j = 0,1$. For $j = 0,1$ and every $k \in \mathbb{N}$, put

$$V_{j,k} = \bigcap_{n=1}^{N_0} \{V \in B(H) : \|Ve_n - A_je_n\| < \frac{1}{k}\}.$$

Then $V_{0,k}$ and $V_{1,k}$ are SOT-open subsets of $B(H)$. Since L_T is mixing on $B(H)$ with the strong operator topology, we have $L_T^{n_k}(V_{0,k}) \cap V_{1,k} \neq \emptyset$ for large enough $n_k \in \mathbb{N}$. Therefore, there exists some $S_k \in V_{0,k}$ and it follows that $T^{n_k}S_k \in V_{1,k}$. Consequently, for $n = 1, \cdots, N_0$ and $k \in \mathbb{N}$, we get

$$\|S_ke_n - A_0e_n\| < \frac{1}{k}; \|T^{n_k}S_ke_n - A_1e_n\| < \frac{1}{k}.$$

Therefore

$$\|S_kE - A_0E\|_2^2 = \sum_{n=1}^{N_0} \|(S_k - A_0)(e_n)\|^2 < \frac{N_0}{k^2}$$

and

$$\|L_T^{n_k}(S_kE) - A_1E\|_2^2 = \sum_{n=1}^{N_0} \|(T^{n_k}S_k - A_1)(e_n)\|^2 < \frac{N_0}{k^2}$$

for $k \in \mathbb{N}$ and we have that S_kE converges to A_0 and $T^{n_k}(S_kE)$ converges to A_1 in $\|.\|_2$. And we can see that $S_kE \in V_0 \cap S(H)$ and $L_T^{n_k}(S_kE) \in V_1 \cap S(H)$ for large enough $n_k \geq 1$.

Besides, we note that S_kE and $L_T^{n_k}(S_kE)$ are finite rank operators and so they are Hilbert-Schmidt operators. Then it follows that $L_T^{n_k}(V_0) \cap V_1 \neq \emptyset$, for large enough $n_k \geq 1$. So the operator L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology. □

Proof. of Theorem 2.10.

By Proposition 2.11, the operator L_T is mixing on $B(H)$ with the strong operator topology if and only if the operator L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology. By Lemma 2.8, we can know that L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology if and only if $\bigoplus\limits_{n=1}^{\infty} T$ is mixing on $\bigoplus\limits_{n=1}^{\infty} H$. Again by using Theorem 2.7, L_T is mixing on $B(H)$ with the strong operator topology. This proof is complete. □

The next corollary is an immediate consequence of Theorem 2.10 and Lemma 2.9.

Corollary 2.12. *If the operator T satisfies the the Hypercyclicity Criterion for some syndetic sequence, then for any nonempty SOT-open sets $U, W \subset H$ and W contains 0, the return sets $N(U,W)$ and $N(W,U)$ are cofinite.*

3. Mixing properties of weighted backward shifts on ℓ^2

In this section, we will prove that the Hypercyclicity Criterion with a syndetic sequence for any weighted backward shift T on ℓ^2 is equivalent to mixing property of the corresponding left multiplication operator L_T in the strong operator topology.

Let $\mathbb{K}$ a real or complex scalar field, and the space of all sequences $\mathbb{K}^{\mathbb{N}} = \{(x_n)_n : x_n \in \mathbb{K}, n \in \mathbb{N}\}$. Let $1 \leq p < \infty$. Then the space

$$\ell^p := \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

of p-summable sequences, endowed with the norm $\|x\| := \left(\sum\limits_{n=1}^{\infty} |x_n|^p \right)^{1/p}$, is a Banach space. In particular, ℓ^2 is a Hilbert space with inner product defined by $\langle x, y \rangle := \sum\limits_{n=1}^{\infty} x_n \overline{y_n}$. Occasionally we let the index start with 0. The finite sequences, that is, sequences of the form $(x_1, \cdots, x_n, 0, \cdots), n \geq 1$, constitute a dense subset. Considering only the finite sequences with entries from $\mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$ we see that any ℓ^p, $1 \leq p < \infty$, is separable. The space $\ell^p(\mathbb{Z})$ of p-summable sequences, indexed over $\mathbb{Z}$, is defined analogously.

In this section, we only consider $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ as separable infinite dimensional complex Hilbert spaces. For simplicity, we use T on ℓ^2 to denote unilateral weighted backward shift on $\ell^2(\mathbb{N})$ or bilateral weighted backward shift on $\ell^2(\mathbb{Z})$.

In the above section, we note that Lemma 2.6 is obvious. Thus, a natural question is whether a converse of this Lemma 2.6 holds. In other words, assuming that T is topologically mixing, must T satisfy the Hypercyclicity Criterion for a syndetic sequence ? This is true when T is a weighted backward shift by the following Lemma.

Lemma 3.1. [7, Theorem 1.2]. *A weighted backward shift T on ℓ^2 is topologically mixing if and only if T satisfies the Hypercyclicity Criterion for a syndetic sequence.*

Now, we will complement Theorem 2.10, when T is a weighted backward shift on ℓ^2.

Theorem 3.2. *For weighted backward shift T on ℓ^2 and the corresponding left multiplication mappings L_T, then the following are equivalent:*

(i) The operator T satisfies the Hypercyclicity Criterion with respect to some syndetic sequence.

(ii) L_T is mixing on $B(H)$ with the strong operator topology.

(iii) L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology.

(iv) For any nonempty SOT-open sets $U, W \subset H$ and W contains 0, the return sets $N_{L_T}(U,W)$ and $N_{L_T}(W,U)$ are cofinite.

Proof. $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. By Proposition 2.11, it is clear. By Theorem 2.10, (i) implies (ii). By Proposition 2.5, Lemma 2.8 and Lemma 3.1, (iii) implies (i). This completes this proof.

□

Theorem 3.3. *For any weighted backward shift T on ℓ^2, the following are equivalent:*

(i) The operator T satisfies the Hypercyclicity Criterion with respect to some syndetic sequence.

(ii) For each pair U, V of nonempty open subsets of H, and each neighborhood W of zero, the return sets $N_T(U, W)$ and $N_T(W, V)$ are cofinite.

Proof. The implication $(i) \to (ii)$ is immediate. To see the implication $(ii) \to (i)$, by Theorem 3.2, it suffices to show that L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology. Suppose that each pair U_1 and V_1 are nonempty $\|.\|_2$-open subsets of $B_2(H)$. Next, for the orthogonal basis $E = \{e_i : i \geq 1\}$ of H, there exist finite rank operator A and B such that $A \in S(H) \cap U_1$ and $B \in S(H) \cap V_1$. For some integer $N \geq 1$, we have $Ae_i = Be_i = 0$ for $i > N$. And there exists certain $\varepsilon > 0$ such that

$$\{S \in S(H) : \|S - A\|_2 < 2\sqrt{N}\varepsilon\} \subseteq U_1$$

and

$$\{S \in S(H) : \|S - B\|_2 < 2\sqrt{N}\varepsilon\} \subseteq V_1.$$

Consider the open sets

$$U_i = \{x \in H : \|x - Ae_n\| < \varepsilon\}, V_i = \{x \in H : \|x - Be_n\| < \varepsilon\}$$

for $i = 1, 2, ..., N$. By the assumption, there exist strictly increasing integers $0 = n_0 < n_1 < n_2 < ... < n_{N-1}$ and $0 = m_0 < m_1 < m_2 < ... < m_{N-1}$ such that

$$U = U_1 \cap T^{-n_1}(U_2) \cap T^{-n_2}(U_3) \cap ... \cap T^{-n_{N-1}}(U_N) \neq \emptyset \tag{3.1}$$

and

$$V = V_1 \cap T^{-m_1}(V_2) \cap T^{-m_2}(V_3) \cap ... \cap T^{-m_{N-1}}(V_N) \neq \emptyset \tag{3.2}$$

Let $\delta = \min\{\frac{\varepsilon}{\|T\|^{n_{i-1}}}, \frac{\varepsilon}{\|T\|^{m_{i-1}}} : i = 1, 2, ..., N\}$ and $W = \{x : \|x\| < \delta\}$. By (ii), there exist $x \in W$ and $y \in U$ such that $T^n x \in V$ and $T^n y \in W$ for large enough integer n. For $i = 1, 2, ..., N$, by (3.1) and (3.2), we show that

$$\|T^{n_{i-1}} y - Ae_i\| < \varepsilon \tag{3.3}$$

and

$$\|T^n (T^{m_{i-1}} x) - Be_i\| < \varepsilon. \tag{3.4}$$

Next, we define $S_1 = \sum_{i=1}^{N} T^{n_{i-1}} y \otimes e_i$ and $S_2 = \sum_{i=1}^{N} T^{m_{i-1}} x \otimes e_i$. Let $S = S_1 + S_2$, since S_1, S_2 are finite rank operators, then S is also a Hilbert-Schmidt operator.

Again by using (3.3) and (3.4), we can obtain

$$\begin{aligned} \|S - A\|_2 &\leq \|S_1 - A\|_2 + \|S_2\|_2 \\ &= \{\sum_{i=1}^{N} \|T^{n_{i-1}} y - Ae_i\|^2\}^{\frac{1}{2}} + \{\sum_{i=1}^{N} \|T^{m_{i-1}} x\|^2\}^{\frac{1}{2}} \\ &< 2\sqrt{N}\varepsilon, \end{aligned}$$

where $x \in W$. Therefore, $S \in U_1$. Similarly, we have that

$$\begin{aligned} \|L_T^n S - B\|_2 &\leq \|L_T^n S_2 - B\|_2 + \|L_T^n S_1\|_2 \\ &= \{\sum_{i=1}^{N} \|T^n S_2 e_i - Be_i\|^2\}^{\frac{1}{2}} + \{\sum_{i=1}^{N} \|T^n S_1 e_i\|^2\}^{\frac{1}{2}} \\ &< 2\sqrt{N}\varepsilon. \end{aligned}$$

where $T^n y \in W$ and large enough n. So it follows that L_T is mixing on $B_2(H)$ with the $\|.\|_2$-topology. By Theorem 3.2, the operator T satisfies the Hypercyclicity Criterion with respect to some syndetic sequence. The proof is complete. □

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Approximation by Spherical Neural Networks with Sigmoidal Functions*

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Abstract

This paper addresses the approximation by feed-forward neural networks (FNNs) with sigmoidal active functions on the unit sphere. Firstly, some nice properties of typical logistic function are derived, and the function and its derivatives are taken as active functions to construct spherical FNNs approximation operators, where the spherical Cesáro mean is employed as a key link in constructing the operators. Subsequently, by using spherical quadrature formula and Marcinkiewicz-Zygmund type inequality, the error of the operators approximating continuous spherical function is estimated, and a Jackson type theorem is established by means of the best polynomial approximation.

Keywords Feed-forward neural networks; Unit sphere; Approximation; Error estimate
MSC 41A10, 41A25, 41A30, 65D30

1 Introduction

Feed-forward neural networks (FNNs) with one hidden layer is a class of basic and important neural networks, which can be described mathematically as

$$\mathcal{N}(x) := \sum_{i=1}^{N} c_i \phi(\omega_i \cdot x + \theta_i), \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the active function, $x := (x_1, x_2, \ldots, x_n)^\intercal \in \mathbb{R}^n$ is the input, $c_i \in \mathbb{R}$ $(i = 1, \ldots, N)$ are the output weights connecting the N nodes, $\omega_i := (\omega_{i1}, \omega_{i2}, \ldots, \omega_{in}) \in \mathbb{R}^n$ are the input weights connecting the i-th hidden node and the input, and $\theta_i \in \mathbb{R}$ $(i = 1, \ldots, N)$ are the biases of the i-th hidden node.

As we know, FNNs are universal approximator. Namely, for any continuous or integrable function defined on a compact set, there exists an FNN that can approximate the function with arbitrary accuracy. In connection with such paradigms there arise mainly three problems: A density problem, a complexity problem, and an algorithmic problem. The density problem deals with the question: Which functions can be approximated and, in particular, can all members of a certain class of functions be approximated in a suitable sense? By now, this problem has been satisfactorily solved. We refer the reader to [9, 13, 16] The complexity problem discusses the relationship between the size of the number of neurons and approximation capacity of networks. Many papers, such as [4, 5, 8, 17, 26], have addressed the solution of the problem, where some operators of FNNs have been constructed to approximate continuous or integrable target functions, and in particular, some estimates of the approximation error have been established. In this paper, we will focus on the complexity problem of FNNs approximation on the unit sphere.

In many practical applications we require modeling by the data collected over the surface of the earth, that is, we need deal with the functions which are defined on the unit sphere with geodesic

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distance. Naturally, it is necessary to consider the approximation by FNNs on the unit sphere, and there have been some studies associated with this topic [6, 7, 18, 21], where some estimates of upper bounds of approximation were built. In [19], Lin, Cao, and Xu studied the optimal rate of approximation for some Sobolev space by spherical feed-forward neural networks (SFNNs) with the square integrable active function. On the other hand, we know that the sigmoidal function defined by

$$\sigma(x) := \frac{1}{1+e^{-x}}. \tag{1.2}$$

is typical logistic function, and is usually used to be an active function in FNNs (see [1, 2, 3, 10, 12, 27]). The main purpose of this paper is to investigate the analysis properties of the sigmoidal function (1.2), and take this function and its derivative as active function to construct SFNNs form as (1.1). Particularly, we will establish the error estimate of approximation using the SFNNs by means of the best polynomial approximation.

This paper is organized as follows. In the next section, we will give some preliminaries concluding related notations and three lemmas. In Section 3, we will prove our main result.

2 Preliminaries

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x|_2 = 1\}$, where $|\cdot|_2$ denotes the Euclidean norm. The surface measure on $\mathbb{S}^2$ will be denoted by μ. Corresponding to the surface measure μ, the space $L^2 := L^2(\mathbb{S}^2)$ is the usual Hilbert space of square-integrable functions on $\mathbb{S}^2$ with the inner product

$$(f, g) := \int_{\mathbb{S}^2} f(x)g(x)d\mu(x), \tag{2.3}$$

and the norm $\|f\|_2 := \sqrt{(f, f)}$. The space of continuous functions on $\mathbb{S}^2$ is denoted by $C(\mathbb{S}^2)$, and is a Banach space with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{S}^2} |f(x)|$. Using polar coordinates for a representation of the sphere $\mathbb{S}^2$, we have for a point $x \in \mathbb{S}^2$ the coordinate relation $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, where $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$.

Let $n \geq 0$ be a fixed integer. The restriction of a harmonic homogeneous polynomial of degree n to the unit sphere $\mathbb{S}^2$ is called a spherical harmonic of degree n. The space of all spherical harmonics of degree at most n is denoted by Π_n. It comprises the restriction to $\mathbb{S}^2$ of all algebraic polynomials in 3 variables of total degree at most n. And $\Pi_n = \oplus_{l=0}^n H_l$, where H_n denotes the space of all spherical harmonics of precise degree n. The spaces H_n are mutually orthogonal with respect to (2.3), and the dimension of H_n is $2n+1$. If we choose an orthogonal basis $\{Y_{n,l} : l = 1, 2, \ldots, 2n+1\}$ for each H_n, then the set $\{Y_{k,l} : k = 0, 1, \ldots,\ l = 1, 2, \ldots, 2k+1\}$ is an orthogonal basis for $L^2(\mathbb{S}^2)$.

The spherical harmonics on $\mathbb{S}^2$ of degree l satisfies the addition formula

$$\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y), \tag{2.4}$$

where $x \cdot y$ denotes the usual inner product on $\mathbb{R}^3$, and P_l is the Legendre polynomial with degree l and $P_l(1) = 1$. For more details of spherical harmonics, we refer the reader to [15], [23], and [25].

For any two points x and y on the sphere $\mathbb{S}^2$, the spherical distance $\mathrm{dist}(x, y)$ is defined to be the geodesic distance, that is, $\mathrm{dist}(x, y) := \arccos(x, y)$. For a point set $X = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^2$ the global mesh norm, defined by

$$h_X := \sup_{x\in\mathbb{S}^2} \inf_{x_j\in X} \mathrm{dist}(x, x_j), \tag{2.5}$$

measures how far away a point $x \in \mathbb{S}^2$ can be from the closest point of the point set X. It is not difficult to see that $\mathbb{S}^2 = \bigcup_{x_j\in X} B(x_j, h_X)$, where $B(x_j, h_X)$ is the closed spherical cap with center x_j and radius h_X.

Now we first discuss the property of function σ and give a lemma. Since

$$\sigma(x)=\frac{1}{1+e^{-x}}=\frac{e^x}{1+e^x}=1-\frac{1}{1+e^x},\quad \frac{1}{1+e^x}=\frac{1}{e^x-1}-\frac{2}{e^{2x}-1},$$

and

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{2}{e^{2x}-1}\right)=\frac{1}{2}=\frac{1}{1+e^x}\Big|_{x=0}.$$

By a known expansion

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n,$$

where B_n is Bernoulli number. Then we have $\frac{x}{e^x-1}-\frac{2x}{e^{2x}-1}=\sum_{n=1}^{\infty}\frac{(1-2^n)B_n}{n!}x^n$, which leads to

$$\frac{1}{e^x-1}-\frac{2}{e^{2x}-1}=\sum_{n=1}^{\infty}\frac{(1-2^n)B_n}{n!}x^{n-1}.$$

Hence,

$$\sigma(x)=(1+B_1)+\frac{(2^2-1)B_2}{2!}x+\cdots+\frac{(2^n-1)B_n}{n!}x^{n-1}+\cdots=\frac{1}{2}+\sum_{k=1}^{\infty}b_{2k-1}x^{2k-1},$$

where $b_{2k-1}\neq 0, k=1,2,\ldots$. By a simple calculation we get

$$\tilde{\sigma}(x):=(\sigma(x))'=\frac{e^x}{(e^x+1)^2}. \tag{2.6}$$

Set

$$\bar{\sigma}(x):=\sigma(x)+\tilde{\sigma}(x)=\frac{3}{4}+\frac{(2^2-1)B_2}{2!}x+\frac{3(2^4-1)B_4}{4!}x^2+\cdots. \tag{2.7}$$

From (2.6) we see

$$\lim_{|x|\to+\infty}\tilde{\sigma}(x)=\lim_{|x|\to+\infty}\frac{e^x}{(e^x+1)^2}=0.$$

So $\bar{\sigma}(x)$ has properties:

(i) $\lim_{x\to+\infty}\bar{\sigma}(x)=1,\ \lim_{x\to-\infty}\bar{\sigma}(x)=0$, that is, $\bar{\sigma}(x)$ is also a sigmoidal function;

(ii) $\bar{\sigma}^{(k)}(0)\neq 0,\quad k=0,1,2,\ldots$.

Thus, from Proposition 1 of [4] we have the following Lemma 1.

Lemma 1. For any polynomial with degree n, $p_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, and K is a compact subset of $\mathbb{R}$. Then for any given $\epsilon>0$, there exist real numbers $b_0,b_1,\ldots,b_n$ and $c_0,c_1,c_2,\ldots,c_n$, such that $\sup_{x\in K}|\sum_{i=0}^n b_i\bar{\sigma}(c_ix)-p_n(x)|\leq\epsilon$.

To get the numerical integration formula on the unit sphere we need introduce the following equiangular grid points $\Gamma_N=\{(\theta_m,\varphi_l),0\leq m\leq 2N,\ 0\leq l\leq 2N\}$, where

$$\theta_m=\frac{m\pi}{2N},\quad \varphi_l=\frac{l\pi}{N}. \tag{2.8}$$

We can prove the following lemma.

Lemma 2. For spherical point set $\Gamma_N\subset\mathbb{S}^2$, its mesh norm h_{Γ_N} satisfies the following relation

$$h_{\Gamma_N}\leq\frac{3}{2N}\pi.$$

Proof. For given $x\in\mathbb{S}^2$, x can be represented as $x=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$. By (2.8), there exists $\theta_i(0\leq i\leq 2N)$ such that $\sin\theta_i\neq 0$, and $-\frac{\pi}{2N}\leq\theta-\theta_i\leq\frac{\pi}{2N}$. So $\cos(\theta-\theta_i)\geq\cos\frac{\pi}{2N}$. Writing $y=(\sin\theta_i\cos\varphi,\sin\theta_i\sin\varphi,\cos\theta_i)$, then we have

$$x\cdot y=\cos\theta\cos\theta_i+\sin\theta\sin\theta_i\cos^2\varphi+\sin\theta\sin\theta_i\sin^2\varphi=\cos(\theta-\theta_i).$$

Therefore, $\arccos(x\cdot y)\le \frac{\pi}{2N}$.

Similarly, for φ, there exists $j(0\le j\le 2N)$ such that $-\frac{\pi}{N}\le \varphi-\varphi_j\le \frac{\pi}{N}$. Hence $\cos(\varphi-\varphi_j)\cos\frac{\pi}{N}$.

Set $z=(\sin\theta_i\cos\varphi_j,\sin\theta_i\sin\varphi_j,\cos\theta_i)$. We get

$$\begin{aligned} y\cdot z &= \cos^2\theta_i+\sin^2\theta_i\cos\varphi\cos\varphi_j+\sin^2\theta_i\sin\varphi\sin\varphi_j \\ &= 1-2\sin^2\theta_i\sin^2\frac{\varphi-\varphi_j}{2}\ge 1-2\sin^2\frac{\varphi-\varphi_j}{2}=\cos(\varphi-\varphi_j), \end{aligned}$$

that is, $\arccos(y\cdot z)\le\frac{\pi}{N}$. By spherical triangle inequality (see Lemma 1 in P. 120 of [11]), we have

$$\arccos(x\cdot z)\le \arccos(x\cdot y)+\arccos(y\cdot z)\le \frac{\pi}{2N}\frac{\pi}{N}=\frac{3}{2N}\pi.$$

This shows that for any $x\in\mathbb{S}^2$ it holds that $\inf_{x_j\in X}\operatorname{dist}(x,x_j)\le\frac{3}{2N}\pi$. Thus, the lemma follows from the definition of mesh norm (2.5).

In what follows we introduce the decompositions of the sphere $\mathbb{S}^2$ coming along with the points in Γ_N. We denote by $\mathcal{R}$ a decomposition of $\mathbb{S}^2$, i.e., $\mathcal{R}$ is a finite collection of closed regions $R\subset\mathbb{S}^2$, having no common interior points and covering the whole sphere, i.e., $\bigcup_{R\in\mathcal{R}}R=\mathbb{S}^2$. For a given set C_0, the decomposition $\mathcal{R}$ is called C_0-compatible if each region $R\in\mathcal{R}$ contains at least one point of C_0 in its interior. In this case each $R\in\mathcal{R}$ can be labeled uniquely by a point $\xi\in C_0$. The set of such points will be denoted by C and called a reduced set.

The further illustration related to the existence and construction of compatible decompositions can be found in [22].

Clearly, C is the essential subset of C_0. For the region uniquely determined by $\xi\in C$ we write R_ξ. Moreover, we define the discrete L^1-norm and ∞-norm of a function f as follows:

$$\|f\|_{C,1}:=\sum_{\xi\in C}|f(\xi)|\mu(R_\xi),$$

and $\|f\|_{C,\infty}:=\sup_{\xi\in C}|f(\xi)|$. In addition, we define the partition norm of the decomposition R: $\|R\|:=\sup_{R\in\mathcal{R}}\operatorname{diam}R=\sup_{R\in\mathcal{R}}\sup_{x,y\in R}d(x,y)$. Below, we give an important result (see Proposition 3.2 of [22], Theorem 4.2 or Theorem 5.1 of [14]).

Proposition For any given set $C_0\subset\mathbb{S}^2$, and a C_0-compatible decomposition of $\mathbb{S}^2$ with reduced set C, $\mathcal{R}$, we have

(1) $2h_{C_0}\le\|\mathcal{R}\|\le 8\sqrt{3}h_{C_0}$;

(2) If $\eta\in(0,1)$ is arbitrarily fixed and $\|\mathcal{R}\|\le\frac{\eta}{84n}$, then for any $p\in\Pi_n$, we have $\|p\|_C\le(1+\eta)\|p\|_1$;

(3) If $\eta\in(0,\frac12)$ is arbitrarily fixed and $\|\mathcal{R}\|\le\frac{\eta}{84m}$, then there exist nonnegative numbers $\{a_\xi:\xi\in C\}$, such that $\int_{\mathbb{S}^2}p(x)d\mu(x)=\sum_{\xi\in C}a_\xi p(\xi),\forall p\in\Pi_m$.

Furthermore, we have

$$\left\|\left(\frac{a_\xi}{\mu(R_\xi)}\right)\right\|_{C,\infty}\le c,$$

here and in the following c is an absolute positive constant, and its value may be different at different occurrences, even within the same formula.

Now we consider spherical point set Γ_N, and choose $\eta=\frac14$. For given n, we set $N\ge 12\sqrt{3}\times 336\pi n$, and obtain

$$\|\mathcal{R}\|\le 8\sqrt{3}h_{\Gamma_N}\le 8\sqrt{3}\cdot\frac{3\pi}{2N}\le\frac{1}{336n}=\frac{\eta}{84n}. \tag{2.9}$$

From above Proposition, (2.9) and Lemma 2, it follows the following lemma.

Lemma 3 Let Γ_N be a spherical point set as above, N satisfies $N\ge 12\sqrt{3}\times 336\pi n$, and C_{Γ_N} denote the reduced set of Γ_N. Then

(1) For any $p\in\Pi_n$, there holds $\|p\|_{C_{\Gamma_N}}\le\frac54\|p\|_1$;

(2) There exist nonnegative numbers $\{a_\xi:\xi\in C_{\Gamma_N}\}$, such that $\int_{\mathbb{S}^2}p(x)d\mu(x)=\sum_{\xi\in C_{\Gamma_N}}a_\xi p(\xi),\ \forall p\in\Pi_n$, and $\left\|\left(\frac{a_\xi}{\mu(R_\xi)}\right)\right\|_{C_{\Gamma_N},\infty}\le c$.

We also need to introduce a de la Vallée-Poussin kernel to construct a spherical integral operator. We thus define a function $h : \mathbb{R} \to \mathbb{R}$ as

$$h(t) := \begin{cases} 1, & x \in [0,1), \\ 1-2(x-1)^2, & x \in [1, \frac{3}{2}), \\ 2(2-x)^2, & x \in [\frac{3}{2}, 2), \\ 0, & x \in [2, +\infty). \end{cases}$$

Then (see [24])

$$\sum_{l=0}^{2L-1} \left| \Delta^3 h\left(\frac{l}{L}\right) \right| = \frac{16}{L^2},$$

where $\Delta^3\left(\frac{l}{L}\right)$ denotes the third order forward difference of the sequence $h\left(\frac{0}{L}\right), h\left(\frac{1}{L}\right), \ldots$. Using addition formula (2.4) we define kernel

$$K_l(x,y) := K_l(x \cdot y) := \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y).$$

Furthermore, we construct a new kernel $H_L(x \cdot y)$ by way of $h(t)$

$$H_L(x \cdot y) := \sum_{l=0}^{2L} h\left(\frac{l}{L}\right) K_l(x \cdot y).$$

For given $f \in L^2(\mathbb{S}^2)$, we take the approximation form:

$$V_L f(x) := (f, H_L(x, \cdot)) = \int_{\mathbb{S}^2} f(y) H_L(x,y) d\mu(y).$$

From [24] we know V_L has properties:

(1) V_L reproduces polynomials with degree up to L, that is,

$$V_L p = p \text{ for all } p \in \Pi_L; \tag{2.10}$$

(2) The linear operator sequence $V_1, V_2, \ldots,$ is bounded uniformly.

3 Main Result and Its Proof

We construct SFNN operators with active function $\bar{\sigma}$ given by (2.7) as follows:

$$\mathcal{N}_N(x) := \sum_{\xi \in C_{\Gamma_N}} a_\xi f(\xi) \sum_{j=1}^{2n} b_j \bar{\sigma}(c_j \xi \cdot x) = \sum_{\xi \in C_{\Gamma_N}} \sum_{j=1}^{2n} a_\xi b_j f(\xi) \bar{\sigma}(c_j \xi \cdot x), \tag{3.11}$$

where b_i, c_i and a_i are defined as Lemma 1 and Lemma 3, respectively. Then we obtain

Theorem. Let $f \in C(\mathbb{S}^2)$, and spherical point set $\Gamma_N \subset \mathbb{S}^2$ defined as lemma 3. Then for FNN operator defined by (3.11) and arbitrary positive number $\varepsilon > 0$, we have

$$|f(x) - \mathcal{N}_N f(x)| \le c(E_n(f) + \|f\|_\infty \varepsilon),$$

where $E_n(f) := \inf_{p_n \in \Pi_n} \max_{x \in \mathbb{S}^2} |f(x) - p_n(x)|$ is the best approximation of degree n of f (see [20]).

Proof. For $f \in C(\mathbb{S}^2)$, we denote by $p_n(x)$ the best approximation polynomial with degree n. Then

$$\begin{aligned} \left| f(x) - \sum_{\xi \in C_{\Gamma_N}} a_\xi f(\xi) H_n(x \cdot \xi) \right| &\le |f(x) - p_n(x)| + \left| p_n(x) - \sum_{\xi \in C_{\Gamma_N}} a_\xi f(\xi) H_n(x \cdot \xi) \right| \\ &\le E_n(f) + \Delta. \end{aligned}$$

By (2.10) and Lemma 3 we have

$$\begin{aligned}
\Delta &= \left| \int_{\mathbb{S}^2} p_n(y) H_n(x\cdot y) d\mu(y) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) \right| \\
&= \left| \sum_{\xi\in C_{\Gamma_N}} a_\xi p_n(\xi) H_n(x\cdot\xi) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) \right| \\
&\le E_n(f) \sum_{\xi\in C_{\Gamma_N}} a_\xi \left| H_n(x\cdot\xi)\right| \le cE_n(f) \sum_{\xi\in C_{\Gamma_N}} \mu(R_\xi) \left| H_n(x\cdot\xi)\right| \\
&= c(1+\eta)E_n(f) \int_{\mathbb{S}^2} \left| H_n(x\cdot y)\right| d\mu(y) = c(1+\eta)E_n(f)\|V_n\| \le cE_n(f),
\end{aligned}$$

where we have used the result (see (4.12) of [24]) $\|V_n\| = \sup_{x\in\mathbb{S}^2}\int_{\mathbb{S}^2} |H_n(x\cdot y)|\, d\mu(y)$. Therefore,

$$\left| f(x) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) \right| \le cE_n(f). \tag{3.12}$$

Since

$$\begin{aligned}
|f(x) - \mathcal{N}_N(x)| &\le \left| f(x) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) \right| \\
&+ \left| \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) \sum_{j=1}^{2n} b_j \bar{\sigma}(c_j\xi\cdot x) \right|.
\end{aligned}$$

By Lemma 1 and the construction of operator $\mathcal{N}_N(x)$ we have

$$\begin{aligned}
&\left| \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) H_n(x\cdot\xi) - \sum_{\xi\in C_{\Gamma_N}} a_\xi f(\xi) \sum_{j=1}^{2n} b_j \bar{\sigma}(c_j\xi\cdot x) \right| \\
&\le \sum_{\xi\in C_{\Gamma_N}} a_\xi |f(\xi)| \left| H_n(x\cdot\xi) - \sum_{j=1}^{2n} b_j \bar{\sigma}(c_j\xi\cdot x) \right| \le \varepsilon\|f\|_\infty \sum_{\xi\in C_{\Gamma_N}} a_\xi = 4\pi\|f\|_\infty\varepsilon. \qquad (3.13)
\end{aligned}$$

Thus, by combining (3.12) with (3.13) we obtain that $|f(x) - \mathcal{N}_N(x)| \le c(E_n(f) + \|f\|_\infty\varepsilon)$.

The proof of Theorem is complete.

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Spectrality of self-affine measures with a four elements collinear digit set

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Abstract In this paper, using the feature of zero-point sets $Z(\widehat{\mu}_{M,D})$, where $\widehat{\mu}_{M,D}$ is the Fourier transform of self-affine measures $\mu_{M,D}$, we discuss the spectrality and non-spectrality of the collinear digit set. We give a method to deal with the spectrality and non-spectrality with collinear case. The results here provides some supportive evidence to the two related conjectures.

Keywords and Phrases: Iterated function system, self-affine measure, collinear digit set, spectral measure, compatible pair

2010 Mathematics Subject Classification: 28A80; 42C05; 46C05.

1 Introduction

We call a probability measure μ a spectral measure if there exists a discrete set $\Lambda \in \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i<\lambda,x>} : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, all the eigenvalues of the integer matrix M have modulus greater than 1. Relating to the IFS $\{\phi_d(x) = M^{-1}(x + d)\}_{d\in D}$, there exists a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu \circ \phi_d^{-1}, \qquad (1.\ 1)$$

where $|D|$ be the cardinality of D. Let Q and P be finite subsets of $\mathbb{R}^n$ of the same cardinality q. We say (Q, P) is a compatible pair if the $q \times q$ matrix $H_{Q,P} := [q^{-1/2}e^{2\pi i\langle b,p\rangle}]_{b\in Q,p\in P}$ is unitary, i.e., $H_{Q,P}H^*_{Q,P} = I_q$. Here we use $*$ to denote the transposed conjugate.

We recall the following related conclusions.

The plane Sierpinski gasket $T(M, D)$ corresponding to

$$M = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

Li [2-4] proved that $\mu_{M,D}$ is a spectral measure or a non-spectral measure.

In the present paper, we will consider expanding integer matrix $M \in M_2(\mathbb{Z})$ and $D \in \mathbb{Z}^2$ as follows,

$$M = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, D = \{0, 1, 3, 4n + 2\}\, \mathbf{v}. \tag{1. 2}$$

where $\mathbf{v} = \{\alpha, \beta, \}^T, \alpha^2 + \beta^2 \neq 0, n \in \mathbb{Z}$.

We will get the following two Theorems.

Theorem 1. For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a collinear digit set D given by (1.2), let $(\alpha, \beta)^T$ is an eigenvector of the matrix M and l is eigenvalue, then,

i) if $l \in 4\mathbb{Z} + \{1, 3\}$, then there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$;

ii) if $l \in 4\mathbb{Z}$, then $\mu_{M,D}$ is a spectral measure.

Theorem 2. For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a collinear digit set D given by (1.2), let $(\alpha, \beta)^T$ is not an eigenvector of the matrix M, if $det(M) \in 4\mathbb{Z}$, then there are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$. If, in addition, $a + c = 0$, then $\mu_{M,D}$ is a spectral measure.

The two related conjectures about the spectrality and non-spectrality as follows:

Conjecture i. Let$M \in M_n(\mathbb{Z})$ be an expanding integer matrix, and $D \subset \mathbb{Z}^n$ a finite digit set with $0 \in D$. If there exists a subset $S \subset \mathbb{Z}^n$, $0 \in S$ such that $(M^{-1}D, S)$ is a compatible pair, then $\mu_{M,D}$ is a spectral measure.

Conjecture ii. For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $|D| \notin W(m)$, where $W(m)$ denote the non-negative integer combination of the divisor of $|det(M)|$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$.

2 Proof of the results

Proof of Theorem 1. Since l is an eigenvalue of the matrix M, then l is a zero of the characteristic polynomial $det(\lambda I - M)$. Since M is expanding and $det(\lambda I - M)$ is a monic polynomial with integer coefficients, we have $l \in \mathbb{Z}\backslash\{0, \pm 1\}$. Since

$$\widehat{\mu}_{M,\omega D}(\xi) = \prod_{j=1}^{\infty} m_{\omega D}(M^{*-j}\xi) = \prod_{j=1}^{\infty} m_D(M^{*-j}\omega\xi) = \widehat{\mu}_{M,D}(\omega\xi),$$

so the two measures $\mu_{M,D}$ and $\mu_{M,\omega D}$ have the same spectrality for any non-zero number $\omega \in \mathbb{R}$. Without loss of generality, we assume that $gcd(\alpha, \beta) = 1$. Then, we have $p, q \in \mathbb{Z}$ such that $p\alpha - q\beta = 1$ and $gcd(p, q) = 1$. Let

$$Q = \begin{pmatrix} \alpha & q \\ \beta & p \end{pmatrix},$$

then $detQ = 1$ and $Q^{-1} = \begin{pmatrix} p & -q \\ -\beta & \alpha \end{pmatrix}$. Now, we consider the pairs $(\widehat{M}, \widehat{D})$ given by

$$\widehat{M} = Q^{-1}MQ = \begin{pmatrix} l & bp^2 - dq^2 + (a-c)pq \\ 0 & a + c - l \end{pmatrix}$$

and

$$\widehat{D} = Q^{-1}D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} (4n+2) \\ 0 \end{pmatrix} \right\}. \tag{2. 1}$$

i) Let $\theta_0 = \{\xi \in \mathbb{R}^2, m_{\widehat{D}}(\xi) = 0\}$, then $\theta_0 = Z_1 \cup Z_2 \cup Z_3$, where

$$Z_1 = \left\{ \begin{pmatrix} \frac{1}{4} + k_1 \\ a_1 \end{pmatrix}, k_1 \in \mathbb{Z}, a_1 \in \mathbb{R} \right\} \subset \mathbb{R}^2,$$

$$Z_2 = \left\{ \begin{pmatrix} \frac{1}{2} + k_2 \\ a_2 \end{pmatrix}, k_2 \in \mathbb{Z}, a_2 \in \mathbb{R} \right\} \subset \mathbb{R}^2.$$

$$Z_3 = \left\{ \begin{pmatrix} \frac{3}{4} + k_3 \\ a_3 \end{pmatrix}, k_3 \in \mathbb{Z}, a_3 \in \mathbb{R} \right\} \subset \mathbb{R}^2. \tag{2. 2}$$

Then

$$\widehat{M}^* Z_1 = \begin{pmatrix} l & 0 \\ bp^2 - dq^2 + (a-c)pq & a + c - l \end{pmatrix} \begin{pmatrix} \frac{1}{4} + \widetilde{k} \\ \widetilde{\rho} \end{pmatrix} = \begin{pmatrix} \frac{l}{4} + l\widetilde{k} \\ \widetilde{\rho}_1 \end{pmatrix}.$$

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$$\widehat{M}^* Z_2 = \begin{pmatrix} l & 0 \\ bp^2 - dq^2 + (a-c)pq & a+c-l \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \widetilde{k} \\ \widetilde{\rho}_2 \end{pmatrix} = \begin{pmatrix} \frac{l}{2} + l\widetilde{k} \\ \widetilde{\rho}_2 \end{pmatrix}.$$

$$\widehat{M}^* Z_3 = \begin{pmatrix} l & 0 \\ bp^2 - dq^2 + (a-c)pq & a+c-l \end{pmatrix} \begin{pmatrix} \frac{3}{4} + \widetilde{k} \\ \widetilde{\rho}_3 \end{pmatrix} = \begin{pmatrix} \frac{3l}{4} + l\widetilde{k} \\ \widetilde{\rho}_3 \end{pmatrix}.$$

Since $l \in 4\mathbb{Z}+1$ or $l \in 4\mathbb{Z}+3$. When $l \in 4\mathbb{Z}+1$, $\widehat{M}^* Z_1 \subset Z_1$, $\widehat{M}^* Z_2 \subset Z_2$, $\widehat{M}^* Z_3 \subset Z_3$; When $l \in 4\mathbb{Z}+3$, $\widehat{M}^* Z_1 \subset Z_3$, $\widehat{M}^* Z_2 \subset Z_2$, $\widehat{M}^* Z_3 \subset Z_1$. We get, $\widehat{M}^*(Z_1 \bigcup Z_2 \bigcup Z_3) \subset Z_1 \bigcup Z_2 \bigcup Z_3$, so

$$Z(\widehat{\mu}_{\widehat{M},\widehat{D}}) = \cup_{j=1}^{\infty} (\widehat{M}^{*j}(Z_1 \cup Z_2 \cup Z_3) \subset Z_1 \cup Z_2 \cup Z_3.$$

If $\lambda_j \in \mathbb{R}^2 (j = 1,2,3,4,5)$ are such that the five exponential functions

$$e^{2\pi i \lambda_1 x}, e^{2\pi i \lambda_2 x}, e^{2\pi i \lambda_3 x}, e^{2\pi i \lambda_4 x}, e^{2\pi i \lambda_5 x}$$

are mutually orthogonal in $L^2(\mu_{\widehat{M},\widehat{D}})$, then the differences

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$$

$$\lambda_3 - \lambda_2, \lambda_4 - \lambda_2, \lambda_5 - \lambda_2$$

$$\lambda_4 - \lambda_3, \lambda_5 - \lambda_3$$

$$\lambda_5 - \lambda_4$$

are in the zero set $Z_1 \bigcup Z_2 \bigcup Z_3$. Now, the four elements $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$ are also in the union of the three sets $Z_i, i = 1,2,3$, this will deduce an impossible result easily. For example, if $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in Z_1$, then

$$\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in Z_1 - Z_1 \in \left\{ \begin{pmatrix} k_4 \\ a_4 \end{pmatrix}, k_4 \in \mathbb{Z}, a_4 \in \mathbb{R} \right\}.$$

Since $\lambda_3 - \lambda_2 \in Z_1 \bigcup Z_2 \bigcup Z_3$ a contradict with

$$(Z_1 \cup Z_2 \cup Z_3) \cap \left\{ \begin{pmatrix} k_4 \\ a_4 \end{pmatrix}, k_4 \in \mathbb{Z}, a_4 \in \mathbb{R} \right\} = \emptyset.$$

Then there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{\widehat{M},\widehat{D}})$, so there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

ii) Since $l \in 4\mathbb{Z}$,

$$\widehat{\mu}_{\widehat{M},\widehat{D}}(\xi) = \prod_{j=1}^{\infty} \frac{1}{4}(1 + e^{\frac{2\pi i \xi_1}{l^j}} + e^{\frac{2\pi i (4n+2)\xi_1}{l^j}} + e^{\frac{2\pi i 3\xi_1}{l^j}}), \tag{2. 3}$$

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where $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$.

Let $\overline{M} = [l], \overline{D} = \{0, 1, 3, 4n+2\}$, then

$$\widehat{\mu}_{\overline{M},\overline{D}}(\xi_1) = \prod_{j=1}^{\infty} \frac{1}{4}(1 + e^{\frac{2\pi i \xi_1}{l^j}} + e^{\frac{2\pi i (4n+2)\xi_1}{l^j}} + e^{\frac{2\pi i 3\xi_1}{l^j}}), \forall \xi_1 \in \mathbb{R}. \tag{2. 4}$$

If $\overline{S} = \{\frac{-p}{4}, 0, \frac{p}{4}, \frac{p}{2}\} \subset [2 - |p|, |p| - 2]$, we can get $(\overline{M}^{-1}\overline{D}, \overline{S})$ is a compatible pair, by Theorem 1.2 in [10], we get

$$\Lambda(\overline{M}, \overline{S}) = \left\{ \sum_{j=0}^{k-1} (p)^j s_j, k \geq 1, s_j \in \overline{S} \right\}$$

is a spectral of $\mu_{\overline{M},\overline{D}}$. Hence

$$\sum_{\lambda \in \Lambda(\overline{M},\overline{S})} |\widehat{\mu}_{\overline{M},\overline{D}}(\xi_1 - \lambda)|^2 = 1, \forall \xi_1 \in \mathbb{R}. \tag{2. 5}$$

Define $\widetilde{\Lambda} = \left\{ \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \lambda \in \Lambda(\overline{M}, \overline{S}) \right\}$, from (2.3),(2.4),(2.5) we get

$$\sum_{\widetilde{\lambda} \in \widetilde{\Lambda}} |\widehat{\mu}_{\widehat{M},\widehat{D}}(\xi - \widetilde{\lambda})|^2 = \sum_{\lambda \in \Lambda(\overline{M},\overline{S})} |\widehat{\mu}_{\overline{M},\overline{D}}(\xi_1 - \lambda)|^2 = 1, \forall \xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2.$$

Then $\widetilde{\Lambda}$ is a spectral of $\mu_{\widehat{M},\widehat{D}}$, hence $Q^{*-1}\widetilde{\Lambda}$ is a spectral of $\mu_{M,D}$. This completes the proof of Theorem 1.

Note that, in the above proof, we have $det(M) = (a + c - l)l$. If $det(M) \notin 4\mathbb{Z}$, then $l \notin 4\mathbb{Z}$, so $\mu_{M,D}$ is a non-spectral measure and there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. If $det(M) \in 4\mathbb{Z}$ and $l \in 4\mathbb{Z}$, then $\mu_{M,D}$ is a spectral measure. If $det(M) \in 4\mathbb{Z}$ and $l \notin 4\mathbb{Z}$, then $\mu_{M,D}$ is a non-spectral measure and there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. For example, let M and D be given by

$$M = \begin{pmatrix} 4 & b \\ 0 & 3 \end{pmatrix} b \in \mathbb{Z}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4n+2 \\ 0 \end{pmatrix} \right\}.$$

Then $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, l = 4$, so $\mu_{M,D}$ is a spectral measure. If M and D be given by

$$M = \begin{pmatrix} 3 & b \\ 0 & 4 \end{pmatrix} b \in \mathbb{Z}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4n+2 \\ 0 \end{pmatrix} \right\}.$$

Then $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}, l = 3$, so $\mu_{M,D}$ is a non-spectral measure and there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

Proof of Theorem 2. Since $(\alpha, \beta)^T$ is not an eigenvector of the matrix M, we can verify the integer matrix B defined by

$$B := \begin{pmatrix} a\alpha + b\beta & \alpha \\ d\alpha + c\beta & \beta \end{pmatrix}$$

is invertible. Now, consider the pairs $(\widetilde{M}, \widetilde{D})$ given by

$$\widetilde{M} = B^{-1}MB = \begin{pmatrix} a+c & 1 \\ bd-ac & 0 \end{pmatrix}$$

and

$$\widetilde{D} = B^{-1}D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4n+2 \end{pmatrix} \right\}. \tag{2. 6}$$

Let

$$\widetilde{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{bd-ac}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{bd-ac}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{ac-bd}{4} \\ 0 \end{pmatrix} \right\} \subset \mathbb{Z}^2,$$

such that $(\widetilde{M}^{-1}\widetilde{D}, \widetilde{S})$ is a compatible pair, so $E(\Lambda(\widetilde{M}, \widetilde{S}))$ is an infinite orthogonal exponentials in $L^2(\mu_{\widetilde{M},\widetilde{D}})$.

If $a + c = 0$, the following we proved $\Lambda(\widetilde{M}, \widetilde{S})$ is a spectrum for $\mu_{\widetilde{M},\widetilde{D}}$.

We need the following lemma due to Strichartz [1].

Lemma 3. Let $M \in M_n(\mathbb{Z})$ be expanding, D and S be finite subsets of $\mathbb{Z}^n$ such that $(M^{-1}D, S)$ is a compatible pair and $0 \in D \cap S$. Suppose that the zero set $Z(m_{M^{-1}D}(x))$ is disjoint from the set $T(M^*, S)$. Then $\Lambda(M, S)$ is a spectrum for $\mu_{M,D}$.

Since,

$$Z(m_{\widetilde{M}^{-1}\widetilde{D}}(x)) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : 1 + e^{\frac{2\pi i x_1}{bd-ac}} + e^{\frac{2\pi i(4n+2)x_1}{bd-ac}} + e^{\frac{2\pi i 3 x_1}{bd-ac}} = 0, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \in (bd-ac)((\frac{1}{4} + \mathbb{Z}) \cup (\frac{1}{2} + \mathbb{Z}) \cup (\frac{3}{4} + \mathbb{Z})), x_2 \in \mathbb{R} \right\},$$

if $x \in Z(m_{\widetilde{M}^{-1}\widetilde{D}}(x))$, we have $|x_1| \geq 1$.

$T(\widetilde{M}^*, \widetilde{S}) = \{\sum_{j=1}^{\infty} \widetilde{M}^{*-j} s_j : s_j \in \widetilde{S}\}$

$$= \left\{ \sum_{j=1}^{\infty} \begin{pmatrix} \nu^{-j} & 0 \\ 0 & \nu^{-j} \end{pmatrix} \begin{pmatrix} s_{1,j} \\ s_{2,j} \end{pmatrix} + \sum_{j=0}^{\infty} \begin{pmatrix} 0 & \nu^{-j-1} \\ \nu^{-j} & 0 \end{pmatrix} \begin{pmatrix} s_{1,j} \\ s_{2,j} \end{pmatrix} : \begin{pmatrix} s_{1,j} \\ s_{2,j} \end{pmatrix} \in \widetilde{S} \right\}$$

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$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \in \widetilde{T}_1, x_2 \in \mathbb{R} \right\}, \nu = bd - ac$, where

$$\widetilde{T}_1 = \frac{bd-ac}{4} \left\{ \sum_{j=1}^{\infty} (bd-ac)^{-j} s_{1,j} : s_{1,j} \in \{0,-1,1,2\} \right\}.$$

Since $det(M) = ac - bd \in 4\mathbb{Z}$, if $x \in T(\widetilde{M}^*, \widetilde{S})$, then $|x_1| \le \frac{2}{3}$, so

$$T(\widetilde{M}^*, \widetilde{S}) \cap Z(m_{\widetilde{M}^{-1}\widetilde{D}}(x)) = \emptyset.$$

Therefore, the conditions of Lemma 3 are satisfied, so $\mu_{\widetilde{M},\widetilde{D}}$ is a spectral measure, then $\mu_{M,D}$ is.

From the proofs of two Theorems, we also yields the following result which we list as Corollary 1.

Corollary 1 For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a collinear digit set D given by (1.2), let $(\alpha, \beta)^T$ is not an eigenvector of the matrix M, if M have two integer eigenvalues, and one of them in $4\mathbb{Z}$, then $\mu_{M,D}$ is a spectral measure.

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A new second-order symmetric duality in multiobjective programming over cones *

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Abstract

In this paper, a new second-order symmetric duality in multiobjective programming over arbitrary cones is fomulated. The weak, strong and converse duality theorems are proved for these programs under η-invexity assumptions. Our results generalize these existing dual formulations which were discussed by the authors in [5, 6, 7, 8, 21].

Key words: Multiobjective programming, symmetric duality, cones, η-invexity.

MR(2000)Subject Classification: 49N15,90C30

1. Introduction

The notation of symmetric dual was first introduced by Dorn [1] for quadratic programming. Subsequently, it was extended to general nonlinear programs for convex\concavity functions by Dantzig [2] and Mond [3]. Later on, another pair of symmetric dual nonlinear programs under weaker convexity assumptions were presented by Mond et al.[4]. Weir et al.[5] as well as Gulati et al.[6] proved multiobjective symmetric duality results. Chandra and Kumar [7] studied Mond-Weir type symmetric duality with cone constraints. His results were extended by Khurana [8] to the case which the objective function has been optimized with respect to a closed convex cone.

Mangasarian [9] introduced the concept of second-order duality for nonlinear programs. He has indicated a possible computational advantage of the second-order dual over the first order dual. Since then, several authors [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] in this field have worked on the second order duality. Recently, Yang et al. [18, 19] studied second-order multiobjective symmetric dual programs and established the duality results under F-convexity assumptions. Gulati et al.[20] studied Wolfe and Mond-Weir type second-order symmetric duality over arbitrary cones under η-bonvexity\η-pseudobonvexity assumptions. Gulati et al.[21] considered a pair of Mond-Weir type second-order symmetric dual programs over arbitrary cones and proved duality results under invexity

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assumptions. Ahmad et al.[22] formulated a pair of mixed symmetric dual programs over arbitrary cones and established duality results by using cone-invexity assumptions.

In this paper, motivated by [21, 22], we consider a new second-order multiobjective symmetric dual programs over arbitrary cones and prove weak, strong, converse duality results under η-invexity assumptions. Our results generalize the work in [5, 6, 7, 8, 21].

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space. For $N = \{1, 2, \cdots, n\}$ and $M = \{1, 2, \cdots, m\}$, let $J_1 \subseteq N,\ K_1 \subseteq M$ and $J_2 = N\backslash J_1,\ K_2 = M\backslash K_1$. Let $\mid J_1 \mid$ denote the number of elements in J_1. The other symbols $\mid J_2 \mid$, $\mid K_1 \mid$ and $\mid K_2 \mid$ are defined similarly.
We consider the following multiobjective programming problem:

$$\begin{array}{lll} (P) & K-minimize & f(x) \\ & \text{s.t.} & -g(x) \in Q, \quad x \in S, \end{array}$$

where $S \subseteq R^{n+m}$ is open, $f : S \to R^k,\ g : S \to R^m$, K and Q are closed convex pointed cones with nonempty interiors in R^k and R^m, respectively.

Let $X^0 = \{x \in S : -g(x) \in Q\}$ be the set of all feasible solution for (P) and f be differentiable on S.

Definition 2.1 *[8]A point $\overline{x} \in X^0$ is an efficient solution of (P) if there exists no $x \in X^0$ such that $f(\overline{x}) - f(x) \in K\backslash\{0\}$.*

Definition 2.2 *[23]The function $h : S \to R$ is η-invex at $u \in S$ with respect to $\eta : S \times S \to R^n$ if for any $x \in S$,*

$$h(x) - h(u) \geq \eta(x, u)^T \nabla h(u).$$

Definition 2.3 *[23]The function $h : S \to R$ is pseudoinvex at $u \in S$ with respect to $\eta : S \times S \to R^n$ if for any $x \in S$,*

$$\eta(x, u)^T \nabla h(u) \geq 0 \Rightarrow h(x) \geq h(u).$$

Definition 2.4 *[24]Let C be a closed convex cone in R^n with nonempty interiors. The positive polar cone C^* of C is defined by*

$$C^* = \{z \in R^n : x^T z \geq 0 \ for\ all\ x \in C\}.$$

3. New second-order symmetric duality

We consider the following pair of second-order multiobjective symmetric dual problem and establish weak, strong and converse duality theorems.
Primal(MP):

$$\begin{array}{ll} K-minimize & f_1(x_1, y_1) + f_2(x_2, y_2) - [y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2, y_2) + \nabla_{y_2 y_2}(\omega^T g_2)(x_2, y_2)p_2)]e \\ \text{s.t.} & -(\nabla_{y_1}(\lambda^T f_1)(x_1, y_1) + \nabla_{y_1 y_1}(\omega^T g_1)(x_1, y_1)p_1) \in C_3^*, \\ & -(\nabla_{y_2}(\lambda^T f_2)(x_2, y_2) + \nabla_{y_2 y_2}(\omega^T g_2)(x_2, y_2)p_2) \in C_4^*, \\ & y_1^T(\nabla_{y_1}(\lambda^T f_1)(x_1, y_1) + \nabla_{y_1 y_1}(\omega^T g_1)(x_1, y_1)p_1) \geq 0, \\ & \lambda \in intK^*, \quad \lambda^T e = 1, \quad x_1 \in C_1, \quad x_2 \in C_2, \end{array}$$

Dual(MD):

$$\begin{array}{ll} K-maximize & f_1(u_1,v_1)+f_2(u_2,v_2)-[u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2)+\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2)]e \\ \text{s.t.} & \nabla_{x_1}(\lambda^T f_1)(u_1,v_1)+\nabla_{x_1x_1}(\omega^T g_1)(u_1,v_1)r_1 \in C_1^*, \\ & \nabla_{x_2}(\lambda^T f_2)(u_2,v_2)+\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2 \in C_2^*, \\ & u_1^T(\nabla_{x_1}(\lambda^T f_1)(u_1,v_1)+\nabla_{x_1x_1}(\omega^T g_1)(u_1,v_1)r_1) \le 0, \\ & \lambda \in intK^*, \quad \lambda^T e=1, \quad v_1\in C_3, \quad v_2 \in C_4, \end{array}$$

where

$(i)e=(1,1,\cdots,1)^T\in R^k$,

$(ii)f_1: R^{|J_1|}\times R^{|K_1|}\to R^k$ is a twice differentiable function,

$(iii)g_1: R^{|J_1|}\times R^{|K_1|}\to R^m$ is a twice differentiable function,

$(iv)f_2: R^{|J_2|}\times R^{|K_2|}\to R^k$ is a twice differentiable function,

$(v)g_2: R^{|J_2|}\times R^{|K_2|}\to R^m$ is a twice differentiable function,

$(vi)\omega\in R^m,\ p_1\in R^{|K_1|},\ p_2\in R^{|K_2|},\ r_1\in R^{|J_1|},\ r_2\in R^{|J_2|}$,

(vii) for $i=1,2,3,4,\ C_i$ is a closed convex cone with nonempty interior in $R^{|J_1|}, R^{|J_2|}, R^{|K_1|}, R^{|K_2|}$ and C_i^* is its positive polar cone, K is a closed convex pointed cone in R^k such that $intK\neq\emptyset$ and K^* is its positive polar cone.

Remark 3.1 *If we set $J_2=\emptyset, K_2=\emptyset$, then (MP) and (MD) reduce to Mond-Weir type symmetric dual programs in [21]. If we set $J_1=\emptyset, K_1=\emptyset$, then (MP) and (MD) reduce to Wolfe type symmetric dual programs.*

Theorem 3.1 *(Weak duality) Let $(x_1,y_1,x_2,y_2,\lambda,\omega,p_1,p_2)$ be feasible for (MP) and $(u_1,v_1,u_2,v_2,\lambda,\omega,r_1,r_2)$ be feasible for (MD). Suppose that*

(i) $(\lambda^T f_2)(\cdot,v_2)$ be η_1-invex at u_2 with respect to η_1 for fixed v_2, $\eta_1(x_2,u_2)+u_2\in C_2$ for all $x_2\in C_2$;

(ii) $-(\lambda^T f_2)(x_2,\cdot)$ be η_2-invex at y_2 with respect to η_2 for fixed x_2, $\eta_2(v_2,y_2)+y_2\in C_4$ for all $v_2\in C_4$;

(iii) $\begin{bmatrix} r_2^T & 0\\ 0 & p_2^T\end{bmatrix}\begin{bmatrix}\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2) & 0\\ 0 & -\nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)\end{bmatrix}\begin{bmatrix}\eta_1(x_2,u_2)\\ \eta_2(v_2,y_2)\end{bmatrix}\le 0$ *;*

(vi) $(\lambda^T f_1)(\cdot,v_1)$ be pseudoinvex at u_1 with respect to η_3 for fixed v_1, $\eta_3(x_1,u_1)+u_1\in C_1$ for all $x_1\in C_1$;

(v) $-(\lambda^T f_1)(x_1,\cdot)$ be pseudoinvex at y_1 with respect to η_4 for fixed x_1, $\eta_4(v_1,y_1)+y_1\in C_3$ for all $v_1\in C_3$;

(vi) $\begin{bmatrix} r_1^T & 0\\ 0 & p_1^T\end{bmatrix}\begin{bmatrix}\nabla_{x_1x_1}(\omega^T g_1)(u_1,v_1) & 0\\ 0 & -\nabla_{y_1y_1}(\omega^T g_1)(x_1,y_1)\end{bmatrix}\begin{bmatrix}\eta_3(x_1,u_1)\\ \eta_4(v_1,y_1)\end{bmatrix}\le 0$ *.*

Then $f_1(u_1,v_1)-f_1(x_1,y_1)+f_2(u_2,v_2)-u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2)+\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2)e-f_2(x_2,y_2)+y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2,y_2)+\nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2)e\notin K\backslash 0$.

Proof. Suppose, to the contrary, that

$$\begin{array}{l} f_1(u_1,v_1)-f_1(x_1,y_1)+f_2(u_2,v_2)-u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2)+\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2)e \\ -f_2(x_2,y_2)+y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2,y_2)+\nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2)e\in K\backslash 0 \end{array}$$

Since $\lambda\in intK^*$, we obtain

$$\begin{aligned} \lambda^T\{-f_1(u_1,v_1) &+ f_1(x_1,y_1)-f_2(u_2,v_2)+u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2)-\nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2)e \\ &+ f_2(x_2,y_2)-y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2,y_2)-\nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2)e\}<0. \end{aligned} \tag{3.1}$$

In view of $\lambda^T e = 1$, one gets

$$\begin{aligned}\lambda^T f_1(x_1,y_1) &- \lambda^T f_1(u_1,v_1) - \lambda^T f_2(u_2,v_2) + u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2) + \nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2) \\ &+ \lambda^T f_2(x_2,y_2) - y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2,y_2) + \nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2) < 0.\end{aligned} \tag{3.2}$$

By η_1-invexity of $\lambda^T f_2(\cdot, v_2)$, η_2-invexity of $-\lambda^T f_2(x_2,\cdot)$ and hypothesis (iii), we have

$$\lambda^T f_2(x_2,v_2) - \lambda^T f_2(u_2,v_2) \geq \eta_1^T(x_2,u_2)\{\nabla_{x_2}(\lambda^T f_2)(u_2,v_2) + \nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2\}, \tag{3.3}$$

$$\lambda^T f_2(x_2,y_2) - \lambda^T f_2(x_2,v_2) \geq -\eta_2^T(v_2,y_2)\{\nabla_{y_2}(\lambda^T f_2)(x_2,y_2) + \nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2\}, \tag{3.4}$$

The second constraint in (MD) and hypothesis (i) implies that

$$\begin{aligned}\eta_1^T(x_2,u_2)\{\nabla_{x_2}(\lambda^T f_2)(u_2,v_2) &+ \nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2\} \\ &\geq -u_2^T\{\nabla_{x_2}(\lambda^T f_2)(u_2,v_2) + \nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2\}.\end{aligned} \tag{3.5}$$

Similarly, by hypothesis (ii) and the second constraint in (MP),

$$\begin{aligned}-\eta_2^T(v_2,y_2)\{\nabla_{y_2}(\lambda^T f_2)(x_2,y_2) &+ \nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2\} \\ &\geq y_2^T\{\nabla_{y_2}(\lambda^T f_2)(x,y) + \nabla_{y_2y_2}(\omega^T g)(x_2,y_2)p_2\}.\end{aligned} \tag{3.6}$$

Finally, the above four inequalities (3.3)(3.4)(3.5)(3.6) yield

$$\begin{aligned}-\lambda^T f_2(u_2,v_2) &+ u_2^T(\nabla_{x_2}(\lambda^T f_2)(u_2,v_2) + \nabla_{x_2x_2}(\omega^T g_2)(u_2,v_2)r_2) \\ &+ \lambda^T f_2(x_2,y_2) - y_2^T(\nabla_{y_2}(\lambda^T f_2)(x_2,y_2) + \nabla_{y_2y_2}(\omega^T g_2)(x_2,y_2)p_2) \geq 0.\end{aligned} \tag{3.7}$$

Similar to the proof of Theorem 3.1 in [21], we obtain

$$\lambda^T f_1(x_1,y_1) - \lambda^T f_1(u_1,v_1) \geq 0. \tag{3.8}$$

The sum of (3.7) and (3.8) contradicts (3.2).

Theorem 3.2 *(Strong Duality)Suppose that $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{\lambda},\overline{\omega},\overline{p}_1,\overline{p}_2)$ be an efficient solution for (M-P). Let*

(i) $\nabla_{y_1y_1}(\overline{\omega}^T g_1)(\overline{x}_1,\overline{y}_1)$ and $\nabla_{y_2y_2}(\overline{\omega}^T g_2)(\overline{x}_2,\overline{y}_2)$ be all nonsingular,
(ii) the columns of $(\nabla_{y_1} f_1(\overline{x}_1,\overline{y}_1), \nabla_{y_2} f_2(\overline{x}_2,\overline{y}_2))$ be linearly independent, and
(iii)$\nabla_{y_1y_1}(\overline{\omega}^T g_1)(\overline{x}_1,\overline{y}_1)\overline{p}_1 \notin span\{\nabla_{y_1} f_{11}(\overline{x}_1,\overline{y}_1),\cdots,\nabla_{y_1} f_{1k}(\overline{x}_1,\overline{y}_1)\}\backslash\{0\}$,
$\nabla_{y_2y_2}(\overline{\omega}^T g_2)(\overline{x}_2,\overline{y}_2)\overline{p}_2 \notin span\{\nabla_{y_2} f_{21}(\overline{x}_2,\overline{y}_2),\cdots,\nabla_{y_2} f_{2k}(\overline{x}_2,\overline{y}_2)\}\backslash\{0\}$,
where $f_1 = (f_{11}, f_{12},\cdots,f_{1k})$ and $f_2 = (f_{21}, f_{22},\cdots,f_{2k})$.

Then, $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{\omega},\overline{r}_1=0,\overline{r}_2=0)$ is feasible for $(MD)_{\overline{\lambda}}$, and the objective function values of (MP) and $(MD)_{\overline{\lambda}}$ are equal. Also, if the hypotheses of the weak duality theorem are satisfied for all feasible solutions of $(MP)_{\overline{\lambda}}$ and $(MD)_{\overline{\lambda}}$, then $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{\omega},\overline{r}_1=0,\overline{r}_2=0)$ is an efficient solution for $(MD)_{\overline{\lambda}}$.

Proof. Since $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{\lambda},\overline{\omega},\overline{p}_1,\overline{p}_2)$ be an efficient solution for (MP), by using the Fritz John type necessary optimality conditions established by Suneja et al. in 2002 (See Lemma 1 in [24]), there exist $\alpha \in K^*$, $\beta_1 \in C_3$, $\beta_2 \in C_4$, $\gamma \in R_+$, $\xi \in R$ such that the following conditions are satisfied at $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{\lambda},\overline{\omega},\overline{p}_1,\overline{p}_2)$:

$$(x_1-\overline{x}_1)^T[\nabla_{x_1} f_1\alpha + (\nabla_{y_1x_1}(\overline{\lambda}^T f_1) + \nabla_{x_1}(\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1))(\beta_1 - \gamma\overline{y}_1)] \geq 0, \quad for\ all\ x_1 \in C_1, \tag{3.9}$$

$$(x_2-\overline{x}_2)^T[\nabla_{x_2} f_2\alpha + (\nabla_{y_2x_2}(\overline{\lambda}^T f_2) + \nabla_{x_2}(\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2))(\beta_2 - \alpha^T e\overline{y}_2)] \geq 0, \quad for\ all\ x_2 \in C_2, \tag{3.10}$$

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$$\begin{aligned}(y_1 \quad - \quad & \overline{y}_1)^T\{\nabla_{y_1} f_1(\alpha-\gamma\overline{\lambda}) + [\nabla_{y_1y_1}(\overline{\lambda}^T f_1) + \nabla_{y_1}(\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1)](\beta_1 - \gamma\overline{y}_1) \\ - \quad & \gamma\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1\} \geq 0,\end{aligned} \tag{3.11}$$

for all $y_1 \in R^{|K_1|}$,

$$\begin{aligned}(y_2 \quad - \quad & \overline{y}_2)^T\{\nabla_{y_2} f_2(\alpha-\alpha^T e\overline{\lambda}) + [\nabla_{y_2y_2}(\overline{\lambda}^T f_2) + \nabla_{y_2}(\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2)](\beta_2 - \alpha^T e\overline{y}_2) \\ - \quad & \alpha^T e\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2\} \geq 0,\end{aligned} \tag{3.12}$$

for all $y_2 \in R^{|K_2|}$,

$$[(\beta_1-\gamma\overline{y}_1)^T\nabla_{y_1} f_1 + (\beta_2-\alpha^T e\overline{y}_2)^T\nabla_{y_2} f_2 - \xi e](\lambda-\overline{\lambda}) \geq 0, \quad for\ all\ \lambda \in intK^*, \tag{3.13}$$

$$(\beta_1-\gamma\overline{y}_1)^T(\nabla_\omega(\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1)) + (\beta_2-\alpha^T e\overline{y}_2)^T(\nabla_\omega(\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2)) = 0, \tag{3.14}$$

$$(\beta_1-\gamma\overline{y}_1)^T\nabla_{y_1y_1}(\overline{\omega}^T g_1) = 0, \tag{3.15}$$

$$(\beta_2-\alpha^T e\overline{y}_2)^T\nabla_{y_2y_2}(\overline{\omega}^T g_2) = 0, \tag{3.16}$$

$$\beta_1^T(\nabla_{y_1}(\overline{\lambda}^T f_1) + \nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1) = 0, \tag{3.17}$$

$$\beta_2^T(\nabla_{y_2}(\overline{\lambda}^T f_2) + \nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2) = 0, \tag{3.18}$$

$$\gamma y_1^T(\nabla_{y_1}(\lambda^T f_1)(\overline{x}_1,\overline{y}_1) + \nabla_{y_1y_1}(\omega^T g_1)(\overline{x}_1,\overline{y}_1)\overline{p}_1) = 0, \tag{3.19}$$

$$\xi(\lambda^T e - 1) = 0, \tag{3.20}$$

$$(\alpha,\ \beta_1,\ \beta_2,\ \gamma,\ \xi) \neq 0. \tag{3.21}$$

(3.11)(3.12) and (3.13) yield the equations

$$\nabla_{y_1} f_1(\alpha-\gamma\overline{\lambda}) + [\nabla_{y_1y_1}(\overline{\lambda}^T f_1) + \nabla_{y_1}(\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1)](\beta_1-\gamma\overline{y}_1) - \gamma\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1 = 0, \tag{3.22}$$

$$\nabla_{y_2} f_2(\alpha-\alpha^T e\overline{\lambda}) + [\nabla_{y_2y_2}(\overline{\lambda}^T f_2) + \nabla_{y_2}(\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2)](\beta_2-\alpha^T e\overline{y}_2) - \alpha^T e\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2 = 0, \tag{3.23}$$

and

$$(\beta_1-\gamma\overline{y}_1)^T\nabla_{y_1} f_1 + (\beta_2-\alpha^T e\overline{y}_2)^T\nabla_{y_2} f_2 - \xi e = 0. \tag{3.24}$$

By hypothesis (i), (3.15) and (3.16), we have

$$\beta_1 = \gamma\overline{y}_1, \quad \beta_2 = \alpha^T e\overline{y}_2. \tag{3.25}$$

Now, we claim that $\alpha^T e \neq 0$. Indeed, if $\alpha^T e = 0$, then $\beta_2 = 0$ from (3.25). Therefore, from (3.23), we get

$$(\nabla_{y_2} f_2)\alpha = 0,$$

which by hypothesis (ii) give $\alpha = 0$. In view of (3.22)(3.24)(3.25) and hypothesis (iii), we conclude that $\gamma = 0,\ \beta_1 = 0,\ \xi = 0$, and contradicts $(\alpha,\ \beta_1,\ \beta_2,\ \gamma,\ \xi) \neq 0$.

Similarly, we also claim that $\gamma \neq 0$.

Substituting (3.25) into (3.22) and (3.23), we have

$$\nabla_{y_1} f_1(\alpha-\gamma\overline{\lambda}) = \gamma\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1, \quad \nabla_{y_2} f_2(\alpha-\alpha^T e\overline{\lambda}) = \alpha^T e\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2.$$

Using hypothesis(iii), the above relation implies

$$\gamma\nabla_{y_1y_1}(\overline{\omega}^T g_1)\overline{p}_1 = 0,\ \alpha^T e\nabla_{y_2y_2}(\overline{\omega}^T g_2)\overline{p}_2 = 0,$$

which in view of hypothesis(i) yields $\overline{p}_1 = 0,\ \overline{p}_2 = 0$. Thus $\nabla_{y_1} f_1(\alpha - \gamma\overline{\lambda}) = 0,\ \nabla_{y_2} f_2(\alpha - \alpha^T e\overline{\lambda}) = 0$. By hypothesis (ii), one gets $\alpha = \gamma\overline{\lambda},\ \alpha = \alpha^T e\overline{\lambda}$. Further, the above equation, (3.9), (3.10)and (3.25) imply

$$(x_1 - \overline{x}_1)^T \nabla_{x_1}(\overline{\lambda}^T f_1) \geq 0 \ \ for\ all\ x_1 \in C_1,$$

$$(x_2 - \overline{x}_2)^T \nabla_{x_2}(\overline{\lambda}^T f_2) \geq 0 \ \ for\ all\ x_2 \in C_2.$$

Let $x_1 \in C_1$, then $\overline{x}_1 + x_1 \in C_1$ and the above inequality implies

$$x_1^T \nabla_{x_1}(\overline{\lambda}^T f_1) \geq 0 \ \ for\ all\ x_1 \in C_1.$$

Therefore $\nabla_{x_1}(\overline{\lambda}^T f_1) \in C_1^*$.

Similarly, we also obtain that $\nabla_{x_2}(\overline{\lambda}^T f_2) \in C_2^*$.

Hence $(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2, \overline{\omega}, \overline{r}_1 = 0, \overline{r}_2 = 0)$ satisfies the constraints of $(MD)_{\overline{\lambda}}$, that is, it is feasible for the dual problem $(MD)_{\overline{\lambda}}$. Moreover, (MP) and $(MD)_{\overline{\lambda}}$ have equal objective function value from the above proof and (3.10)(3.18).

Now, suppose $(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2, \overline{\omega}, \overline{r}_1 = 0, \overline{r}_2 = 0)$ is not an efficient solution for $(MD)_{\overline{\lambda}}$, then there exists a feasible solution $(u_1, u_2, v_1, v_2, \omega, r_1, r_2)$ for $(MD)_{\overline{\lambda}}$, such that

$$\begin{aligned} f_1(u_1, v_1) &- f_1(\overline{x}_1, \overline{y}_1) + f_2(u_2, v_2) - u_2^T(\nabla_{x_2}(\overline{\lambda}^T f_2)(u_2, v_2) + \nabla_{x_2x_2}(\omega^T g_2)(u_2, v_2) r_2)e \\ &- f_2(\overline{x}_2, \overline{y}_2) + \overline{y}_2^T(\nabla_{y_2}(\overline{\lambda}^T f_2)(\overline{x}_2, \overline{y}_2) + \nabla_{y_2y_2}(\overline{\omega}^T g_2)(\overline{x}_2, \overline{y}_2)\overline{p}_2)e \in K \backslash 0. \end{aligned}$$

which contradicts the weak duality theorem. Hence $(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2, \overline{\omega}, \overline{r}_1 = 0, \overline{r}_2 = 0)$ is an efficient solution for $(MD)_{\overline{\lambda}}$.

Theorem 3.3 *(Converse Duality)Suppose that* $(\overline{u}_1, \overline{u}_2, \overline{v}_1, \overline{v}_2, \overline{\lambda}, \overline{\omega}, \overline{r}_1, \overline{r}_2)$ *be an efficient solution for (MD). Let*

(i) $\nabla_{x_1x_1}(\overline{\omega}^T g_1)(\overline{u}_1, \overline{v}_1)$ *and* $\nabla_{x_2x_2}(\overline{\omega}^T g_2)(\overline{u}_2, \overline{v}_2)$ *be all nonsingular,*

(ii) the columns of $(\nabla_{x_1} f_1(\overline{u}_1, \overline{v}_1), \nabla_{x_2} f_2(\overline{u}_2, \overline{v}_2))$ *be linearly independent, and*

(iii)$\nabla_{x_1x_1}(\overline{\omega}^T g_1)(\overline{u}_1, \overline{v}_1)\overline{r}_1 \notin span\{\nabla_{x_1} f_{11}(\overline{u}_1, \overline{v}_1), \cdots, \nabla_{x_1} f_{1k}(\overline{u}_1, \overline{v}_1)\} \backslash \{0\}$,

$\nabla_{x_2x_2}(\overline{\omega}^T g_2)(\overline{u}_2, \overline{v}_2)\overline{r}_2 \notin span\{\nabla_{x_2} f_{21}(\overline{u}_2, \overline{v}_2), \cdots, \nabla_{x_2} f_{2k}(\overline{u}_2, \overline{v}_2)\} \backslash \{0\}$.

where $f_1 = (f_{11}, f_{12}, \cdots, f_{1k})$ *and* $f_2 = (f_{21}, f_{22}, \cdots, f_{2k})$.

Then, $(\overline{u}_1, \overline{u}_2, \overline{v}_1, \overline{v}_2, \overline{\omega}, \overline{p}_1 = 0, \overline{p}_2 = 0)$ *is feasible for* $(MP)_{\overline{\lambda}}$*, and the objective function values of (MD) and* $(MP)_{\overline{\lambda}}$ *are equal. Also, if the hypotheses of the weak duality theorem are satisfied for all feasible solutions of* $(MP)_{\overline{\lambda}}$ *and* $(MD)_{\overline{\lambda}}$*, then* $(\overline{u}_1, \overline{u}_2, \overline{v}_1, \overline{v}_2, \overline{\omega}, \overline{p}_1 = 0, \overline{p}_2 = 0)$ *is an efficient solution for* $(MP)_{\overline{\lambda}}$.

Proof. Follows on the lines of Theorem 3.2.

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On the second kind Barnes-type multiple twisted zeta function and twisted Euler polynomials

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Abstract : In this paper we introduce the second kind Barnes-type multiple twisted Euler numbers and polynomials, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$.

Key words : the second kind twisted Euler numbers and polynomials, the second kind Barnes-type multiple twisted Euler numbers and polynomials

1 Introduction

Several mathematicians have studied the Euler numbers and polynomials and the Barnes-type multiple twisted Euler numbers and polynomials(see [1-9]). In this paper, we construct the second kind Barnes-type multiple twisted Euler polynomials, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim defined the fermionic p-adic invariant integral on $\mathbb{Z}_p$,

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 2, 3].} \tag{1.1}$$

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From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduced the second kind Euler numbers E_n. The second kind Euler numbers E_n are defined by the generating function:

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$\left(\frac{2e^t}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

In [4], we studied the second kind Euler numbers E_n and polynomials $E_n(x)$ and investigate their properties. The main aim of this paper is to study the second kind Barnes-type multiple twisted Euler polynomials, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$.

2 The second kind Barnes-type multiple twisted Euler polynomials

In this section, we use the notation

$$\sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1 \cdots k_n=0}^{m}.$$

We assume that $w_1, \ldots, w_k \in \mathbb{Z}_p$. Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$. We introduce the second kind Barnes-type multiple twisted Euler polynomials, $E_{n,\omega}(w_1, \ldots, w_k \mid x)$.

For $k \in \mathbb{N}$, we define the second kind Barnes-type multiple twisted Euler polynomials as follows:

$$\begin{aligned}
&F_\omega(w_1, \ldots, w_k \mid x, t) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k} e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \frac{2^k e^{kt}}{(\omega e^{2w_1t}+1)(\omega e^{2w_2t}+1)\cdots(\omega e^{2w_kt}+1)} e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,\omega}(w_1, \ldots, w_k \mid x) \frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

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In the special case, $x = 0$, $E_{n,\omega}(w_1, \ldots, w_k \mid 0) = E_{n,\omega}(w_1, \ldots, w_k)$ are called the second kind n-th Barnes-type multiple twisted Euler numbers.

Theorem 1. For positive integers n and k, we have

$$E_{n,\omega}(w_1, \ldots, w_k \mid x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k}(x + 2w_1x_1 + \cdots + 2w_kx_k + k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).$$

By using the above Theorem 1, we have the following corollary.

Corollary 2. For positive integers n, we have

$$E_{n,\omega}(w_1, \ldots, w_k) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{\sum_{i=1}^k x_i}(2w_1x_1 + \cdots + 2w_kx_k + k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.2}$$

By Theorem 1 and (2.2), we obtain

$$E_{n,\omega}(w_1, \ldots, w_k \mid x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,\omega}(w_1, \ldots, w_k), \tag{2.3}$$

where $\binom{n}{k}$ is a binomial coefficient.

In the special case, $(\underbrace{w_1, \ldots, w_k}_{k-\text{times}}) = (1, \ldots, 1)$, we have $E_n(w_1, \ldots, w_k \mid x) = E_n^{(k)}(x)$, where $E_n^{(k)}(x)$ denotes the second kind twisted Euler polynomials of higher order(see [5]).

We define distribution relation of the second kind Barnes-type multiple twisted Euler polynomials as follows: For $m \in \mathbb{N}$ with $m \equiv 1(\bmod 2)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,\omega}(w_1, \ldots, w_k \mid x) \frac{t^n}{n!} \\
&= \frac{2^k e^{kmt}}{(\omega^m e^{2w_1mt} + 1)(\omega^m e^{2w_2mt} + 1) \cdots (\omega^m e^{2w_kmt} + 1)} \\
&\quad \times \sum_{l_1, \ldots, l_k=0}^{m-1} (-1)^{l_1+\cdots+l_k} \omega^{\sum_{i=1}^k l_i} e^{\left(\frac{x + 2w_1l_1 + \cdots + 2w_kl_k + k - mk}{m}\right)(mt)}.
\end{aligned}$$

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From the above, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,\omega}(w_1,\ldots,w_k \mid x)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty} m^n \sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{l_1+\cdots+l_k}\\&\qquad\times E_{n,\omega^m}\left(w_1,\ldots,w_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 3 (Distribution relation). For $m \in \mathbb{N}$ with $m \equiv 1(\bmod 2)$, we have

$$\begin{aligned}&E_{n,\omega}(w_1,\ldots,w_k \mid x)\\&=m^n \sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{l_1+\cdots+l_k}\\&\qquad\times E_{n,\omega^m}\left(w_1,\ldots,w_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right).\end{aligned}$$

From (2.1), we derive

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}\omega^{x_1+\cdots+x_k}e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\&=2^k\sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{m_1+\cdots+m_k}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.\end{aligned} \tag{2.4}$$

From (2.1) and (2.4), we have the following theorem.

Theorem 4. For positive integers n and k, we have

$$\begin{aligned}&E_{n,\omega}(w_1,\ldots,w_k \mid x)\\&=2^k\sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{m_1+\cdots+m_k}(x+2w_1m_1+\cdots+2w_km_k+k)^n.\end{aligned} \tag{2.5}$$

From (2.2) and (2.5), we have the following corollary.

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Theorem 5. For positive integers n and k, we have

$$E_{n,\omega}(w_1,\ldots,w_k) = 2^k \sum_{m_1,\ldots m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k}\omega^{m_1+\cdots+m_k}(2w_1m_1+\cdots+2w_km_k+k)^n. \tag{2.6}$$

By using binomial expansion and (2.1), we have the following addition theorem.

Theorem 6(Addition theorem). The second kind Barnes-type multiple twisted Euler polynomials $E_{n,\omega}(w_1,\ldots,w_k \mid x)$ satisfies the following relation:

$$E_{n,\omega}(w_1,\ldots,w_k \mid x+y) = \sum_{l=0}^{n}\binom{n}{l}E_{l,\omega}(w_1,\ldots,w_k \mid x)y^{n-l}.$$

3 The second kind Barnes-type multiple twisted Euler zeta function

In this section, we assume that the parameters $w_1,\ldots,w_k$ are positive. Let ω be the p^N-th root of unity. By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the generating function of the second kind Barnes-type multiple twisted Euler polynomials, $E_{n,\omega}(w_1,\ldots,w_k \mid x)$, we define the second kind Barnes-type multiple twisted Euler zeta function. This function interpolates the second kind Barnes-type multiple twisted Euler polynomials at negative integers.

By (2.1), we obtain

$$\begin{aligned} F_\omega(w_1,\ldots,w_k \mid x,t) &= \frac{2^k e^{kt}}{(\omega e^{2w_1t}+1)\cdots(\omega e^{2w_kt}+1)}e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\omega}(w_1,\ldots,w_k \mid x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Hence, by (3.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,\omega}(w_1,\ldots,w_k \mid x)\frac{t^n}{n!} = 2^k \sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{m_1+\cdots+m_k}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.$$

By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the above equation, we have

$$E_{n,\omega}(w_1,\ldots,w_k \mid x) = 2^k \sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^{k} m_i}(x+2w_1m_1+\cdots+2w_km_k+k)^n. \tag{3.2}$$

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By (3.2), we define the second kind Barnes-type multiple twisted Euler zeta function $\zeta_\omega(w_1, \ldots, w_k \mid s, x)$ as follows:

Definition 7. For $s, x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$, we define

$$\zeta_\omega(w_1, \ldots, w_k \mid s, x) = 2^k \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}}{(x + 2w_1m_1 + \cdots + 2w_km_k + k)^s}, \quad (3.3)$$

For $s = -l$ in (3.3) and using (3.2), we arrive at the following theorem.

Theorem 8. For positive integer l, we have

$$\zeta_\omega(w_1, \ldots, w_k \mid -l, x) = E_{l,\omega}(w_1, \ldots, w_k \mid x).$$

By (2.6), we define the second kind multiple twisted Euler zeta function $\zeta_\omega(w_1, \ldots, w_k \mid s)$ as follows:

Definition 9. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, we define

$$\zeta_\omega(w_1, \ldots, w_k \mid s) = 2^k \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}}{(2w_1m_1 + \cdots + 2w_km_k + k)^s}. \quad (3.4)$$

For $s = -l$ in (3.4) and using (2.6), we arrive at the following theorem.

Theorem 10. For positive integer l, we have

$$\zeta_\omega(w_1, \ldots, w_k \mid -l) = E_{l,\omega}(w_1, \ldots, w_k).$$

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The Dynamics and the Solutions of some Rational Difference Equations

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Abstract

This paper is devoted to find the form of the solutions of the following rational difference equations:

$$x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(\pm 1 \pm x_n x_{n-4})}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solutions.

Keywords: local stability, global attractor, solution of difference equations.
Mathematics Subject Classification: 39A10

1 Introduction

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an

important role in mathematics as a whole. For some results in the direction of this study, see for example [1-32] and the papers therein.
Touafek [24] dealt with the behavior of the second order rational difference equation

$$x_{n+1} = \frac{ax_n^4+bx_nx_{n-1}^3+cx_n^2x_{n-1}^2+dx_n^3x_{n-1}+ex_{n-1}^4}{Ax_n^4+Bx_nx_{n-1}^3+Cx_n^2x_{n-1}^2+Dx_n^3x_{n-1}+Ex_{n-1}^4}.$$

In [29] Yalçınkaya dealt with the behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed [33] studied the dynamics of the nonlinear rational difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n+x_{n-k}}{q+x_{n-k}}.$$

Elsayed et al. [16] dealt with properties of the local stability, global attractor and boundedness of the solutions of the following difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_n+dx_{n-2}}.$$

Also, they gave the form of the solutions of some special cases from this equation.
Obaid et al. [23] investigated the global stability character, boundedness and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-1}+cx_{n-2}+dx_{n-3}}{\alpha x_{n-1}+\beta x_{n-2}+\gamma x_{n-3}}.$$

We obtain in this paper the form of the solutions of the following difference equations

$$x_{n+1} = \frac{x_nx_{n-4}}{x_{n-3}(\pm 1 \pm x_nx_{n-4})}, \quad n = 0, 1, ..., \tag{1}$$

where the initial values $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary real numbers. Also, we study the behavior of the solutions.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [21].
Theorem A [21]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1x_{n+k-1} + ... + p_kx_n = 0, \quad n = 0, 1, ...$$

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2 On the Recursive Sequence $x_{n+1}=\frac{x_n x_{n-4}}{x_{n-3}(1+x_n x_{n-4})}$

In this section we give a specific form of the solution of the equation in the form

$$x_{n+1}=\frac{x_n x_{n-4}}{x_{n-3}(1+x_n x_{n-4})},\quad n=0,1,..., \tag{3}$$

where the initial values are arbitrary positive real numbers.

Theorem 1 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(3). Then for $n=0,1,...$*

$$\begin{aligned}
x_{8n-4} &= x_{-4}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i)x_0x_{-4}}{1+(8i+4)x_0x_{-4}}\right),\quad x_{8n-3}=x_{-3}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+1)x_0x_{-4}}{1+(8i+5)x_0x_{-4}}\right),\\
x_{8n-2} &= x_{-2}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+2)x_0x_{-4}}{1+(8i+6)x_0x_{-4}}\right),\quad x_{8n-1}=x_{-1}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+3)x_0x_{-4}}{1+(8i+7)x_0x_{-4}}\right),\\
x_{8n} &= x_0\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+4)x_0x_{-4}}{1+(8i+8)x_0x_{-4}}\right),\ x_{8n+1}=\tfrac{x_0x_{-4}}{x_{-3}(1+x_0x_{-4})}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+5)x_0x_{-4}}{1+(8i+9)x_0x_{-4}}\right),\\
x_{8n+2} &= \tfrac{x_0x_{-4}}{x_{-2}(1+2x_0x_{-4})}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+6)x_0x_{-4}}{1+(8i+10)x_0x_{-4}}\right),\\
x_{8n+3} &= \tfrac{x_0x_{-4}}{x_{-1}(1+3x_0x_{-4})}\prod_{i=0}^{n-1}\left(\tfrac{1+(8i+7)x_0x_{-4}}{1+(8i+11)x_0x_{-4}}\right).
\end{aligned}$$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{8n-12} &= x_{-4}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i)x_0x_{-4}}{1+(8i+4)x_0x_{-4}}\right),\ x_{8n-11}=x_{-3}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+1)x_0x_{-4}}{1+(8i+5)x_0x_{-4}}\right),\\
x_{8n-10} &= x_{-2}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+2)x_0x_{-4}}{1+(8i+6)x_0x_{-4}}\right),\ x_{8n-9}=x_{-1}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+3)x_0x_{-4}}{1+(8i+7)x_0x_{-4}}\right),\\
x_{8n-8} &= x_0\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+4)x_0x_{-4}}{1+(8i+8)x_0x_{-4}}\right),\ x_{8n-7}=\tfrac{x_0x_{-4}}{x_{-3}(1+x_0x_{-4})}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+5)x_0x_{-4}}{1+(8i+9)x_0x_{-4}}\right),\\
x_{8n-6} &= \tfrac{x_0x_{-4}}{x_{-2}(1+2x_0x_{-4})}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+6)x_0x_{-4}}{1+(8i+10)x_0x_{-4}}\right),\\
x_{8n-5} &= \tfrac{x_0x_{-4}}{x_{-1}(1+3x_0x_{-4})}\prod_{i=0}^{n-2}\left(\tfrac{1+(8i+7)x_0x_{-4}}{1+(8i+11)x_0x_{-4}}\right).
\end{aligned}$$

Now, it follows from Eq.(3) that

$$x_{8n-4}=\frac{x_{8n-5}x_{8n-9}}{x_{8n-8}(1+x_{8n-5}x_{8n-9})}$$

$$=\frac{\left(\frac{x_0x_{-4}}{x_{-1}(1+3x_0x_{-4})}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+7)x_0x_{-4})}{(1+(8i+11)x_0x_{-4})}\right)x_{-1}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+3)x_0x_{-4})}{(1+(8i+7)x_0x_{-4})}\right)\right)}{\left(x_0\prod_{i=0}^{n-2}\left(\frac{(1+(8i+4)x_0x_{-4})}{(1+(8i+8)x_0x_{-4})}\right)\right)\left(1+\frac{x_0x_{-4}x_{-1}}{x_{-1}(1+3x_0x_{-4})}\prod_{i=0}^{n-2}\frac{1+(8i+7)x_0x_{-4}}{1+(8i+11)x_0x_{-4}}\frac{1+(8i+3)x_0x_{-4}}{1+(8i+7)x_0x_{-4}}\right)}$$

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$$= \frac{\frac{x_{-4}}{(1+3x_0x_{-4})}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+3)x_0x_{-4})}{(1+(8i+11)x_0x_{-4})}\right)}{\left(\prod_{i=0}^{n-2}\left(\frac{1+(8i+4)x_0x_{-4}}{1+(8i+8)x_0x_{-4}}\right)\right)\left(1+\frac{x_0x_{-4}}{(1+3x_0x_{-4})}\prod_{i=0}^{n-2}\left(\frac{1+(8i+3)x_0x_{-4}}{1+(8i+11)x_0x_{-4}}\right)\right)}$$

$$= \prod_{i=0}^{n-2}\left(\frac{(1+(8i+8)x_0x_{-4})}{(1+(8i+4)x_0x_{-4})}\right)\frac{x_{-4}\left(\frac{1}{(1+(8n-5)x_0x_{-4})}\right)}{\left(1+\frac{x_0x_{-4}}{(1+(8n-5)x_0x_{-4})}\right)}$$

$$= \prod_{i=0}^{n-2}\left(\frac{(1+(8i+8)x_0x_{-4})}{(1+(8i+4)x_0x_{-4})}\right)\frac{x_{-4}}{(1+(8n-4)x_0x_{-4})}.$$

Hence, we have

$$x_{8n-4} = x_{-4}\prod_{i=0}^{n-1}\left(\frac{(1+(8i)x_0x_{-4})}{(1+(8i+4)x_0x_{-4})}\right).$$

Also, we see from Eq.(1) that

$$x_{8n-3} = \frac{x_{8n-4}x_{8n-8}}{x_{8n-7}(1+x_{8n-4}x_{8n-8})}$$

$$= \frac{\left(x_{-4}\prod_{i=0}^{n-1}\left(\frac{(1+(8i)x_0x_{-4})}{(1+(8i+4)x_0x_{-4})}\right)\right)\left(x_0\prod_{i=0}^{n-2}\left(\frac{(1+(8i+4)x_0x_{-4})}{(1+(8i+8)x_0x_{-4})}\right)\right)}{\left(\frac{x_0x_{-4}}{x_{-3}(1+x_0x_{-4})}\prod_{i=0}^{n-2}\frac{1+(8i+5)x_0x_{-4}}{1+(8i+9)x_0x_{-4}}\right)\left(1+x_{-4}\prod_{i=0}^{n-1}\frac{1+(8i)x_0x_{-4}}{1+(8i+4)x_0x_{-4}}x_0\prod_{i=0}^{n-2}\frac{1+(8i+4)x_0x_{-4}}{1+(8i+8)x_0x_{-4}}\right)}$$

$$= \frac{\left(x_0x_{-4}\prod_{i=0}^{n-1}\frac{1}{(1+(8i+4)x_0x_{-4})}\right)\left(\prod_{i=0}^{n-2}(1+(8i+4)x_0x_{-4})\right)}{\left(\frac{x_0x_{-4}}{x_{-3}(1+x_0x_{-4})}\prod_{i=0}^{n-2}\frac{1+(8i+5)x_0x_{-4}}{1+(8i+9)x_0x_{-4}}\right)\left(1+x_0x_{-4}\prod_{i=0}^{n-1}\frac{1}{(1+(8i+4)x_0x_{-4})}\prod_{i=0}^{n-2}(1+(8i+4)x_0x_{-4})\right)}$$

$$= \frac{\left(\frac{1}{(1+(8n-4)x_0x_{-4})}\right)}{\left(\frac{1}{x_{-3}(1+x_0x_{-4})}\right)\left(1+\frac{x_0x_{-4}}{(1+(8n-4)x_0x_{-4})}\right)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+9)x_0x_{-4})}{(1+(8i+5)x_0x_{-4})}\right)$$

$$= \frac{x_{-3}(1+x_0x_{-4})}{(1+(8n-3)x_0x_{-4})}\prod_{i=0}^{n-2}\left(\frac{1+(8i+9)x_0x_{-4}}{1+(8i+5)x_0x_{-4}}\right) = x_{-3}\prod_{i=0}^{n-1}\left(\frac{1+(8i+1)x_0x_{-4}}{1+(8i+5)x_0x_{-4}}\right)$$

Similarly, we can prove the other relations. Thus, the proof is completed.

Theorem 2 *Eq.(3) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof: For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^2\right)}.$$

Then we have

$$\overline{x}^2\left(1+\overline{x}^2\right)=\overline{x}^2, \quad \Rightarrow \quad \overline{x}^2\left(1+\overline{x}^2-1\right)=0, \quad \Rightarrow \quad \overline{x}^4=0.$$

Thus the equilibrium point of Eq.(3) is $\overline{x}=0$.
Let $f:(0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{vw}{u(1+vw)}.$$

Therefore it follows that

$$f_u(u,v,w)=-\frac{vw}{u^2(1+vw)},\ f_v(u,v,w)=\frac{w}{u\left(1+vw\right)^2},\ f_w(u,v,w)=\frac{v}{u\left(1+vw\right)^2},$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x})=-1, \quad f_v(\overline{x},\overline{x},\overline{x})=1, \quad f_w(\overline{x},\overline{x},\overline{x})=1.$$

The proof follows by using Theorem A.
Example 1. We assume an interesting example for Eq.(1) where $x_{-4}=.4$, $x_{-3}=.2$, $x_{-2}=1.3$, $x_{-1}=7$, $x_0=.5$. See Fig. 1.

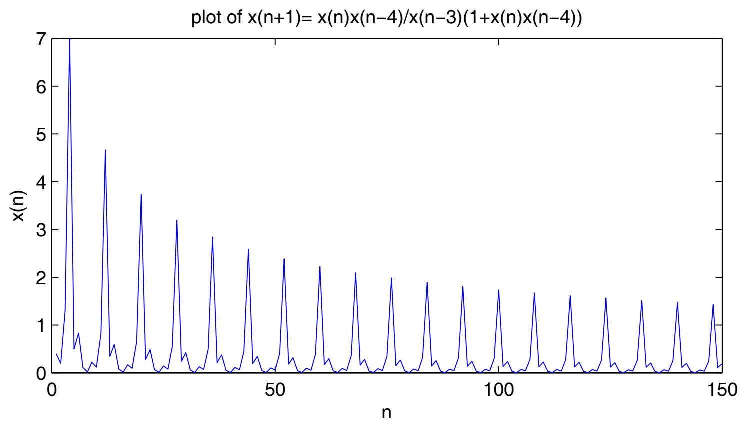

Figure 1.

3 On the Recursive Sequence $x_{n+1}=\frac{x_n x_{n-4}}{x_{n-3}(-1+x_n x_{n-4})}$

In this section we obtain the solution of the difference equation in the form

$$x_{n+1}=\frac{x_n x_{n-4}}{x_{n-3}(-1+x_n x_{n-4})}, \quad n=0,1,..., \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_0 x_{-4}\neq 1$.

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Theorem 3 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(4). Then the solution of Eq.(4) is bounded and periodic with period eight and given by the following formula for* $n = 0, 1, 2, ...$

$$\begin{aligned} x_{8n-4} &= x_{-4},\ \ x_{8n-3} = x_{-3},\ \ x_{8n-2} = x_{-2},\ \ x_{8n-1} = x_{-1},\ \ x_{8n} = x_0, \\ x_{8n+1} &= \frac{x_0 x_{-4}}{x_{-3}\left(-1 + x_0 x_{-4}\right)},\ \ x_{8n+2} = \frac{x_0 x_{-4}}{x_{-2}},\ \ x_{8n+3} = \frac{x_0 x_{-4}}{x_{-1}\left(-1 + x_0 x_{-4}\right)}. \end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{8n-12} &= x_{-4},\ \ x_{8n-11} = x_{-3},\ \ x_{8n-10} = x_{-2},\ \ x_{8n-9} = x_{-1},\ x_{8n-8} = x_0, \\ x_{8n-7} &= \frac{x_0 x_{-4}}{x_{-3}\left(-1 + x_0 x_{-4}\right)},\ \ x_{8n-6} = \frac{x_0 x_{-4}}{x_{-2}},\ \ x_{8n-5} = \frac{x_0 x_{-4}}{x_{-1}\left(-1 + x_0 x_{-4}\right)}. \end{aligned}$$

Now, it follows from Eq.(4) that

$$\begin{aligned} x_{8n-4} &= \frac{x_{8n-5}x_{8n-9}}{x_{8n-8}(-1 + x_{8n-5}x_{8n-9})} = \frac{\frac{x_0 x_{-4}}{x_{-1}(-1+x_0 x_{-4})}x_{-1}}{x_0(-1+\frac{x_0 x_{-4}}{x_{-1}(-1+x_0 x_{-4})}x_{-1})} = x_{-4}, \\ x_{8n-3} &= \frac{x_{8n-4}x_{8n-8}}{x_{8n-7}(-1 + x_{8n-4}x_{8n-8})} = \frac{x_{-4}x_0}{\left(\frac{x_0 x_{-4}}{x_{-3}(-1+x_0 x_{-4})}\right)(-1+x_{-4}x_0)} = x_{-3}, \\ x_{8n-2} &= \frac{x_{8n-3}x_{8n-7}}{x_{8n-6}(-1+x_{8n-3}x_{8n-7})} = \frac{x_{-3}\left(\frac{x_0 x_{-4}}{x_{-3}(-1+x_0 x_{-4})}\right)}{\left(\frac{x_0 x_{-4}}{x_{-2}}\right)\left(-1+\frac{x_{-3}x_0 x_{-4}}{x_{-3}(-1+x_0 x_{-4})}\right)} = x_{-2}. \end{aligned}$$

Similarly, we can obtain the other relations. Thus, the proof is completed.

Theorem 4 *Eq.(4) has three equilibrium points which are $0, \pm\sqrt{2}$.and these equilibrium points are not locally asymptotically stable.*

Proof: As the proof of Theorem 2 and will be omitted.
Example 2. Fig. 2. shows the solutions when $x_{-4} = 4,\ x_{-3} = -2,\ x_{-2} = 3,\ x_{-1} = .7,\ x_0 = -5.$

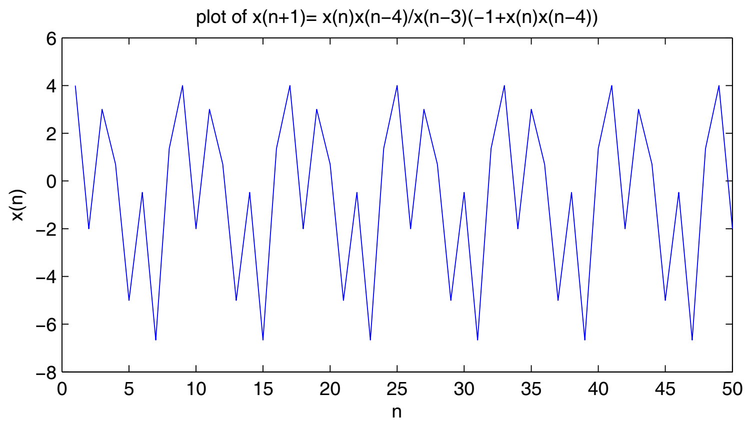

Figure 2.

The following cases can be proved similarly.

6

4 On the Recursive Sequence $x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(1 - x_n x_{n-4})}$

In this section we get the solution of the third following equation

$$x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(1 - x_n x_{n-4})}, \quad n = 0, 1, ..., \tag{5}$$

where the initial values are arbitrary positive real numbers.

Theorem 5 *The solution of Eq.(5). can be in the form for $n = 0, 1, ...$*

$$\begin{aligned}
x_{8n-4} &= x_{-4} \prod_{i=0}^{n-1} \left(\frac{1-(8i)x_0 x_{-4}}{1-(8i+4)x_0 x_{-4}} \right), \quad x_{8n-3} = x_{-3} \prod_{i=0}^{n-1} \left(\frac{1-(8i+1)x_0 x_{-4}}{1-(8i+5)x_0 x_{-4}} \right), \\
x_{8n-2} &= x_{-2} \prod_{i=0}^{n-1} \left(\frac{1-(8i+2)x_0 x_{-4}}{1-(8i+6)x_0 x_{-4}} \right), \quad x_{8n-1} = x_{-1} \prod_{i=0}^{n-1} \left(\frac{1-(8i+3)x_0 x_{-4}}{1-(8i+7)x_0 x_{-4}} \right), \\
x_{8n} &= x_0 \prod_{i=0}^{n-1} \left(\frac{1-(8i+4)x_0 x_{-4}}{1-(8i+8)x_0 x_{-4}} \right), \; x_{8n+1} = \frac{x_0 x_{-4}}{x_{-3}(1-x_0 x_{-4})} \prod_{i=0}^{n-1} \left(\frac{1-(8i+5)x_0 x_{-4}}{1-(8i+9)x_0 x_{-4}} \right), \\
x_{8n+2} &= \frac{x_0 x_{-4}}{x_{-2}(1-2x_0 x_{-4})} \prod_{i=0}^{n-1} \left(\frac{1-(8i+6)x_0 x_{-4}}{1-(8i+10)x_0 x_{-4}} \right), \\
x_{8n+3} &= \frac{x_0 x_{-4}}{x_{-1}(1-3x_0 x_{-4})} \prod_{i=0}^{n-1} \left(\frac{1-(8i+7)x_0 x_{-4}}{1-(8i+11)x_0 x_{-4}} \right).
\end{aligned}$$

Theorem 6 *Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Example 3. See Fig. 3. where we take the initials $x_{-4} = .9,\ x_{-3} = 2,\ x_{-2} = .1,\ x_{-1} = .2,\ x_0 = 6.$

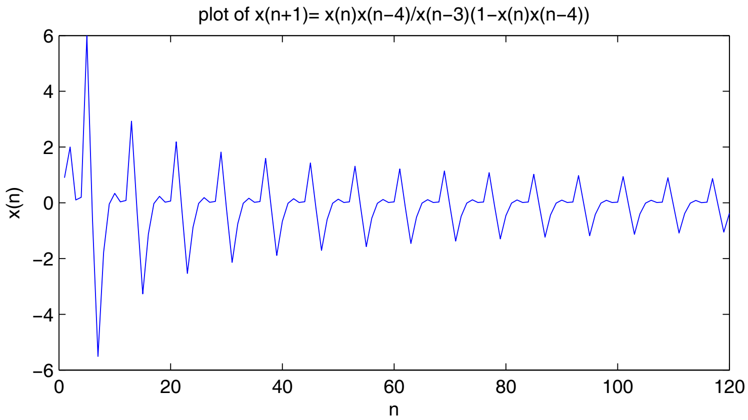

Figure 3.

7

5 On the Recursive Sequence $x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(-1-x_n x_{n-4})}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(-1 - x_n x_{n-4})}, \quad n = 0, 1, ..., \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_0 x_{-4} \neq -1$.

Theorem 7 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(6). Then for $n = 0, 1, 2, ...$ the solution of Eq.(6) is bounded and periodic with period eight and given by*

$$\begin{aligned} x_{8n-4} &= x_{-4}, \quad x_{8n-3} = x_{-3}, \quad x_{8n-2} = x_{-2}, \quad x_{8n-1} = x_{-1}, \quad x_{8n} = x_0, \\ x_{8n+1} &= \frac{x_0 x_{-4}}{x_{-3}(-1 + x_0 x_{-4})}, \quad x_{8n+2} = \frac{x_0 x_{-4}}{x_{-2}}, \quad x_{8n+3} = \frac{x_0 x_{-4}}{x_{-1}(-1 + x_0 x_{-4})}. \end{aligned}$$

Theorem 8 *Eq.(6) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Example 4. Fig. 3. shows the periodicity of the solutions when $x_{-4} = 4$, $x_{-3} = 2$, $x_{-2} = -1$, $x_{-1} = .2$, $x_0 = -6$.

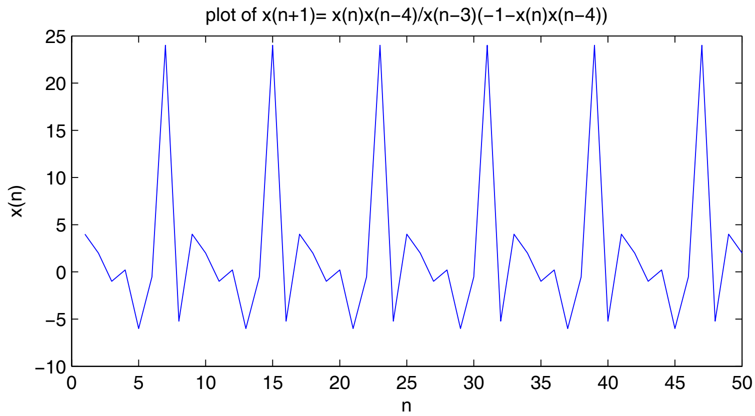

Figure 4.

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Approximation by complex Stancu type Durrmeyer operators in compact disks

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Abstract. In this paper we introduce a class of complex Stancu type Durrmeyer operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: Complex Stancu type Durrmeyer operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25 , 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [14]. Very recently, the problem of the approximation of complex operators has been causing great concern, which has become a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [3] Also, in [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer polynomials, complex genuine Durrmeyer Stancu polynomials and complex Bernstein-Durrmeyer operators based on Jacobi weights were obtained.

The aim of the present article is to obtain approximation results for complex Stancu type Durrmeyer operators which are defined for $f : [0, 1] \to \mathbf{C}$ continuous

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on $[0,1]$ by

$$D_n^{(\alpha,\beta)}(f;z) := (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_0^1 p_{n,k}(t)f(\frac{nt+\alpha}{n+\beta})dt, \tag{1}$$

where α, β are two given real parameters satisfying the condition $0 \le \alpha \le \beta$, $z \in \mathbf{C}, n = 1, 2, \ldots$, and $p_{n,k}(z) = \left(\begin{array}{c} n \\ k \end{array}\right) z^k(1-z)^{n-k}$.

Note that, for $\alpha = \beta = 0$, these operators become the complex Bernstein-Durrmeyer operators $D_n(f;z) = D_n^{(0,0)}(f;z)$, this case has been investigated in [2].

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, then we have that $D_n^{(\alpha,\beta)}(e_m;z)$ is a polynomial of degree less than or equal to min (m,n) and*

$$D_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array}\right) \frac{n^j \alpha^{m-j}}{(n+\beta)^m} D_n(e_j;z).$$

Proof By the definition given by (1) , the proof is easy, here the proof is omitted.

Let $m = 0, 1, 2$, by a simple computation, we have

$$\begin{aligned} D_n^{(\alpha,\beta)}(e_0;z) &= 1; \\ D_n^{(\alpha,\beta)}(e_1;z) &= \frac{n}{n+\beta}(z+\frac{1-2z}{n+2}) + \frac{\alpha}{n+\beta}; \\ D_n^{(\alpha,\beta)}(e_2;z) &= \frac{n^2}{(n+\beta)^2}\left[z^2 + 2\frac{1+2nz-3(n+1)z^2}{(n+2)(n+3)}\right] \\ &\quad + \frac{2n\alpha}{(n+\beta)^2}(z+\frac{1-2z}{n+2}) + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Lemma 2 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, for all $|z| \le r$, $r \ge 1$, we have $|D_n^{(\alpha,\beta)}(e_m;z)| \le r^m$.*

Proof The proof follows directly Lemma 1 and [2, Corollary 2.2].

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Lemma 3 *Let $e_m(z)=z^m$, $m,n\in \mathrm{N}$, $z\in \mathbf{C}$ and $0\le\alpha\le\beta$, then we have*

$$\begin{aligned}D_n^{(\alpha,\beta)}(e_{m+1};z)=&\frac{z(1-z)n}{(n+\beta)(m+n+2)}(D_n^{(\alpha,\beta)}(e_m;z))'\\&+\frac{(m+1+nz)n+\alpha(2m+2+n)}{(n+\beta)(m+n+2)}D_n^{(\alpha,\beta)}(e_m;z)\\&-\frac{\alpha m(n+\alpha)}{(n+\beta)^2(m+n+2)}D_n^{(\alpha,\beta)}(e_{m-1};z).\end{aligned}\tag{2}$$

Proof Let

$$\begin{aligned}T_{n,k}^{(\alpha,\beta)}(f)&:=\int_0^1 p_{n,k}(t)f(\frac{nt+\alpha}{n+\beta})dt,\\\widetilde{T}_{n,k}^{(\alpha,\beta)}(f)&:=\int_0^1 p_{n,k}(t)tf(\frac{nt+\alpha}{n+\beta})dt,\\\widehat{T}_{n,k}^{(\alpha,\beta)}(f)&:=\int_0^1 p_{n,k}(t)t^2f(\frac{nt+\alpha}{n+\beta})dt,\end{aligned}$$

then we have

$$D_n^{(\alpha,\beta)}(f;z)=(n+1)\sum_{k=0}^{n}p_{n,k}(z)T_{n,k}^{(\alpha,\beta)}(f),$$

$$\begin{aligned}\widetilde{T}_{n,k}^{(\alpha,\beta)}(e_m)&=\int_0^1 p_{n,k}(t)\frac{n+\beta}{n}(\frac{nt+\alpha}{n+\beta}-\frac{\alpha}{n+\beta})(\frac{nt+\alpha}{n+\beta})^m dt\\&=\frac{n+\beta}{n}T_{n,k}^{(\alpha,\beta)}(e_{m+1})-\frac{\alpha}{n}T_{n,k}^{(\alpha,\beta)}(e_m),\end{aligned}$$

$$\begin{aligned}\widehat{T}_{n,k}^{(\alpha,\beta)}(e_m)&=\int_0^1 p_{n,k}(t)(\frac{n+\beta}{n})^2(\frac{nt+\alpha}{n+\beta}-\frac{\alpha}{n+\beta})^2(\frac{nt+\alpha}{n+\beta})^m dt\\&=(\frac{n+\beta}{n})^2T_{n,k}^{(\alpha,\beta)}(e_{m+2})-\frac{2\alpha(n+\beta)}{n^2}T_{n,k}^{(\alpha,\beta)}(e_{m+1})+(\frac{\alpha}{n})^2T_{n,k}^{(\alpha,\beta)}(e_m).\end{aligned}$$

By a simple calculation, we obtain

$$z(1-z)p'_{n,k}(z)=(k-nz)p_{n,k}(z),\ (k-nt)p_{n,k}(t)=t(1-t)p'_{n,k}(t),$$

it follows that

$$\begin{aligned}&z(1-z)(D_n^{(\alpha,\beta)}(e_m;z))'\\&=(n+1)\sum_{k=0}^{n}(k-nz)\,p_{n,k}(z)\int_0^1 p_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt\\&=(n+1)\sum_{k=0}^{n}p_{n,k}(z)\int_0^1(k-nt+nt)p_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt-nzD_n^{(\alpha,\beta)}(e_m;z).\end{aligned}$$

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Since

$$\begin{aligned}
&(n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_0^1 (k-nt+nt)p_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt \\
&= (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_0^1 t(1-t)p'_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt \\
&\quad + n(n+1)\sum_{k=0}^{n} p_{n,k}(z)\widetilde{T}_{n,k}^{(\alpha,\beta)}(e_m) \\
&= (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_0^1 t(1-t)p'_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt + (n+\beta)D_n^{(\alpha,\beta)}(e_{m+1};z) \\
&\quad - \alpha D_n^{(\alpha,\beta)}(e_m;z),
\end{aligned}$$

also

$$\begin{aligned}
&\int_0^1 t(1-t)p'_{n,k}(t)(\frac{nt+\alpha}{n+\beta})^m dt \\
&= -\int_0^1 p_{n,k}(t)(1-2t)(\frac{nt+\alpha}{n+\beta})^m dt - \frac{mn}{n+\beta}\int_0^1 p_{n,k}(t)t(1-t)(\frac{nt+\alpha}{n+\beta})^{m-1} dt \\
&= -T_{n,k}^{(\alpha,\beta)}(e_m) + 2\widetilde{T}_{n,k}^{(\alpha,\beta)}(e_m) - \frac{mn}{n+\beta}\widetilde{T}_{n,k}^{(\alpha,\beta)}(e_{m-1}) + \frac{mn}{n+\beta}\widehat{T}_{n,k}^{(\alpha,\beta)}(e_{m-1}) \\
&= \frac{n+\beta}{n}(m+2)T_{n,k}^{(\alpha,\beta)}(e_{m+1}) - (1+\frac{2\alpha}{n}+m+\frac{2\alpha m}{n})T_{n,k}^{(\alpha,\beta)}(e_m) \\
&\quad + \frac{\alpha m(\alpha+n)}{n(n+\beta)}T_{n,k}^{(\alpha,\beta)}(e_{m-1}).
\end{aligned}$$

So, in conclusion, we have

$$\begin{aligned}
z(1-z)(D_n^{(\alpha,\beta)}(e_m;z)) &= \frac{n+\beta}{n}(m+n+2)D_n^{(\alpha,\beta)}(e_{m+1};z) \\
&\quad - (1+\frac{2\alpha}{n}+m+\frac{2\alpha m}{n}+\alpha+nz)D_n^{(\alpha,\beta)}(e_m;z) \\
&\quad + \frac{\alpha m(\alpha+n)}{n(n+\beta)}D_n^{(\alpha,\beta)}(e_{m-1};z),
\end{aligned}$$

which implies the recurrence in the statement.

Lemma 4 *Let* $n \in \mathbf{N}$, $m = 2,3,\cdots$, $e_m(z) = z^m$, $S_{n,m}^{(\alpha,\beta)}(z) := D_n^{(\alpha,\beta)}(e_m;z) -$

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z^m , $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, we have

$$\begin{aligned} S_{n,m}^{(\alpha,\beta)}(z) = & \frac{z(1-z)n}{(n+\beta)(m+n+1)}(D_n^{(\alpha,\beta)}(e_{m-1};z))' \\ & + \frac{(m+nz)n+\alpha(m+n)}{(n+\beta)(m+n+1)}S_{n,m-1}^{(\alpha,\beta)}(z) \\ & + \frac{\alpha m}{(n+\beta)(m+n+1)}D_n^{(\alpha,\beta)}(e_{m-1};z) \\ & - \frac{\alpha(m-1)(n+\alpha)}{(n+\beta)^2(m+n+1)}D_n^{(\alpha,\beta)}(e_{m-2};z) \\ & + \frac{(m+nz)n+\alpha(m+n)}{(n+\beta)(m+n+1)}z^{m-1} - z^m. \end{aligned} \tag{3}$$

Proof Using the recurrence formula (2), by a simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1 *Let $0 \le \alpha \le \beta$, $1 \le r \le R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.*

(i) For *all $|z| \le r$ and $n \in \mathbf{N}$, we have*

$$|D_n^{(\alpha,\beta)}(f;z) - f(z)| \le \frac{K_r^{(\alpha,\beta)}(f)}{n+\beta},$$

where $K_r^{(\alpha,\beta)}(f) = (1+r)\sum_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1} < \infty$.

(ii) (Simultaneous approximation) *If $1 \le r < r_1 < R$ are arbitrary fired, then for all $|z| \le r$ and $n, p \in \mathbf{N}$ we have*

$$|(D_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z)| \le \frac{K_{r_1}^{(\alpha,\beta)}(f)p!r_1}{(n+\beta)(r_1-r)^{p+1}},$$

where $K_{r_1}^{(\alpha,\beta)}(f)$ is defined as in the above point (i).

Proof Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$D_n^{(\alpha,\beta)}(f;z) = \sum_{m=0}^{\infty} c_m D_n^{(\alpha,\beta)}(e_m;z).$$

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Therefore, we get

$$\begin{aligned}|D_n^{(\alpha,\beta)}(f;z)-f(z)| &\leq \sum_{m=0}^{\infty}|c_m|\cdot|D_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|\\ &=\sum_{m=1}^{\infty}|c_m|\cdot|D_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|,\end{aligned}$$

as $D_n^{(\alpha,\beta)}(e_0;z)=e_0(z)=1$.

(i) For $m\in\mathbf{N}$, taking into account that $D_n^{(\alpha,\beta)}(e_{m-1};z)$ is a polynomial degree $\leq min(m-1,n)$, by the well-known Bernstein inequality and Lemma 2 we get

$$|(D_n^{(\alpha,\beta)}(e_{m-1};z))'|\leq\frac{m-1}{r}max\{|D_n^{(\alpha,\beta)}(e_{m-1};z)|:|z|\leq r\}\leq(m-1)r^{m-2}.$$

On the one hand, when $m=1$, for $|z|\leq r$, by Lemma 1 we have

$$|D_n^{(\alpha,\beta)}(e_1;z)-e_1(z)|=|\frac{n}{n+\beta}(z+\frac{1-2z}{n+2})+\frac{\alpha}{n+\beta}-z|\leq\frac{1+r}{n+\beta}(2+\alpha+\beta).$$

When $m\geq 2$, for $n\in\mathbf{N}$, $|z|\leq r$, $0\leq\alpha\leq\beta$, in view of $|(m+nz)n+\alpha(m+n)|\leq(n+\beta)(m+n+1)r$, using the recurrence formula (3) and the above inequality, we have

$$\begin{aligned}|D_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &= |S_{n,m}^{(\alpha,\beta)}(z)|\\ &\leq\frac{r(1+r)}{n+\beta}\cdot(m-1)r^{m-2}+r|S_{n,m-1}^{(\alpha,\beta)}(z)|\\ &\quad+\frac{\alpha}{n+\beta}r^{m-1}+\frac{\alpha}{n+\beta}r^{m-2}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1}\\ &\leq\frac{m-1}{n+\beta}(1+r)r^{m-1}+r|S_{n,m-1}^{(\alpha,\beta)}(z)|\\ &\quad+\frac{\alpha}{n+\beta}(1+r)r^{m-1}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1}\\ &=r|S_{n,m-1}^{(\alpha,\beta)}(z)|+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}.\end{aligned}$$

By writing the last inequality, for $m=2,\cdots$, we easily obtain step by step the following

$$\begin{aligned}|D_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &\leq r\left(r|S_{n,m-2}^{(\alpha,\beta)}(z)|+\frac{2(m-1)+\alpha+\beta}{n+\beta}(1+r)r^{m-2}\right)\\ &\quad+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}\\ &=r^2|S_{n,m-2}^{(\alpha,\beta)}(z)|+\frac{2(m-1+m)+2(\alpha+\beta)}{n+\beta}(1+r)r^{m-1}\\ &\leq\ldots\leq\frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1}.\end{aligned}$$

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In conclusion, for any $m, n \in \mathbf{N}$, $|z| \le r$, $0 \le \alpha \le \beta$, we have

$$|D_n^{(\alpha,\beta)}(e_m; z) - e_m(z)| \le \frac{1+r}{n+\beta} m(m+1+\alpha+\beta) r^{m-1},$$

from which it follows that

$$|D_n^{(\alpha,\beta)}(f; z) - f(z)| \le \frac{1+r}{n+\beta} \sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z) = \sum\limits_{m=2}^{\infty} c_m m(m-1) z^{m-2}$ and the series is absolutely convergent in $|z| \le r$, so we get $\sum\limits_{m=2}^{\infty} |c_m| m(m-1) r^{m-2} < \infty$, which implies $K_r^{(\alpha,\beta)}(f) = (1+r) \sum\limits_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1} < \infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} |(D_n^{(\alpha,\beta)}(f; z))^{(p)} - f^{(p)}(z)| &= \frac{p!}{2\pi} \left| \int_\Gamma \frac{D_n^{(\alpha,\beta)}(f; v) - f(v)}{(v-z)^{p+1}} dv \right| \\ &\le \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}}, \end{aligned}$$

which proves the theorem.

Remark 1 Taking now $\alpha = \beta = 0$ in Theorem 1 (i), we get the estimates of [2, Corollary 2.2 (ii)].

Theorem 2 *Let $0 \le \alpha \le \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For any fixed $r \in [1, R]$ and all $n \in \mathbf{N}$, $|z| \le r$, we have*

$$\begin{aligned} &\left| D_n^{(\alpha,\beta)}(f; z) - f(z) - \frac{(1+\alpha) - (2+\beta)z}{n} f'(z) - \frac{z(1-z)}{n} f''(z) \right| \\ &\qquad \le \frac{M_r(f)}{n^2} + \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}, \end{aligned} \tag{4}$$

where $M_r(f) = \sum\limits_{k=1}^{\infty} |c_k| k B_{k,r} r^{k-1} < \infty$ with $B_{k,r} = 2(k-1)^3 + r(4k^3 + 6k^2 + 6k + 2) + 2r^2 k(k^2+1) + 4k(k-1)^2(1+r)$, $M_{r,1}^{(\alpha,\beta)}(f) = \sum\limits_{k=1}^{\infty} |c_k|[2k^2(k-1)\alpha + 2k^2(k+$

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$1)\beta r+k^2\alpha\beta+k^2\beta^2 r]r^{k-1}<\infty,\ M_{r,2}^{(\alpha,\beta)}(f)=\sum\limits_{k=1}^{\infty}|c_k|\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2}r^{k-2}<\infty.$

Proof For all $z\in D_R$, we have

$$
\begin{aligned}
&D_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{(1+\alpha)-(2+\beta)z}{n}f'(z)-\frac{z(1-z)}{n}f''(z)\\
&=D_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{1-2z}{n}f'(z)-\frac{z(1-z)}{n}f''(z)-\frac{\alpha-\beta z}{n}f'(z)\\
&=\left[D_n(f;z)-f(z)-\frac{[z(1-z)f'(z)]'}{n}\right]+\left[D_n^{(\alpha,\beta)}(f;z)-D_n(f;z)-\frac{\alpha-\beta z}{n}f'(z)\right]\\
&:=I_1+I_2.
\end{aligned}
$$

By [2, Theorem 2.4], we have $|I_1|\le\frac{M_r(f)}{n^2}$, where $M_r(f)=\sum\limits_{k=1}^{\infty}|c_k|kB_{k,r}r^{k-1}<\infty$ with $B_{k,r}=2(k-1)^3+r(4k^3+6k^2+6k+2)+2r^2k(k^2+1)+4k(k-1)^2(1+r)$.

Next, let us estimate $|I_2|$.

By f is analytic in D_R, i.e. $f(z)=\sum\limits_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$, we have

$$
\begin{aligned}
|I_2|&=\left|\sum_{k=1}^{\infty}c_k(D_n^{(\alpha,\beta)}(e_k;z)-D_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1})\right|\\
&\le\sum_{k=1}^{\infty}|c_k|\left|D_n^{(\alpha,\beta)}(e_k;z)-D_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\right|.
\end{aligned}
$$

Since $\frac{n^k}{(n+\beta)^k}-1=-\sum\limits_{j=0}^{k-1}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}$, by Lemma 1, we obtain

$$
\begin{aligned}
&D_n^{(\alpha,\beta)}(e_k;z)-D_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}D_n(e_j;z)+\left[\frac{n^k}{(n+\beta)^k}-1\right]D_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}D_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}D_n(e_{k-1};z)\\
&\quad-\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}D_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}D_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[D_n(e_{k-1};z)-e_{k-1}(z)\right]\\
&\quad+\frac{kn^{k-1}\alpha}{(n+\beta)^k}z^{k-1}-\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}D_n(e_k;z)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{kn^{k-1}\beta}{(n+\beta)^k}\left[D_n(e_k;z)-e_k(z)\right]-\frac{kn^{k-1}\beta}{(n+\beta)^k}z^k-\frac{\alpha-\beta z}{n}kz^{k-1}.\\
=&\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}D_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[D_n(e_{k-1};z)-e_{k-1}(z)\right]\\
&-\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}D_n(e_k;z)-\frac{kn^{k-1}\beta}{(n+\beta)^k}\left[D_n(e_k;z)-e_k(z)\right]\\
&-\left[\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}\right]k\alpha z^{k-1}+\left[\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}\right]k\beta z^k.
\end{aligned}
$$

By [2, Corollary 2.2], for any $k\in\mathbf{N}$, $|z|\le r$, $r\ge 1$, we have

$$|D_n(e_k;z)|\le r^k,\ \ |D_n(e_k;z)-e_k|\le\frac{2k(k+1)}{n}r^k.$$

Hence, we can get

$$
\begin{aligned}
&\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}D_n(e_j;z)\right|\\
&\le\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}r^{k-2}\\
&=\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}\cdot\frac{\alpha^2}{(n+\beta)^2}r^{k-2}\\
&\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}r^{k-2}\\
&\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}r^{k-2}
\end{aligned}
$$

and

$$\left|\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[D_n(e_{k-1};z)-e_{k-1}(z)\right]\right|\le\frac{2k^2(k-1)\alpha}{n(n+\beta)}r^{k-1}.$$

Also, using

$$\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}=\frac{\sum_{j=0}^{k-1}\binom{k}{j}n^j\beta^{k-j}}{n(n+\beta)^k}\le\frac{k\beta}{n(n+\beta)},$$

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we get

$$\begin{aligned}
&|D_n^{(\alpha,\beta)}(e_k;z) - D_n(e_k;z) - \frac{\alpha-\beta z}{n} k z^{k-1}| \\
&\le \frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2} r^{k-2} + \frac{2k^2(k-1)\alpha}{n(n+\beta)} r^{k-1} + \frac{k(k-1)}{2}\cdot\frac{\beta^2}{(n+\beta)^2} r^k \\
&\quad + \frac{2k^2(k+1)\beta}{n(n+\beta)} r^k + \frac{k^2\alpha\beta}{n(n+\beta)} r^{k-1} + \frac{k^2\beta^2}{n(n+\beta)} r^k \\
&= \frac{r^{k-1}}{n(n+\beta)} \left[2k^2(k-1)\alpha + 2k^2(k+1)\beta r + k^2\alpha\beta + k^2\beta^2 r\right] \\
&\quad + \frac{r^{k-2}}{(n+\beta)^2}\cdot\frac{k(k-1)(\alpha^2+\beta^2 r^2)}{2}.
\end{aligned}$$

So, we have

$$|I_2| \le \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2},$$

where $M_{r,1}^{(\alpha,\beta)}(f) = \sum_{k=1}^{\infty} |c_k|[2k^2(k-1)\alpha + 2k^2(k+1)\beta r + k^2\alpha\beta + k^2\beta^2 r] r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f) = \sum_{k=1}^{\infty} |c_k| \frac{k(k-1)(\alpha^2+\beta^2r^2)}{2} r^{k-2}$.

In conclusion, we obtain

$$\left| D_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{(1+\alpha)-(2+\beta)z}{n} f'(z) - \frac{z(1-z)}{n} f''(z) \right|$$

$$\le |I_1| + |I_2| \le \frac{M_r(f)}{n^2} + \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}.$$

In the following theorem, we obtain the exact order of approximation.

Theorem 3 *Let $0 \le \alpha \le \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|D_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \ge \frac{C_r^{(\alpha,\beta)}(f)}{n},\ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \le r\}$ and the constant $C_r^{(\alpha,\beta)}(f) > 0$ depends on f, r and α, β, but it is independent of n.

Proof Denote $e_1(z) = z$ and

$$H_n^{(\alpha,\beta)}(f;z) = D_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{(1+\alpha)-(2+\beta)z}{n} f'(z) - \frac{z(1-z)}{n} f''(z).$$

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For all $z \in D_R$ and $n \in \mathbf{N}$, we have

$$D_n^{(\alpha,\beta)}(f;z) - f(z)$$
$$= \frac{1}{n}\left\{[(1+\alpha)-(2+\beta)z]f'(z) + z(1-z)f''(z) + \frac{1}{n}\left[n^2 H_n^{(\alpha,\beta)}(f;z)\right]\right\}.$$

In view of the property $\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$, it follows

$$\|D_n^{(\alpha,\beta)}(f;\cdot) - f\|_r$$
$$\geq \frac{1}{n}\left\{\|[(1+\alpha)-(2+\beta)e_1]f' + e_1(1-e_1)f''\|_r - \frac{1}{n}\left[n^2||H_n^{(\alpha,\beta)}(f;\cdot)||_r\right]\right\}.$$

Considering the hypothesis that f is not a polynomial of degree 0 in D_R, we have

$$\|[(1+\alpha)-(2+\beta)e_1]f' + e_1(1-e_1)f''\|_r > 0.$$

Indeed, supposing the contrary, it follows that

$$[(1+\alpha)-(2+\beta)z]f'(z) + z(1-z)f''(z) = 0, \text{ for all } z \in \overline{D_r}.$$

Denoting $y(z) = f'(z)$ and looking for the analytic function $y(z)$ under the form $y(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, after replacement in the differential equation, the coefficients identification method immediately leads to $a_k = 0$, for all $k \in \mathbb{N}\bigcup\{0\}$. This implies that $y(z) = 0$ for all $z \in \overline{D_r}$ and therefore f is constant on $\overline{D_r}$, a contradiction with the hypothesis.

Using inequality (4), we get

$$n^2\|H_n^{(\alpha,\beta)}(f;\cdot)\|_r \leq N_r^{(\alpha,\beta)}(f), \tag{5}$$

where $N_r^{(\alpha,\beta)}(f) = M_r(f) + M_{r,1}^{(\alpha,\beta)}(f) + M_{r,2}^{(\alpha,\beta)}(f)$.

Therefore, there exists an index n_0 depending only on f, r and α, β, such that for all $n \geq n_0$, we have

$$\|[(1+\alpha)-(2+\beta)e_1]f' + e_1(1-e_1)f''\|_r - \frac{1}{n}\left[n^2||H_n^{(\alpha,\beta)}(f;\cdot)||_r\right]$$
$$\geq \frac{1}{2}\left\|[(1+\alpha)-(2+\beta)e_1]f' + e_1(1-e_1)f''\right\|_r,$$

which implies

$$\|D_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{1}{2n}\left\|[(1+\alpha)-(2+\beta)e_1]f' + e_1(1-e_1)f''\right\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1, 2, \cdots, n_0 - 1\}$, we have

$$\|D_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{W_{r,n}^{(\alpha,\beta)}(f)}{n},$$

where $W_{r,n}^{(\alpha,\beta)}(f) = n\|D_n^{(\alpha,\beta)}(f;\cdot) - f\|_r > 0$.

As a conclusion, we have

$$\|D_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n}, \text{ for all } n\in\mathbf{N},$$

where

$$\begin{aligned}C_r^{(\alpha,\beta)}(f) =& min\left\{W_{r,1}^{(\alpha,\beta)}(f), W_{r,2}^{(\alpha,\beta)}(f),\ldots,W_{r,n_0-1}^{(\alpha,\beta)}(f),\right.\\ &\left.\frac{1}{2}\|[(1+\alpha)-(2+\beta)e_1]f'+e_1(1-e_1)f''\|_r\right\},\end{aligned}$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary *Let $0\leq\alpha\leq\beta$, $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r\in[1,R)$ we have*

$$\|D_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \asymp \frac{1}{n}, \quad n\in\mathbf{N},$$

where $\|f\|_r=max\{|f(z)|;|z|\leq r\}$ and the constants in the equivalence depend on f, r and α, β, but they are independent of n.

Theorem 4 *Let $0\leq\alpha\leq\beta$, $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$ is analytic in D_R. Also, let $1\leq r<r_1<R$ and $p\in\mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$, then we have*

$$\|(D_n^{(\alpha,\beta)}(f;\cdot))^{(p)}-f^{(p)}\|_r \asymp \frac{1}{n},\ n\in\mathbf{N},$$

where $\|f\|_r=max\{|f(z)|;|z|\leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, α and β, but they are independent of n.

Proof Taking into account the upper estimate in Theorem 1 , it remains to prove the lower estimate only. Denoting by Γ the circle of radius $r_1>r$ and center 0 , by Cauchy's formula, it follows that for all $|z|\leq r$ and $n\in\mathbf{N}$, we have

$$(D_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\Gamma \frac{D_n^{(\alpha,\beta)}(f;v)-f(v)}{(v-z)^{p+1}}dv.$$

Keeping the notation there for $H_n^{(\alpha,\beta)}(f;z)$, for all $n\in\mathbf{N}$ we have

$$\begin{aligned}&D_n^{(\alpha,\beta)}(f;z)-f(z)\\ &=\frac{1}{n}\left\{[(1+\alpha)-(2+\beta)z]f'(z)+z(1-z)f''(z)+\frac{1}{n}\left[n^2H_n^{(\alpha,\beta)}(f;z)\right]\right\}.\end{aligned}$$

By using Cauchy's formula, for all $v\in\Gamma$, we get

$$(D_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)=\frac{1}{n}\left\{[((1+\alpha)-(2+\beta)z)f'(z)+z(1-z)f''(z)]^{(p)}\right.$$

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$$+\frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2H_n^{(\alpha,\beta)}(f;v)}{(v-z)^{p+1}}dv\Bigg\},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z)=z$, it follows

$$\left\|(D_n^{(\alpha,\beta)}(f;\cdot))^{(p)}-f^{(p)}\right\|_r \geq \frac{1}{n}\left[\left\|[((1+\alpha)-(2+\beta)e_1)f'+e_1(1-e_1)f'']^{(p)}\right\|_r\right.$$
$$\left.-\frac{1}{n}\left\|\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r\right].$$

Since for any $|z|\leq r$ and $v\in\Gamma$ we have $|v-z|\geq r_1-r$, so, by inequality (5), we get

$$\left\|\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2\|H_n^{(\alpha,\beta)}(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}}$$
$$\leq \frac{N_{r_1}^{(\alpha,\beta)}(f)p!r_1}{(r_1-r)^{p+1}},$$

where $N_{r_1}^{(\alpha,\beta)}(f)=M_{r_1}(f)+M_{r_1,1}^{(\alpha,\beta)}(f)+M_{r_1,2}^{(\alpha,\beta)}(f)$.

Taking into account the function f is analytic in D_R, by following exactly the lines in Gal [5], seeing also the book Gal [13, pp. 77-78], (where it is proved that $\left\|[(\alpha-\beta e_1)f'+\frac{e_1(1-e_1)}{2}f'']^{(p)}\right\|_r>0$), we have

$$\left\|[((1+\alpha)-(2+\beta)e_1)f'+e_1(1-e_1)f'']^{(p)}\right\|_r>0.$$

In continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

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Convergence Theorems of Iterative Sequences for Generalized Equilibrium Problems Involving Strictly Pseudocontractive Mappings in Hilbert Spaces

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Abstract. In this paper, we consider the problem of finding a common element of the solution set of generalized equilibrium problems, of the solution set of variational inequalities and of the fixed point set of strictly pseudocontractive mappings by the shrinking projection method. Strong convergence theorems of common elements are established in real Hilbert spaces.

Keywords: equilibrium problem; variational inequality; strictly pseudocontractive mapping; nonexpansive mapping; inverse-strongly monotone mapping.

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1 Introduction and Preliminaries

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and C is a nonempty closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping. We denote $F(S)$ the set of fixed points of S.

Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be κ-strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Note that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings. It is also said to be pseudocontractive if $\kappa = 1$, that is,

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

S is said to be strongly pseudocontractive if there exists a positive constant $\lambda \in (0, 1)$ such that $S - \lambda I$ is pseudocontractive. Clearly, the class of strict pseudocontractions

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falls into the one between classes of nonexpansive mappings and pseudocontractions. We remark that the class of strongly pseudocontractive mappings is independent of the class of strict pseudocontractions (see [1-3]).

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-strongly monotone. A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone. A is said to be Lipschitz if there exits a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be L-Lipschitz.

A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply

$$\langle x - y, f - g\rangle > 0.$$

A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let F be a bifunction of $C \times C$ into R, where R denotes the set of real numbers and $A : C \to H$ an inverse-strongly monotone mapping.

We consider the following generalized equilibrium problem: Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $EP(F, A)$, *i.e.,*

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C\}.$$

To study generalized equilibrium problem (1.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Now, we give two special cases of problem (1.1).

(I) If $A \equiv 0$, then generalized equilibrium problem (1.1) is reduced to the following equilibrium problem: Find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by $EP(F)$, *i.e.,*

$$EP(F) = \{x \in C : F(x,y) \geq 0, \quad \forall y \in C\}.$$

(II) If $F \equiv 0$, then problem (1.1) is reduced to the following classical variational inequality: Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

We denote $VI(C,A)$ the solution set of (1.3). It is known that $x \in C$ is a solution to (1.3) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant, I is the identity mapping, and P_C is the metric projection from H onto C.

Recently, many authors studied the problems (1.1), (1.2) and (1.3) by iterative methods; see, for examples, [5-10,16,18-21,23,26].

In 2007, Tada and Takahashi [24] considered problem (1.2) and proved the following theorem.

Theorem TT *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4) and let S be a nonexpansive mapping of C into H such that*

$$\mathcal{F} := EP(F) \cap F(S) \neq \emptyset.$$

Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x - x_n\rangle\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \ \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \subset [a,1]$ for some $a \in (0,1)$ and $\{r_n\} \subset [0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x$.

Recently, Lin and Takahashi [16] and Kim, Cho and Qin ([13],[15]) further improved Theorem TT by considering generalized equilibrium problem (1.1). And, Cho, Qin and Kang [8] considered the generalized problem and a strictly pseudocontractive mapping.

Theorem CQK. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and B a β-inverse-strongly monotone mapping of C into H. Let $S : C \to C$ be a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume that*

$$\mathcal{F} := EP(F,A) \cap VI(C,B) \cap F(S) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_1 \in C = C_1, \\ F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) S_k z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\alpha_n \subset [0,1]$, $\{\lambda_n\} \subset (0, 2\beta)$ *and* $\{r_n\} \subset (0, 2\alpha)$ *satisfy the following conditions:* $0 \leq \alpha_n \leq a < 1$, $0 < b \leq \lambda_n \leq c < 2\beta$, $0 < d \leq r_n \leq e < 2\alpha$ *for some* $a, b, c, d, e \in R$. *Then sequence* $\{x_n\}$ *defined by algorithm* (1.5) *converges strongly to a point* $\bar{x} = P_{\mathcal{F}} x_1$.

In this paper, we consider generalized equilibrium problem (1.1) and a strictly pseudocontractive mapping based on the shrinking projection algorithm which was first introduced by Takahashi, Takeuchi and Kubota [25]. A strong convergence theorem of common elements of the fixed point set of strictly pseudocontractive mappings, of the solution set of the variational inequality (1.3) and of the solution set of the generalized equilibrium problem (1.1) is established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Cho, Qin and Kang [8], Kumam [9], Kim, Anh and Nam [12], Kim and Buong ([11],[14]), Lin and Takahshi [16], and Tada and Takahashi [24].

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([17]). *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ a κ-strict pseudocontraction. Then S is $\frac{1+\kappa}{1-\kappa}$-Lipschitz and $I - S$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 1.2. ([4],[5]) *Let C be a nonempty closed convex subset of a Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 1.3 ([27]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ a k-strict pseudocontraction with a fixed point. Define $S_a : C \to C$ by $S_a x = ax + (1 - a)Sx$ for each $x \in C$. If $a \in [k, 1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.*

Lemma 1.4. ([22]). *Let A be a monotone hemicontinuous mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$$

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and define a mapping M on C by

$$Mv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then M is maximal monotone and $0 \in Mv$ if and only if $\langle Av, u - v\rangle \geq 0$ for all $u \in C$.

2 Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to R which satisfies (A1)-(A4) and $A_m : C \to H$ a λ_m-inverse-strongly monotone mapping for each $1 \leq m \leq N$, where N is some positive integer. Let $S : C \to C$ be a κ-strict pseudocontraction and $B : C \to H$ a β-inverse-strongly monotone mapping. Assume that*

$$\mathcal{F} := \bigcap_{m=1}^{N} EP(F_m, A_m) \cap VI(C, B) \cap F(S) \neq \emptyset.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots,$ and $\{\gamma_{n,N}\}$ be sequences in $[0, 1]$. Let $\{\rho_n\}$ be a positive sequence in $[0, 2\beta]$ and $\{r_{n,m}\}$ a positive sequence in $[0, 2\lambda_m]$ for each $1 \leq m \leq N$. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C = C_1, \\ z_n = P_C(\sum_{m=1}^{N} \gamma_{n,m} u_{n,m} - \rho_n B \sum_{m=1}^{N} \gamma_{n,m} u_{n,m}), \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n) S z_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases} \tag{2.1}$$

where $u_{n,m}$ is such that

$$F_m(u_{n,m}, u_m) + \langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \geq 0, \quad \forall u_m \in C$$

for each $1 \leq m \leq N$. Assume that $\sum_{m=1}^{N} \gamma_{n,m} = 1$ for each $n \geq 1$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots,$ $\{\gamma_{n,N}\}$, $\{r_{n,1}\}, \ldots,$ $\{r_{n,N}\}$ and $\{\rho_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 \leq \kappa \leq \beta_n < b < 1$;

(c) $0 \leq c \leq \gamma_{n,m} < 1$ *for each* $1 \leq m \leq N$;

(d) $0 < d \leq \rho_n \leq e < 2\beta$ *and* $0 < d' \leq r_{n,m} \leq e' < 2\lambda_m$ *for each* $1 \leq m \leq N$.

Then sequence $\{x_n\}$ generated by (2.1) converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Proof. Note that $u_{n,m}$ can be rewritten as

$$u_{n,m} = T_{r_{n,m}}(x_n - r_{n,m} A_m x_n), \; \forall\, 1 \leq m \leq N.$$

Fix $p \in \mathcal{F}$. It follows that for all $n \geq 1$,

$$p = Sp = P_C(I - \rho_n B)p = T_{r_{n,1}}(p - r_{n,1} A_1 p) = \cdots = T_{r_{n,N}}(p - r_{n,N} A_N p).$$

The proof is divided into several steps.

Step 1. We prove that $I - r_{n,1}A_1$ is nonexpansive for each $n \geq 1$.
Indeed, for any $x, y \in C$, we see from the condition (d) that

$$\begin{aligned}
& \|(I - r_{n,1}A_1)x - (I - r_{n,1}A_1)y\|^2 \\
= \;& \|(x-y) - r_{n,1}(A_1x - A_1y)\|^2 \\
= \;& \|x-y\|^2 - 2r_{n,1}\langle x-y, A_1x - A_1y\rangle + r_{n,1}^2\|A_1x - A_1y\|^2 \\
\leq \;& \|x-y\|^2 - r_{n,1}(2\lambda_1 - r_{n,1})\|A_1x - A_1y\|^2 \\
\leq \;& \|x-y\|^2.
\end{aligned}$$

This shows that $I - r_{n,1}A_1$ is nonexpansive for each $n \geq 1$. In a similar way, we can obtain that $I - r_{n,2}A_2$, $I - r_{n,3}A_3, \ldots, I - r_{n,N}A_N$ and $I - \rho_n B$ are nonexpansive for each $n \geq 1$.

Step 2. Now, we prove that C_n is closed and convex for each $n \geq 1$.
From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_i is closed and convex for some $i \geq 1$. We show that C_{i+1} is closed and convex for same i. Indeed, for any $v \in C_i$, we see that $\|y_i - v\| \leq \|x_i - v\|$ is equivalent to

$$\|y_i\|^2 - \|x_i\|^2 - 2\langle v, y_i - x_i\rangle \leq 0.$$

Thus C_{i+1} is closed and convex. This shows that C_n is closed and convex for each $n \geq 1$.

Step 3. Next, we show that $\mathcal{F} \subset C_n$ for each $n \geq 1$.
From the assumption, we see that $\mathcal{F} \subset C = C_1$. Suppose that $\mathcal{F} \subset C_i$ for some $i \geq 1$. For any $v \in \mathcal{F} \subset C_i$, we see that

$$\begin{aligned}
\|y_i - v\| &= \|\alpha_i x_i + (1-\alpha_i)S_i z_i - v\| \\
&\leq \alpha_i\|x_i - v\| + (1-\alpha_i)\|z_i - v\| \\
&\leq \alpha_i\|x_i - v\| + (1-\alpha_i)\sum_{m=1}^{N}\gamma_{i,m}\|u_{i,m} - v\| \\
&\leq \alpha_i\|x_i - v\| + (1-\alpha_i)\sum_{m=1}^{N}\gamma_{i,m}\|T_{r_{i,m}}(I - r_{i,m}A_m)x_n - v\| \\
&\leq \|x_i - v\|.
\end{aligned}$$

This shows that $v \in C_{i+1}$. This proves that $\mathcal{F} \subset C_n$ for each $n \geq 1$.

Step 4. Now, we prove that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{2.2}$$

For each $v \in \mathcal{F} \subset C_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - v\|.$$

In particular, we have

$$\|x_1 - x_n\| \leq \|x_1 - P_{\mathcal{F}}x_1\|.$$

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This implies that $\{x_n\}$ is bounded. Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} 0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|.$$

This proves that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Notice that

$$\begin{aligned} &\|x_n - x_{n+1}\|^2 \\ = \ &\|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ = \ &\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ = \ &\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ = \ &\|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ \le \ &\|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.3}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

This implies that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \le 2\|x_n - x_{n+1}\|.$$

From (2.3), we obtain that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Step 5. Next, we show that $\{u_{n_i,m}\}$ converges weakly to ξ for each $1 \le m \le N$. Putting

$$S_n = \beta_n I + (1 - \beta_n)S, \quad \forall n \ge 1,$$

then we see from Lemma 1.3 that S_n is a nonexpansive mapping with $F(S_n) = F(S)$ for each $n \ge 1$. On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n)S_n z_n\| = (1 - \alpha_n)\|x_n - S_n z_n\|.$$

From the restriction (a) and (2.2), we have

$$\lim_{n\to\infty} \|x_n - S_n z_n\| = 0. \tag{2.4}$$

For any $p \in \mathcal{F}$, we have for all $1 \le m \le N$,

$$\begin{aligned} &\|u_{n,m} - p\|^2 \\ = \ &\|T_{r_{n,m}}(I - r_{n,m}A_m)x_n - T_{r_{n,m}}(I - r_{n,m}A_m)p\|^2 \qquad (2.5) \\ \le \ &\|(x_n - p) - r_{n,m}(A_m x_n - A_m p)\|^2 \\ = \ &\|x_n - p\|^2 - 2r_{n,m}\langle x_n - p, A_m x_n - A_m p\rangle + r_{n,m}^2\|A_m x_n - A_m p\|^2 \\ \le \ &\|x_n - p\|^2 - r_{n,m}(2\lambda_m - r_{n,m})\|A_m x_n - A_m p\|^2. \end{aligned}$$

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It follows that

$$\begin{aligned}
& \|y_n - p\|^2 = \|\alpha_n x_n + (1-\alpha_n)S_n z_n - p\|^2 \qquad (2.6)\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_n z_n - p\|^2\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\sum_{m=1}^{N}\gamma_{n,m}\|u_{n,m} - p\|^2\\
\le\ & \|x_n - p\|^2 - (1-\alpha_n)\sum_{m=1}^{N}\gamma_{n,m}r_{n,m}(2\lambda_m - r_{n,m})\|A_m x_n - A_m p\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
& (1-\alpha_n)\gamma_{n,m}r_{n,m}(2\lambda_m - r_{n,m})\|A_m x_n - A_m p\|^2\\
\le\ & \|x_n - p\|^2 - \|y_n - p\|^2\\
\le\ & (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|, \quad \forall 1 \le m \le N.
\end{aligned}$$

In view of the conditions (a), (c) and (d), we obtain from (2.2) that

$$\lim_{n\to\infty}\|A_m x_n - A_m p\| = 0, \quad \forall 1 \le m \le N. \qquad (2.7)$$

On the other hand, we have from Lemma 1.2 that for all $1 \le m \le N$,

$$\begin{aligned}
& \|u_{n,m} - p\|^2\\
=\ & \|T_{r_{n,m}}(I - r_{n,m}A_m)x_n - T_{r_{n,m}}(I - r_{n,m}A_m)p\|^2\\
\le\ & \langle (I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p, u_{n,m} - p\rangle\\
=\ & \frac{1}{2}(\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p\|^2 + \|u_{n,m} - p\|^2\\
& -\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p - (u_{n,m} - p)\|^2)\\
\le\ & \frac{1}{2}\big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \|x_n - u_{n,m} - r_{n,m}(A_m x_n - A_m p)\|^2\big)\\
=\ & \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \big(\|x_n - u_{n,m}\|^2\\
& -2r_{n,m}\langle x_n - u_{n,m}, A_m x_n - A_m p\rangle + r_{n,m}^2\|A_m x_n - A_m p\|^2\big)\Big).
\end{aligned}$$

This implies that for all $1 \le m \le N$,

$$\|u_{n,m} - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_{n,m}\|^2 + 2r_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\|,$$

from which it follows that

$$\begin{aligned}
& \|y_n - p\|^2\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_n z_n - p\|^2\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \qquad (2.8)\\
\le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\sum_{m=1}^{N}\gamma_{n,m}\|u_{n,m} - p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - p\|^2 + (1-\alpha_n)\sum_{m=1}^{N} \gamma_{n,m} 2r_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\| \\
&\quad -(1-\alpha_n)\sum_{m=1}^{N} \gamma_{n,m}\|x_n - u_{n,m}\|^2 \\
&\leq \|x_n - p\|^2 + \sum_{m=1}^{N} 2r_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\| \\
&\quad -(1-\alpha_n)\sum_{m=1}^{N} \gamma_{n,m}\|x_n - u_{n,m}\|^2.
\end{aligned}$$

Hence, we get that for all $1 \leq m \leq N$,

$$\begin{aligned}
(1-\alpha_n)\gamma_{n,m}\|x_n - u_{n,m}\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad + \sum_{m=1}^{N} 2r_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\| \\
&\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\
&\quad + \sum_{m=1}^{N} 2r_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\|.
\end{aligned} \tag{2.9}$$

In view of the restrictions (a) and (c), we obtain from (2.2) and (2.7) that

$$\lim_{n\to\infty} \|x_n - u_{n,m}\| = 0, \quad \forall 1 \leq m \leq N. \tag{2.10}$$

Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ. It follows from (2.10) that $\{u_{n_i,m}\}$ converges weakly to ξ for each $1 \leq m \leq N$.

Step 6. Next, we show that

$$\xi \in \bigcap_{m=1}^{N} EP(F_m, A_m). \tag{2.11}$$

Since $u_{n,m} = T_{r_{n,m}}(x_n - r_{n,m}A_m x_n)$, we have for all $u_m \in C$,

$$F_m(u_{n,m}, u_m) + \langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \geq 0.$$

From the condition (A2), we see that for all $u_m \in C$,

$$\langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \geq F_m(u_m, u_{n,m}). \tag{2.12}$$

Replacing n by n_i, for all $u_m \in C$, we arrive at

$$\langle A_m x_{n_i}, u_m - u_{n_i,m}\rangle + \langle u_m - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\rangle \geq F_m(u_m, u_{n_i,m}). \tag{2.13}$$

For t_m with $0 < t_m \leq 1$ and $u_m \in C$, let $u_{t_m} = t_m u_m + (1-t_m)\xi$ for each $1 \leq m \leq N$. Since $u_m \in C$ and $\xi \in C$, we have $u_{t_m} \in C$ for each $1 \leq m \leq N$. It follows from

(2.13) that

$$\begin{aligned} & \langle u_{t_m} - u_{n_i,m}, A_m u_{t_m}\rangle \\ \geq\ & \langle u_{t_m} - u_{n_i,m}, A_m u_{t_m}\rangle - \langle A_m x_{n_i,m}, u_{t_m} - u_{n_i,m}\rangle \\ & -\langle u_{t_m} - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\rangle + F_m(u_{t_m}, u_{n_i,m}) \\ =\ & \langle u_{t_m} - u_{n_i,m}, A_m u_{t_m} - A_m u_{n_i,m}\rangle + \langle u_{t_m} - u_{n_i,m}, A_m u_{n_i,m} - A_m x_{n_i}\rangle \\ & -\langle u_{t_m} - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\rangle + F_m(u_{t_m}, u_{n_i,m}). \end{aligned} \tag{2.14}$$

From (2.10), we have $Au_{n_i,m} - Ax_{n_i} \to 0$ as $i \to \infty$ for each $1 \leq m \leq N$. On the other hand, we obtain from the monotonicity of A_m that $\langle u_{t_m} - u_{n_i,m}, A_m u_{t_m} - A_m u_{n_i,m}\rangle \geq 0$. It follows from (A4) that

$$\langle u_{t_m} - \xi, A_m u_{t_m}\rangle \geq F_m(u_{t_m}, \xi), \quad \forall 1 \leq m \leq N. \tag{2.15}$$

From (A1), (A4) and (2.15), we obtain that

$$\begin{aligned} 0 &= F_m(u_{t_m}, u_{t_m}) \\ &\leq t_m F_m(u_{t_m}, u_m) + (1-t_m) F_m(u_{t_m}, \xi) \\ &\leq t_m F_m(u_{t_m}, u_m) + (1-t_m)\langle u_{t_m} - \xi, A_m u_{t_m}\rangle \\ &= t_m F_m(u_{t_m}, u_m) + (1-t_m) t_m \langle u_m - \xi, A_m u_{t_m}\rangle, \end{aligned}$$

which yields that

$$F_m(u_{t_m}, u_m) + (1-t_m)\langle u_m - \xi, A_m u_{t_m}\rangle \geq 0, \quad \forall 1 \leq m \leq N.$$

Letting $t_m \to 0$ in the above inequality for each $1 \leq m \leq N$, we arrive at

$$F_m(\xi, u_m) + \langle u_m - \xi, A_m \xi\rangle \geq 0, \quad \forall 1 \leq m \leq N.$$

This shows that $\xi \in EP(F_m, A_m)$ for each $1 \leq m \leq N$, that is,

$$\xi \in \bigcap_{m=1}^{N} EP(F_m, A_m).$$

Step 7. Next, we show that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{2.16}$$

Putting $w_n = \sum_{m=1}^{N} \gamma_{n,m} u_{n,m}$ for each $n \geq 1$, we see that

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n) S_n z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|S_n z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|(I - \rho_n B) w_n - p\|^2 \\ &\leq \|x_n - p\|^2 - (1-\alpha_n)\rho_n(2\beta - \rho_n)\|Bw_n - Bp\|^2. \end{aligned} \tag{2.17}$$

This in turn implies that

$$\begin{aligned} (1-\alpha_n)\rho_n(2\beta - \rho_n)\|Bw_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|. \end{aligned} \tag{2.18}$$

In view of the conditions (a) and (d), we obtain from (2.2) that

$$\lim_{n\to\infty} \|Bw_n - Bp\| = 0. \tag{2.19}$$

On the other hand, we have from the firmly nonexpansivity of P_C that

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(I-\rho_n B)w_n - P_C(I-\rho_n B)p\|^2 \\
&\le \langle (I-\rho_n B)w_n - (I-\rho_n B)p, z_n - p\rangle \\
&= \frac{1}{2}\big(\|(I-\rho_n B)w_n - (I-\rho_n B)p\|^2 + \|z_n - p\|^2 \\
&\quad -\|(I-\rho_n B)w_n - (I-\rho_n B)p - (z_n - p)\|^2\big) \\
&\le \frac{1}{2}\big(\|w_n - p\|^2 + \|z_n - p\|^2 - \|w_n - z_n - \rho_n(Bw_n - Bp)\|^2\big) \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|w_n - z_n\|^2 \\
&\quad +2\rho_n\langle w_n - z_n, Bw_n - Bp\rangle - \rho_n^2\|Bw_n - Bp\|^2\big).
\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|w_n - z_n\|^2 + 2\rho_n\|w_n - z_n\|\|Bw_n - Bp\|,$$

from which it follows that

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_n z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \|x_n - p\|^2 - (1-\alpha_n)\|w_n - z_n\|^2 + 2\rho_n\|w_n - z_n\|\|Bw_n - Bp\|.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&(1-\alpha_n)\|w_n - z_n\|^2 \\
\le\ & \|x_n - p\|^2 - \|y_n - p\|^2 + 2\rho_n\|w_n - z_n\|\|Bw_n - Bp\| \\
\le\ & (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + 2\rho_n\|w_n - z_n\|\|Bw_n - Bp\|.
\end{aligned}$$

In view of the restriction (a), we obtain from (2.2) and (2.19) that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{2.20}$$

Note that

$$\|z_n - x_n\| \le \|z_n - w_n\| + \|w_n - x_n\| \le \|z_n - w_n\| + \sum_{m=1}^{N} \gamma_{n,m}\|u_{n,m} - x_n\|.$$

In view of (2.10) and (2.20), we obtain that $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Step 8. Next, we show that

$$\xi \in VI(C,B). \tag{2.21}$$

In fact, let T be the maximal monotone mapping defined by

$$Tx = \begin{cases} Bx + N_C x, & x \in C \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, we have $y - Bx \in N_C x$. Since $z_n \in C$, by definition of N_C, we have

$$\langle x - z_n, y - Bx\rangle \geq 0. \tag{2.22}$$

In view of $z_n = P_C(I - \rho_n B)w_n$, we obtain that

$$\langle x - z_n, z_n - (I - \rho_n B)w_n\rangle \geq 0,$$

and hence

$$\langle x - z_n, \frac{z_n - w_n}{\rho_n} + Bw_n\rangle \geq 0. \tag{2.23}$$

From (2.22) and (2.23), we obtain from the monotonicity of B that

$$\begin{aligned}
\langle x - z_{n_i}, y\rangle &\geq \langle x - z_{n_i}, Bx\rangle \\
&\geq \langle x - z_{n_i}, Bx\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - w_{n_i}}{\rho_{n_i}} + Bw_{n_i}\rangle \\
&= \langle x - z_{n_i}, Bx - Bz_{n_i}\rangle + \langle x - z_{n_i}, Bz_{n_i} - Bw_{n_i}\rangle \\
&\quad - \langle x - z_{n_i}, \frac{z_{n_i} - w_{n_i}}{\rho_{n_i}}\rangle \\
&\geq \langle x - z_{n_i}, Bz_{n_i} - Bw_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - w_{n_i}}{\rho_{n_i}}\rangle.
\end{aligned}$$

It follows from (2.16) that $z_{n_i} \rightharpoonup \xi$. On the other hand, we know that B is $\frac{1}{\beta}$-Lipschitz continuous. It follows from (2.20) that $\langle x - \xi, y\rangle \geq 0$. From the maximality of T, we get $0 \in T\xi$. This means that $\xi \in VI(C, B)$.

Step 9. Next, we show that

$$\xi \in F(S). \tag{2.24}$$

Note that

$$\|Sz_n - x_n\| \leq \frac{\|x_n - S_n z_n\|}{1 - \beta_n} + \frac{\beta_n\|z_n - x_n\|}{1 - \beta_n}.$$

In view of (2.4) and (2.16), we obtain from the restriction (b) that

$$\lim_{n\to\infty} \|Sz_n - x_n\| = 0. \tag{2.25}$$

On the other hand, we see from Lemma 1.1 that

$$\begin{aligned}
\|Sx_n - x_n\| &\leq \|Sx_n - Sz_n\| + \|Sz_n - x_n\| \\
&\leq \frac{1 + \kappa}{1 - \kappa}\|x_n - z_n\| + \|Sz_n - x_n\|.
\end{aligned}$$

It follows from (2.16) and (2.25) that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. We can conclude from Lemma 1.1 that $\xi \in F(S)$.

Step 10. Finally, we prove that sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

From (2.11), (2.21) and (2.24), we know that

$$\xi \in \mathcal{F} = \bigcap_{m=1}^{N} EP(F_m, A_m) \cap VI(C, B) \cap F(S).$$

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Put $\bar{x} = P_{\mathcal{F}}x_1$. Since $\bar{x} = P_{\mathcal{F}}x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x_1$, we have

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \bar{x}\|.$$

On the other hand, we have

$$\|x_1 - \bar{x}\| \leq \|x_1 - \xi\| \leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - \bar{x}\|.$$

We, therefore, obtain that

$$\|x_1 - \xi\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \bar{x}\|.$$

This implies $x_{n_i} \to \xi = \bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we obtain that $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof.

If $F_m = F$, $A_m = A$ and $r_{n,m} = r_n$ for each $1 \leq m \leq N$, then Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4) and $A : C \to H$ a λ-inverse-strongly monotone mapping. Let $S : C \to C$ be a κ-strict pseudocontraction and $B : C \to H$ a β-inverse-strongly monotone mapping. Assume that*

$$\mathcal{F} := EP(F, A) \cap VI(C, B) \cap F(S) \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. Let $\{\rho_n\}$ be a positive sequence in $[0, 2\beta]$ and $\{r_n\}$ a positive sequence in $[0, 2\lambda]$. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C = C_1, \\ z_n = P_C(u_n - \rho_n Bu_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n)Sz_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \tag{2.26}$$

where u_n is such that for all $u \in C$,

$$F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{\rho_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 \leq \kappa \leq \beta_n < b < 1$;

(c) $0 < d \leq \rho_n \leq e < 2\beta$ *and* $0 < d' \leq r_n \leq e' < 2\lambda$.

Then sequence $\{x_n\}$ generated by (2.26) converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}}x_1$.

Remark 2.3. Corollary 2.2 is reduced to Theorem CQK if $\beta_n = \kappa$ for each $n \geq 1$.

If $B = 0$, then Corollary 2.2 is reduced to the following.

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4), $A : C \to H$ a λ-inverse-strongly monotone mapping and $S : C \to C$ a κ-strict pseudocontraction. Assume that*

$$\mathcal{F} := EP(F, A) \cap F(S) \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ and $\{r_n\}$ a positive sequence in $[0, 2\lambda]$. Let $\{x_n\}$ be a sequence generated in the following:

$$\begin{cases} x_1 \in C = C_1, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n u_n + (1 - \beta_n)Su_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases} \tag{2.27}$$

where u_n is such that

$$F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 \leq \kappa \leq \beta_n < b < 1$;

(c) $0 < d' \leq r_n \leq e' < 2\lambda$.

Then sequence $\{x_n\}$ generated by (2.27) converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Remark 2.5. Corollary 2.4 can be viewed as an improvement of the result in [16]. To be more precise, if S is nonexpansive and $\beta_n = 0$ for each $n \geq 1$, then Corollary 2.4 is reduced to the result in [16].

3 Applications

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_m : C \to H$ a λ_m-inverse-strongly monotone mapping for each $1 \leq m \leq N$, where N is some positive integer. Let $S : C \to C$ be a κ-strict pseudocontraction and $B : C \to H$ a β-inverse-strongly monotone mapping. Assume that*

$$\mathcal{F} := \bigcap_{m=1}^{N} VI(C, A_m) \cap VI(C, B) \cap F(S) \neq \emptyset.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots,$ and $\{\gamma_{n,N}\}$ be sequences in $[0, 1]$. Let $\{\rho_n\}$ be a positive sequence in $[0, 2\beta]$ and $\{r_{n,m}\}$ a positive sequence in $[0, 2\lambda_m]$ for each $1 \leq m \leq N$. Let $\{x_n\}$ be a sequence generated in the following:

$$\begin{cases} x_1 \in C = C_1, \\ z_n = P_C(\sum_{m=1}^{N} \gamma_{n,m} u_{n,m} - \rho_n B \sum_{m=1}^{N} \gamma_{n,m} u_{n,m}), \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n)Sz_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1 \end{cases} \tag{3.1}$$

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where $u_{n,m} = P_C(x_n - r_{n,m}A_m x_n)$ *for each* $1 \le m \le N$. *Assume that* $\sum_{m=1}^N \gamma_{n,m} = 1$ *for each* $n \ge 1$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots, \{\gamma_{n,N}\}$, $\{r_{n,1}\}, \ldots, \{r_{n,N}\}$ *and* $\{\rho_n\}$ *satisfy the following restrictions:*

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le \kappa \le \beta_n < b < 1$;

(c) $0 \le c \le \gamma_{n,m} < 1$ *for each* $1 \le m \le N$;

(d) $0 < d \le \rho_n \le e < 2\beta$ *and* $0 < d' \le r_{n,m} \le e' < 2\lambda_m$ *for each* $1 \le m \le N$.

Then sequence $\{x_n\}$ *generated by (3.1) converges strongly to some point* $\bar{x}$, *where* $\bar{x} = P_{\mathcal{F}} x_1$.

Proof. Putting $F_m \equiv 0$ for each $1 \le m \le N$, we see that

$$\langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C$$

is equivalent to

$$\langle x_n - r_{n,m}A_m x_n - u_{n,m}, u_{n,m} - u_m\rangle \ge 0, \forall u_m \in C.$$

This implies that $u_{n,m} = P_C(x_n - r_{n,m}A_m x_n)$ for each $1 \le m \le N$. Then we can obtain from Theorem 2.1 the desired result immediately.

Next, we consider the following optimization problem (OP): Find an x^* such that

$$\varphi_1(x^*) = \min_{x\in C}\varphi_1(x),\ \varphi_2(x^*) = \min_{x\in C}\varphi_2(x),\ \cdots, \varphi_N(x^*) = \min_{x\in C}\varphi_N(x), \tag{3.2}$$

where $\varphi_m : C \to R$ a convex and lower semicontinuous function for each $1 \le m \le N$, where $N \ge 1$ is some positive integer.

Theorem 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* φ_m *be a proper convex and lower semicontinuous function for each* $1 \le m \le N$, *where* N *is some positive integer. Assume that*

$$sol(OP) \ne \emptyset,$$

where $sol(OP)$ *denotes the solution set of problem (3.2). Let* $\{\alpha_n\}$, $\{\gamma_{n,1}\}, \ldots,$ *and* $\{\gamma_{n,N}\}$ *be sequences in* $(0,1)$. *Let* $\{r_{n,m}\}$ *be a positive sequence in* $(0,\infty)$ *for each* $1 \le m \le N$. *Let* $\{x_n\}$ *be a sequence generated in the following:*

$$\begin{cases} x_1 \in C = C_1, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^N \gamma_{n,m}u_{n,m}, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{3.3}$$

where $u_{n,m}$ *is such that*

$$\varphi_m(u_m) - \varphi_m(u_{n,m}) + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C$$

for each $1 \le m \le N$. *Assume that* $\sum_{m=1}^N \gamma_{n,m} = 1$ *for each* $n \ge 1$ *and the sequences* $\{\alpha_n\}$, $\{\gamma_{n,1}\}, \ldots, \{\gamma_{n,N}\}$, $\{r_{n,1}\}, \ldots,$ *and* $\{r_{n,N}\}$ *satisfy the following restrictions:*

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(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le c \le \gamma_{n,m} < 1$ *for each* $1 \le m \le N$;

(c) $0 < d' \le r_{n,m} \le e' < \infty$ *for each* $1 \le m \le N$.

Then sequence $\{x_n\}$ *generated by (3.3) converges strongly to some point* $\bar{x}$, *where* $\bar{x} = P_{sol(OP)}x_1$.

Proof. Putting $S = I$, $A_m = B = 0$ and $F_m(x, y) = \varphi_m(y) - \varphi_m(x)$ for each $1 \le m \le N$, we can obtained the desired conclusion easily.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to R which satisfies (A1)-(A4) and $T_m : C \to C$ a t_m-strict pseudocontraction for each $1 \le m \le N$, where N is some positive integer. Let $S : C \to C$ be a κ-strict pseudocontraction and $D : C \to C$ a ν-strict pseudocontraction. Assume that*

$$\mathcal{F} := \bigcap_{m=1}^{N} EP(F_m, I - T_m) \cap F(D) \cap F(S) \neq \emptyset.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots,$ and $\{\gamma_{n,N}\}$ be sequences in $[0, 1]$. Let $\{\rho_n\}$ be a positive sequence in $[0, 1-\nu]$ and $\{r_{n,m}\}$ a positive sequence in $[0, 1-t_m]$ for each $1 \le m \le N$. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C = C_1, \\ z_n = (1-\rho_n)\sum_{m=1}^{N}\gamma_{n,m}u_{n,m} + \rho_n D\sum_{m=1}^{N}\gamma_{n,m}u_{n,m}, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n z_n + (1-\beta_n)Sz_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{3.4}$$

where $u_{n,m}$ is such that for all $u_m \in C$,

$$F_m(u_{n,m}, u_m) + \langle x_n - T_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0,$$

for each $1 \le m \le N$. Assume that $\sum_{m=1}^{N}\gamma_{n,m} = 1$ for each $n \ge 1$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \ldots, \{\gamma_{n,N}\}$, $\{r_{n,1}\}, \ldots, \{r_{n,N}\}$ and $\{\rho_n\}$ satisfy the following restrictions:

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le \kappa \le \beta_n < b < 1$;

(c) $0 \le c \le \gamma_{n,m} < 1$ *for each* $1 \le m \le N$;

(d) $0 < d \le \rho_n \le e < 1 - \nu$ *and* $0 < d' \le r_{n,m} \le e' < 1 - t_m$ *for each* $1 \le m \le N$.

Then sequence $\{x_n\}$ *generated by (3.4) converges strongly to some point* $\bar{x}$, *where* $\bar{x} = P_{\mathcal{F}}x_1$.

Proof. Put $A_m = I - T_m$ for each $1 \le m \le N$ and $B = I - D$. Then D is $\frac{1-\nu}{2}$-inverse-strongly monotone. We have $F(D) = VI(C, B)$ and

$$\begin{aligned} &P_C(\sum_{m=1}^{N}\gamma_{n,m}u_{n,m} - \rho_n B\sum_{m=1}^{N}\gamma_{n,m}u_{n,m}) \\ &= (1-\rho_n)\sum_{m=1}^{N}\gamma_{n,m}u_{n,m} + \rho_n D\sum_{m=1}^{N}\gamma_{n,m}u_{n,m}. \end{aligned}$$

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Hence, we can obtain from Theorem 2.1 the desired result immediately.

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APPROXIMATION OF THE JENSEN TYPE FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN C^*-ALGEBRAS

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Abstract. In this paper, we approximate homomorphisms in non-Archimedean C^*-algebras and non-Archimedean Lie C^*-algebras and of derivations on non-Archimedean C^*-algebras and non-Archimedean Lie C^*-algebras for the following Jensen type functional equation

$$f(\frac{x+y}{2})+f(\frac{x-y}{2})=f(x).$$

1. Introduction and preliminaries

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r|\,|s|$, and $|r+s|\le\max\{|r|,|s|\}$ for all $r,s\in K$. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0|=0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:X\to[0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\|=0$ if and only if $x=0$;
(ii) for any $r\in K, x\in X$, $\|rx\|=|r|\|x\|$;
(iii) the strong triangle inequality (ultrametric) holds; namely,

$$\|x+y\|\le\max\{\|x\|,\|y\|\}\quad(x,y\in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n-x_m\|\le\max\{\|x_{j+1}-x_j\|:m\le j\le n-1\}\qquad(n>m),$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x\in\mathbb{Z}$ such that $x=\frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p:=p^{-n_x}$ defines

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a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

A non-Archimedean Banach algebra is a complete non-Archimedaen algebra $\mathcal{A}$ which satisfies $\|ab\| \le \|a\|.\|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [8, 33].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is a mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$;

(ii) $(\alpha s + \beta t)^* = \overline{\alpha} s^* + \overline{\beta} t^*$;

(iii) $(st)^* = t^* s^*$ for $s, t \in \mathcal{U}$.

If, in addition $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*–algebra.

The stability problem of functional equations originated from a question of Ulam [35] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Th.M. Rassias [26] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. [26] *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [11] generalized the Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4], [5, 6] [10], [13] [17], [21], [26]–[32], [34]).

Theorem 1.2. [23, 24, 25] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}||x||^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

We recall two fundamental results in fixed point theory.

Theorem 1.3. [1] *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \leq Lf(x, y), \qquad \forall x, y \in X$$

for some Lipschitz constant $L < 1$. Then

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, i.e.,*

$$\lim_{n\to\infty} J^n x = x^*$$

for any starting point $x \in X$;

(3) *one has the following estimation inequalities:*

$$\begin{aligned} d(J^n x, x^*) &\leq L^n d(x, x^*), \\ d(J^n x, x^*) &\leq \frac{1}{1-L} d(J^n x, J^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Jx) \end{aligned}$$

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [7] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

After Isac and Rassias [14] began to provide applications of the new fixed point theorems in the proof of stability theory of functional equations at 1996, the stability problems of functional equations have been extensively investigated by a number of authors (see [2, 3, 7, 9, 16, 22]).

This paper is organized as follows: In Sections 2 and 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean C^*-algebras and of derivations on non-Archimedean C^*-algebras for the Jensen type functional equation.

In Sections 4 and 5, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean Lie C^*-algebras and of derivations on non-Archimedean Lie C^*-algebras for the Jensen type functional equation.

2. Stability of homomorphisms in non-Archimedean C^*-algebras

Throughout this section, assume that A is a non-Archimedean C^*-algebra with norm $\|\cdot\|_A$ and that B is a non-Archimedean C^*-algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x,y) := \mu f(\frac{x+y}{2}) + \mu f(\frac{x-y}{2}) - f(\mu x) \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in non-Archimedean C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0,\infty)$ and $\eta : A \to [0,\infty)$ such that*

$$\|D_\mu f(x,y)\|_B \leq \varphi(x,y), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \psi(x,y), \tag{2.3}$$

$$\|f(x^*) - f(x)^*\|_B \leq \eta(x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $|2| < 1$ and

$$\varphi(2x,2y) \leq |2|L\varphi(x,y), \tag{2.5}$$

$$\psi(2x,2y) \leq |4|L\psi(x,y), \tag{2.6}$$

$$\eta(2x) \leq |2|L\eta(x) \tag{2.7}$$

for all $x, y \in A$, then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{L}{1-L}\varphi(x,0) \tag{2.8}$$

for all $x \in A$.

Proof. It follows from (2.5), (2.6), (2.7) and $L < 1$ that

$$\lim_{n\to\infty} \frac{1}{|2|^n}\varphi(2^n x, 2^n y) = 0, \tag{2.9}$$

$$\lim_{n\to\infty} \frac{1}{|2|^{2n}}\psi(2^n x, 2^n y) = 0, \tag{2.10}$$

$$\lim_{n\to\infty} \frac{1}{|2|^n}\eta(2^n x) = 0, \tag{2.11}$$

for all $x, y \in A$.

Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x,0), \quad \forall x \in A\}.$$

It is easy to show that (X,d) is complete.

Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

By Theorem 3.1 of [1],

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g,h \in X$.

Letting $\mu = 1$ and $y = 0$ in (2.2), we get

$$\|2f(\frac{x}{2}) - f(x)\|_B \leq \varphi(x,0) \tag{2.12}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{2}f(2x)\|_B \leq \frac{1}{|2|}\varphi(2x,0) \leq L\varphi(x,0)$$

for all $x \in A$. Hence $d(f, Jf) \leq L$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.13}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.13) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x,0)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.14}$$

for all $x \in A$.

(3) $d(f,H) \leq \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,H) \leq \frac{L}{1-L}.$$

This implies that the inequality (2.8) holds.

It follows from (2.5) and (2.14) that

$$\begin{aligned}&\|H(\frac{x+y}{2})+H(\frac{x-y}{2})-H(x)\|_B\\&=\lim_{n\to\infty}\frac{1}{|2|^n}\|f(2^{n-1}(x+y))+f(2^{n-1}(x-y))-f(2^n x)\|_B\\&\leq\lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^n x,2^n y)=0\end{aligned}$$

for all $x, y \in A$. Then,

$$H(\frac{x+y}{2})+H(\frac{x-y}{2})=H(x) \tag{2.15}$$

for all $x, y \in A$. Letting $z=\frac{x+y}{2}$ and $w=\frac{x-y}{2}$ in (2.15), we get

$$H(z)+H(w)=H(z+w)$$

for all $z, w \in A$. So the mapping $H : A \to B$ is Cauchy additive, i.e., $H(z+w) = H(z)+H(w)$ for all $z, w \in A$.

Letting $y = x$ in (2.2), we get

$$\mu f(x)=f(\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By a similar method to above, we get

$$\mu H(x)=H(\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.6) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty}\frac{1}{|4|^n}\|f(4^n xy)-f(2^n x)f(2^n y)\|_B\\&\leq \lim_{n\to\infty}\frac{1}{|4|^n}\psi(2^n x,2^n y)=0\end{aligned}$$

for all $x, y \in A$. Then,

$$H(xy)=H(x)H(y)$$

for all $x, y \in A$.

It follows from (2.7) that

$$\begin{aligned}\|H(x^*)-H(x)^*\|_B &= \lim_{n\to\infty}\frac{1}{|2|^n}\|f(2^n x^*)-f(2^n x)^*\|_B\\&\leq \lim_{n\to\infty}\frac{1}{|2|^n}\eta(2^n x)=0\end{aligned}$$

for all $x \in A$. Then,

$$H(x^*)=H(x)^*$$

for all $x \in A$.

Thus $H : A \to B$ is a non-Archimedean C^*-algebra homomorphism satisfying (2.8), as desired. □

Corollary 2.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\|D_\mu f(x,y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.16}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.17}$$

$$\|f(x^*) - f(x)^*\|_B \leq \theta\|x\|_A^r \tag{2.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{|2|^r\theta}{|2| - |2|^r}\|x\|_A^r \tag{2.19}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) = \psi(x,y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

and

$$\eta(x) := \theta(\|x\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{r-1}$ and we get the desired result. □

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0, \infty)$ and $\eta : A \to [0, \infty)$ satisfying (2.2), (2.3) and (2.4). If there exists an $L < 1$ such that $|2| < 1$ and*

$$|2|\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq L\varphi(x,y), \tag{2.20}$$

$$|4|\psi\left(\frac{x}{2}, \frac{y}{2}\right) \leq L\psi(x,y), \tag{2.21}$$

$$|2|\eta\left(\frac{x}{2}\right) \leq L\eta(x) \tag{2.22}$$

for all $x, y \in A$, then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{L}{|2| - |2|L}\varphi(x, 0) \tag{2.23}$$

for all $x \in A$.

Proof. It follows from (2.20), (2.21), (2.22) and $L < 1$ that

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0,$$
$$\lim_{n\to\infty} |2|^{2n} \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0,$$
$$\lim_{n\to\infty} |2|^n \eta\left(\frac{x}{2^n}\right) = 0,$$

for all $x, y \in A$.

We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g(\frac{x}{2})$$

for all $x \in A$.

It follows from (2.12) that

$$\|f(x) - 2f(\frac{x}{2})\|_B \le \varphi(\frac{x}{2}, 0) \le \frac{L}{|2|}\varphi(x, 0)$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{L}{|2|}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.24}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.24) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x, 0)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f(\frac{x}{2^n}) = H(x)$$

for all $x \in A$.

(3) $d(f, H) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \le \frac{L}{|2| - |2|L},$$

which implies that the inequality (2.23) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.16), (2.17) and (2.18). Then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{|2|^r - |2|}\|x\|_A^r \tag{2.25}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) = \psi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

and

$$\eta(x) := \theta\|x\|_A^r$$

for all $x, y \in A$. Then $L = |2|^{1-r}$ and we get the desired result. □

Theorem 2.5. *Let $f : A \to B$ be an odd mapping for which there are functions $\varphi, \psi : A^2 \to [0, \infty)$ and $\eta : A \to [0, \infty)$ satisfying (2.2), (2.3) and (2.4). If there exists an $L < 1$ such that $\varphi(x, 3x) \leq |2|L\varphi(\frac{x}{2}, \frac{3x}{2})$ for all $x \in A$ and (2.5), (2.6) and (2.7) hold then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{1}{|2| - |2|L}\varphi(x, 3x) \tag{2.26}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x, 3x), \quad \forall x \in A\}.$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

By Theorem 3.1 of [1],

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in X$.

Letting $\mu = 1$ and replacing y by $3x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_B \leq \varphi(x, 3x) \tag{2.27}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{2}f(2x)\|_B \leq \frac{1}{|2|}\varphi(x, 3x)$$

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for all $x \in A$. Hence $d(f, Jf) \leq \frac{1}{|2|}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.28}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.28) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, 3x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x)$$

for all $x \in A$.

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{|2| - |2|L}.$$

This implies that the inequality (2.26) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd mapping such that*

$$\|D_\mu f(x, y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.29}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.30}$$

$$\|f(x^*) - f(x)^*\|_B \leq \theta\|x\|_A^{2r} \tag{2.31}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{3^r\theta}{|2| - |2|^{2r}} \|x\|_A^{2r} \tag{2.32}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y) = \psi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

and

$$\eta(x) := \theta \cdot \|x\|_A^r$$

for all $x, y \in A$. Then $L = |2|^{2r-1}$ and we get the desired result. □

Theorem 2.7. *Let $f : A \to B$ be an odd mapping for which there are functions $\varphi, \psi : A^2 \to [0,\infty)$ and $\eta : A \to [0,\infty)$ satisfying* (2.2), (2.3), (2.4). *If there exists an $L < 1$ such that $\varphi(x, 3x) \le \frac{1}{|2|} L\varphi(2x, 6x)$ for all $x \in A$, also* (2.20), (2.21) *and* (2.22) *hold, then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{L}{|2| - |2|L}\varphi(x, 3x) \tag{2.33}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g(\frac{x}{2})$$

for all $x \in A$.

It follows from (2.27) that

$$\|f(x) - 2f(\frac{x}{2})\|_B \le \varphi(\frac{x}{2}, \frac{3x}{2}) \le \frac{L}{|2|}\varphi(x, 3x)$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.34}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.34) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x, 3x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f(\frac{x}{2^n}) = H(x)$$

for all $x \in A$.

(3) $d(f,H) \leq \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,H) \leq \frac{L}{2-2L},$$

which implies that the inequality (2.33) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd mapping satisfying (2.29), (2.30) and (2.31). Then there exists a unique non-Archimedean C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{|2|^{2r} - |2|} \|x\|_A^{2r} \tag{2.35}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x,y) = \psi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

and

$$\eta(x) := \theta \cdot \|x\|_A^r$$

for all $x, y \in A$. Then $L = |2|^{1-2r}$ and we get the desired result. □

3. Stability of derivations on non-Archimedean C^*-algebras

Throughout this section, assume that A is a non-Archimedean C^*-algebra with norm $\|\cdot\|_A$.

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0,\infty)$ such that*

$$\|D_\mu f(x,y)\|_A \leq \varphi(x,y), \tag{3.1}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \psi(x,y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,0) \leq |2|L\varphi(\frac{x}{2},0)$ for all $x \in A$ and also (2.5) and (2.6) hold. Then there exists a unique non-Archimedean derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{L}{1-L}\varphi(x,0) \tag{3.3}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned} \|\delta(xy) - \delta(x)y - x\delta(y)\|_A &= \lim_{n\to\infty} \frac{1}{|4|^n} \|f(4^n xy) - f(2^n x) \cdot 2^n y - 2^n x f(2^n y)\|_A \\ &\le \lim_{n\to\infty} \frac{1}{|4|^n} \psi(2^n x, 2^n y) = 0 \end{aligned}$$

for all $x, y \in A$. Then,

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\|D_\mu f(x,y)\|_A \le \theta(\|x\|_A^r + \|y\|_A^r), \tag{3.4}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \theta(\|x\|_A^r + \|y\|_A^r) \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{|2|^r \theta}{|2| - |2|^r} \|x\|_A^r \tag{3.6}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) = \psi(x,y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0,\infty)$ satisfying (3.1) and (3.2). If there exists an $L < 1$ such that $\varphi(x,0) \le \frac{1}{|2|} L\varphi(2x,0)$ for all $x \in A$, and also (2.20) and (2.21) hold, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{L}{|2| - |2|L} \varphi(x,0) \tag{3.7}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.4) *and* (3.5)*. Then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{|2|^r - |2|}\|x\|_A^r \tag{3.8}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) = \psi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{1-r}$ and we get the desired result. □

Remark 3.5. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8.

4. Stability of homomorphisms in non-Archimedean Lie C^*-algebras

A non-Archimedean C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *non-Archimedean Lie C^*-algebra* (see [18], [20], [19]).

Definition 4.1. Let A and B be non-Archimedean Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *non-Archimedean Lie C^*-algebra homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a non-Archimedean Lie C^*-algebra with norm $\|\cdot\|_A$ and that B is a non-Archimedean Lie C^*-algebra with norm $\|\cdot\|_B$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean Lie C^*-algebras for the functional equation $D_\mu f(x, y) = 0$.

Theorem 4.2. *Let $f : A \to B$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0, \infty)$ satisfying* (2.2) *such that*

$$\|f([x, y]) - [f(x), f(y)]\|_B \leq \psi(x, y) \tag{4.1}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, 0) \leq |2|L\varphi(\frac{x}{2}, 0)$ for all $x \in A$, and also (2.5) *and* (2.6) *hold, then there exists a unique non-Archimedean Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.8)*.*

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (2.8). The mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned}\|H([x,y])-[H(x),H(y)]\|_B &= \lim_{n\to\infty}\frac{1}{|4|^n}\|f(4^n[x,y])-[f(2^nx),f(2^ny)]\|_B\\ &\le \lim_{n\to\infty}\frac{1}{|4|^n}\psi(2^nx,2^ny)=0\end{aligned}$$

for all $x, y \in A$. Then,

$$H([x,y]) = [H(x),H(y)]$$

for all $x, y \in A$.

Thus $H : A \to B$ is a non-Archimedean Lie C^*-algebra homomorphism satisfying (2.8), as desired. □

Corollary 4.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.16) *such that*

$$\|f([x,y])-[f(x),f(y)]\|_B \le \theta(\|x\|_A^r+\|y\|_A^r) \tag{4.2}$$

for all $x, y \in A$. Then there exists a unique non-Archimedean Lie C^*-algebra homomorphism $H : A \to B$ satisfying (2.19).

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x,y)=\psi(x,y):=\theta(\|x\|_A^r+\|y\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{r-1}$ and we get the desired result. □

Theorem 4.4. *Let $f : A \to B$ be a mapping for which there are functions $\varphi, \psi : A^2 \to [0,\infty)$ and $\eta : A \to [0,\infty)$ satisfying* (2.2), (2.5), (2.6) *and* (4.1). *If there exists an $L < 1$ such that $\varphi(x,0) \le \frac{1}{|2|}L\varphi(2x,0)$ for all $x \in A$, then there exists a unique non-Archimedean Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.23).

Proof. The proof is similar to the proofs of Theorems 2.1 and 4.2. □

Corollary 4.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.16) *and* (4.2). *Then there exists a unique non-Archimedean Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.25).

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x,y)=\psi(x,y):=\theta(\|x\|_A^r+\|y\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{1-r}$ and we get the desired result. □

Remark 4.6. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and their corollaries.

5. Stability of non-Archimedean Lie derivations on C^*-algebras

Definition 5.1. Let A be a non-Archimedean Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *Lie derivation* if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a non-Archimedean Lie C^*-algebra with norm $\|\cdot\|_A$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean Lie C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 5.2. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi, \psi : A^2 \to [0,\infty)$ satisfying (2.5), (2.6) and (3.1) such that*

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_A \leq \psi(x,y) \tag{5.1}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,0) \leq |2|L\varphi(\frac{x}{2}, 0)$ for all $x \in A$. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying (3.3).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned}
&\|\delta([x,y]) - [\delta(x), y] - [x, \delta(y)]\|_A \\
&= \lim_{n\to\infty} \frac{1}{|4|^n} \|f(4^n[x,y]) - [f(2^n x), 2^n y] - [2^n x, f(2^n y)]\|_A \\
&\leq \lim_{n\to\infty} \frac{1}{|4|^n} \psi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. Then,

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 5.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.4) such that*

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r) \tag{5.2}$$

for all $x, y \in A$. Then there exists a unique non-Archimedean Lie derivation $\delta : A \to A$ satisfying (3.6).

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x,y) = \psi(x,y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

and

$$\eta(x) := \theta\|x\|_A^r$$

for all $x, y \in A$. Then $L = |2|^{r-1}$ and we get the desired result. □

Theorem 5.4. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi, \psi : A^2 \to [0, \infty)$ and $\eta : A \to [0, \infty)$ satisfying* (2.20)*,* (2.21)*,* (2.22)*,* (3.1) *and* (5.1)*. If there exists an $L < 1$ such that $\varphi(x, 0) \leq \frac{1}{|2|}L\varphi(2x, 0)$ for all $x \in A$, then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.7)*.*

Proof. The proof is similar to the proofs of Theorems 2.3 and 4.2. □

Corollary 5.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.4) *and* (5.2)*. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.8)*.*

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x, y) = \psi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then $L = |2|^{1-r}$ and we get the desired result. □

Remark 5.6. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and their corollaries.

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Possibility interval-valued fuzzy soft set and its application in decision making

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Abstract

In this paper, we introduce the concept of possibility interval-valued fuzzy soft set which is an extension to possibility fuzzy soft set. Some operations on a possibility interval-valued fuzzy soft set are investigated, such as complement operation, union and intersection operations, "AND" and "OR" operations. We have further studied the similarity between two possibility interval-valued fuzzy soft sets. Finally, application of possibility interval-valued fuzzy soft sets in decision making problem has been shown.

Key words: Interval-valued fuzzy set; Possibility fuzzy soft set; Possibility interval-valued fuzzy soft set; Similarity measure

1 Introduction

Molodtsov [1] initiated a novel concept called soft sets as a new mathematical tool for dealing with uncertainties. The soft set theory is free from many difficulties that have troubled the usual theoretical approaches. It has been found that fuzzy sets, rough sets, and soft sets are closely related concepts [2]. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operational research, Perron integration, probability theory, and measurement theory [1,3].

Research works on soft sets are very active and progressing rapidly in these years. Maji et al. [4] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Jun [5] introduced the notion of soft BCK/BCI-algebras. Jun and Park [6]

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discussed the applications of soft sets in ideal theory of BCK/BCI-algebras. Feng et al. [7] applied soft set theory to the study of semirings and initiated the notion of soft semirings. Furthermore, based on [4], Ali et al. [8] introduced some new operations on soft sets and improved the notion of complement of soft set. They proved that certain De-Morgan's laws hold in soft set theory. Qin and Hong [9] introduced the notion of soft equality and established lattice structures and soft quotient algebras of soft sets. Yang et al. [10] presented the concept of the interval-valued fuzzy soft sets by combining interval-valued fuzzy set [11–13] and soft set models. Feng et al. [14] provided a framework to combine fuzzy sets, rough sets and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. Park et al [15] discussed some properties of equivalence soft set relations. Shabir [16] presented a new approach to soft rough sets by combining the rough set and soft set. By combining fuzzy set and soft set models, Maji et al [17] presented the notion of generalized fuzzy soft sets theory. Zhou [18] and Alkhazaleh [19] presented the notion of generalized interval-valued fuzzy soft sets theory by combining generalized fuzzy soft set and interval-valued fuzzy set respectively. Alkhazaleh et al. [20] defined and studied the possibility fuzzy soft sets where a possibility of each element in the universe is attached with the parameterization of fuzzy sets while defining a fuzzy soft set. The purpose of this paper is to combine the interval-valued fuzzy set and possibility fuzzy soft soft, from which we can obtain a new soft set model: possibility interval-valued fuzzy soft set theory. Intuitively, possibility interval-valued fuzzy soft set theory presented in this paper is an extension of interval-valued fuzzy soft set and possibility fuzzy soft set. We have further studied the similarity between two possibility interval-valued fuzzy soft sets. We finally present examples which show that the decision making method of possibility interval-valued fuzzy soft set can be successfully applied to many problems that contain uncertainties.

The rest of this paper is organized as follows. The following section briefly reviews some backgrounds on interval-valued fuzzy sets, soft sets, fuzzy soft sets and possibility fuzzy soft softs. In section 3, the concept of possibility interval-valued fuzzy soft set is presented. The complement, union, intersection, sum, "AND" and "OR" operations on the possibility interval-valued fuzzy soft set are then defined. Also their some interesting properties have been investigated. In section 4, similarity between two possibility interval-valued fuzzy soft sets has been discussed. An application of possibility interval-valued fuzzy soft set in decision making problem has been shown in section 5. Section 6 concludes the paper.

2 Preliminaries

In this section, we briefly review the concepts of interval-valued fuzzy sets, soft sets, possibility fuzzy soft sets, interval-valued fuzzy soft sets, and so on. Further details could be found in [1, 10–13, 18–20]. Throughout this paper, unless otherwise stated, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$.

Definition 2.1 (**[11]**) *An interval-valued fuzzy set $\hat{X}$ on a universe U is a mapping such that*

$$\hat{X} : U \to Int([0,1]),$$

where Int([0,1]) stands for the set of all closed subintervals of [0,1].

For the sake of convenience, the set of all interval-valued fuzzy sets on U is denoted by $IVF(U)$. Suppose that $\hat{X} \in IVF(U), \forall x \in U, \mu_{\hat{X}}(x) = [\mu^-_{\hat{X}}(x), \mu^+_{\hat{X}}(x)]$ is called the degree of membership an element x to $\hat{X}$. $\mu^-_{\hat{X}}(x)$ and $\mu^+_{\hat{X}}(x)$ are referred to as the lower and upper degrees of membership an element x to $\hat{X}$ where $0 \leq \mu^-_{\hat{X}}(x) \leq \mu^+_{\hat{X}}(x) \leq 1$.

The basic operations on $IVF(U)$ are defined as follows : for all $\hat{X}, \hat{Y} \in IVF(U)$, then

(1) the complement of $\hat{X}$ is denoted by $\hat{X}^c$ where
$\mu^c_{\hat{X}}(x) = 1 - \mu_{\hat{X}}(x) = [1 - \mu^+_{\hat{X}}(x), 1 - \mu^-_{\hat{X}}(x)]$;

(2) the intersection of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \cap \hat{Y}$ where
$\mu_{\hat{X}\cap\hat{Y}}(x) = inf[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [inf(\mu^-_{\hat{X}}(x), \mu^-_{\hat{Y}}(x)), inf(\mu^+_{\hat{X}}(x), \mu^+_{\hat{Y}}(x))]$;

(3) the union of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \cup \hat{Y}$ where
$\mu_{\hat{X}\cup\hat{Y}}(x) = sup[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [sup(\mu^-_{\hat{X}}(x), \mu^-_{\hat{Y}}(x)), sup(\mu^+_{\hat{X}}(x), \mu^+_{\hat{Y}}(x))]$;

(4) the sum of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \oplus \hat{Y}$ where
$\mu_{\hat{X}\oplus\hat{Y}}(x) = [inf\{1, (\mu^-_{\hat{X}}(x) + \mu^-_{\hat{Y}}(x))\}, inf\{1, (\mu^+_{\hat{X}}(x) + \mu^+_{\hat{Y}}(x))\}]$;

Definition 2.2 (**[1]**) *A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2.3 (**[21]**) *A pair (F, A) is called a fuzzy soft set over U if $A \subseteq E$ and $F : A \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

Definition 2.4 (**[20]**) *Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \cdots, e_m\}$ be the universal set of parameters. The pair (U, E) is called a soft universe. Let $F : E \to F(U)$, and μ be a fuzzy subset of E, i.e.$\mu : E \to F(U)$. Let $F_\mu : E \to F(U) \times F(U)$ be a function defined as follows:*

$F_\mu(e) = (F(e)(x), \mu(e)(x)), \forall x \in U$.

Then F_μ is called a possibility fuzzy soft set (PFSS in short) over the soft universe (U, E).

For each parameter e_i, $F_\mu(e_i) = (F(e_i)(x), \mu(e_i)(x))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$ but also the degree of possibility of belongingness of the elements of U in $F(e_i)$ which is represented by $\mu(e_i)$.

Definition 2.5 (**[22]**) *A t-norm is an increasing, associative, and commutative mapping $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition: $T(a, 1) = a$ for all $a \in [0,1]$.*

Definition 2.6 ([22]) *A t-conorm is an increasing, associative, and commutative mapping $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition: $S(a,0) = a$ for all $a \in [0,1]$.*

3 Possibility interval-valued fuzzy soft sets

3.1 Concept of possibility interval-valued fuzzy soft set

In this subsection, we generalized the the concept of possibility fuzzy soft sets as introduced by S.Alkhazaleh et al. [20]. In our generalization of possibility fuzzy soft set, a possibility of each element in the universe is attached with the parameterization of interval-valued fuzzy sets while defining an interval-valued fuzzy soft set.

Definition 3.1 *Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \cdots, e_m\}$ be the universal set of parameters. The pair (U, E) is called a soft universe. Suppose that $\tilde{F} : E \to IVF(U)$, and $\tilde{f}$ is an interval-valued fuzzy subset of E, i.e.$\tilde{f} : E \to IVF(U)$. We say that $\tilde{F}_{\tilde{f}}$ is a possibility interval-valued fuzzy soft set(PIVFSS, in short) over the soft universe (U, E) if and only if $\tilde{F}_{\tilde{f}}$ is a mapping given by*

$$\tilde{F}_{\tilde{f}} : E \to IVF(U) \times IVF(U),$$

where $\tilde{F}_{\tilde{f}}(e) = (\tilde{F}(e)(x), \tilde{f}(e)(x)), \forall x \in U$.

For each parameter $e_i, \tilde{F}_{\tilde{f}}(e_i) = (\tilde{F}(e_i)(x), \tilde{f}(e_i)(x))$ indicates not only the range of belongingness of the elements of U in $F(e_i)$ but also the range of possibility of such belongingness of the elements of U in $F(e_i)$, which is represented by $\tilde{f}(e_i)$. So we can write $\tilde{F}_{\tilde{f}}(e_i)$ as follows:

$\tilde{F}_{\tilde{f}}(e_i) = \{(x, [\mu^-_{\tilde{F}(e_i)}(x), \mu^+_{\tilde{F}(e_i)}(x)], [\mu^-_{\tilde{f}(e_i)}(x), \mu^+_{\tilde{f}(e_i)}(x)]) : x \in U\}$.

Sometimes we write $\tilde{F}_{\tilde{f}}$ as $(\tilde{F}_{\tilde{f}}, E)$. If $A \subseteq E$, we can also have a PIVFSS $(\tilde{F}_{\tilde{f}}, A)$.

Remark 3.2 *A possibility interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $IVF(U) \times IVF(U)$. If $\forall e \in E, \forall x_i \in U, \mu^-_{\tilde{F}(e)}(x_i) = \mu^+_{\tilde{F}(e)}(x_i)$, and $\mu^-_{\tilde{f}(e)}(x_i) = \mu^+_{\tilde{f}(e)}(x_i)$, then $\tilde{F}_{\tilde{f}}$ will be degenerated to be a possibility fuzzy soft set [20].*

Example 3.3 *Let U be a set of three houses under consideration of a decision maker to purchase, which is denoted by $U = \{x_1, x_2, x_3\}$. Let E be a parameter set, where $E = \{e_1, e_2, e_3\}$=\{expensive; beautiful; in the green surroundings\}. Let $\tilde{F}_{\tilde{f}} : E \to IVF(U) \times IVF(U)$ be a function given by as follows:*

$\tilde{F}_{\tilde{f}}(e_1) = \{(x_1, [0.2, 0.6], [0.1, 0.8]), (x_2, [0.6, 0.8], [0.7, 0.8]), (x_3, [0.5, 0.7], [0.8, 0.9])\}$,

$\tilde{F}_{\tilde{f}}(e_2) = \{(x_1, [0.7, 0.8], [0.1, 0.3]), (x_2, [0.2, 0.4], [0.5, 0.8]), (x_3, [0.4, 0.7], [0.6, 0.8])\}$,

$\tilde{F}_{\tilde{f}}(e_3) = \{(x_1, [0.4, 0.6], [0.5, 0.7]), (x_2, [0.4, 0.8], [0.5, 0.7]), (x_3, [0.1, 0.6], [0.3, 0.4])\}$.
Then $\tilde{F}_{\tilde{f}}$ is a PIVFSS over (U, E).

In matrix notation, we write

$$\tilde{F}_{\tilde{f}} = \begin{pmatrix} [0.2,0.6],[0.1,0.8] & [0.6,0.8],[0.7,0.8] & [0.5,0.7],[0.8,0.9] \\ [0.7,0.8],[0.1,0.3] & [0.2,0.4],[0.5,0.8] & [0.4,0.7],[0.6,0.8] \\ [0.4,0.6],[0.5,0.7] & [0.4,0.8],[0.5,0.7] & [0.1,0.6],[0.3,0.4] \end{pmatrix}$$

Definition 3.4 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSS over (U, E). Now $\tilde{F}_{\tilde{f}}$ is said to be a possibility interval-valued fuzzy soft subset of $\tilde{G}_{\tilde{g}}$ if and only if*
(1) $\tilde{f}(e)$ is an interval-valued fuzzy subset of $\tilde{g}(e)$, for all $e \in E$,
(2) $\tilde{F}(e)$ is also an interval-valued fuzzy subset of $\tilde{G}(e)$, for all $e \in E$.
In this case, we write $\tilde{F}_{\tilde{f}} \sqsubseteq \tilde{G}_{\tilde{g}}$.

Example 3.5 *Consider the PIVFSS $\tilde{F}_{\tilde{f}}$ over (U, E) given in Example 3.3. Let $\tilde{G}_{\tilde{g}}$ be another PIVFSS over (U, E) defined as follows:*
$\tilde{G}_{\tilde{g}}(e_1) = \{(x_1, [0.1, 0.5], [0.1, 0.5]), (x_2, [0.3, 0.6], [0.5, 0.7]), (x_3, [0.2, 0.5], [0.6, 0.7])\}$,
$\tilde{G}_{\tilde{g}}(e_2) = \{(x_1, [0.3, 0.6], [0.1, 0.2]), (x_2, [0.1, 0.3], [0.4, 0.7]), (x_3, [0.3, 0.6], [0.4, 0.7])\}$,
$\tilde{G}_{\tilde{g}}(e_3) = \{(x_1, [0.1, 0.4], [0.3, 0.4]), (x_2, [0.2, 0.5], [0.3, 0.4]), (x_3, [0.1, 0.3], [0.2, 0.3])\}$.
Clearly,we have $\tilde{G}_{\tilde{g}} \sqsubseteq \tilde{F}_{\tilde{f}}$.

Definition 3.6 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSSs over (U, E). Now $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ are said to be a possibility interval-valued fuzzy soft equal if and only if*
(1) $\tilde{F}_{\tilde{f}}$ is a possibility interval-valued fuzzy soft subset of $\tilde{G}_{\tilde{g}}$;
(2) $\tilde{G}_{\tilde{g}}$ is a possibility interval-valued fuzzy soft subset of $\tilde{F}_{\tilde{f}}$,
which can be denoted by $\tilde{F}_{\tilde{f}} = \tilde{G}_{\tilde{g}}$.

3.2 Operations on possibility interval-valued fuzzy soft set

Definition 3.7 *The complement of $\tilde{F}_{\tilde{f}}$, denoted by $\tilde{F}^c_{\tilde{f}}$, is defined by $\tilde{F}^c_{\tilde{f}} = \tilde{G}_{\tilde{g}}$, where $\tilde{G}(e) = \tilde{F}^c(e), \tilde{g}(e) = \tilde{f}^c(e)$.*

From the above definition, we can see that $(\tilde{F}^c_{\tilde{f}})^c = \tilde{F}_{\tilde{f}}$.

Example 3.8 *Consider the PIVFSS $\tilde{G}_{\tilde{g}}$ over (U, E) defined in Example 3.5. Thus, by Definition 3.7, we have*

$$\tilde{G}^c_{\tilde{g}} = \begin{pmatrix} [0.5,0.9],[0.5,0.9] & [0.4,0.7],[0.3,0.5] & [0.5,0.8],[0.3,0.4] \\ [0.4,0.7],[0.8,0.9] & [0.7,0.9],[0.3,0.6] & [0.4,0.7],[0.3,0.6] \\ [0.6,0.9],[0.6,0.7] & [0.5,0.8],[0.6,0.7] & [0.7,0.9],[0.7,0.8] \end{pmatrix}$$

Definition 3.9 *The union operation on the two PIVFSSs* $\tilde{F}_{\tilde{f}}$ *and* $\tilde{G}_{\tilde{g}}$*, denoted by* $\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}$*, is defined by a mapping given by* $\tilde{H}_{\tilde{h}} : E \to IVF(U) \times IVF(U)$*, such that* $\tilde{H}_{\tilde{h}}(e) = (\tilde{H}(e)(x), \tilde{h}(e)(x))$*, where* $\tilde{H}(e) = S(\tilde{F}(e), \tilde{G}(e)) = [S(\mu^-_{\tilde{F}(e)}(x), \mu^-_{\tilde{G}(e)}(x)), S(\mu^+_{\tilde{F}(e)}(x), \mu^+_{\tilde{G}(e)}(x))]$ *and* $\tilde{h}(e) = [S(\mu^-_{\tilde{f}(e)}(x), \mu^-_{\tilde{g}(e)}(x)), S(\mu^+_{\tilde{f}(e)}(x), \mu^+_{\tilde{g}(e)}(x))]$.

Definition 3.10 *The intersection operation on the two PIVFSSs* $\tilde{F}_{\tilde{f}}$ *and* $\tilde{G}_{\tilde{g}}$*, denoted by* $\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}$*, is defined by a mapping given by* $\tilde{H}_{\tilde{h}} : E \to IVF(U) \times IVF(U)$*, such that* $\tilde{H}_{\tilde{h}}(e) = (\tilde{H}(e)(x), \tilde{h}(e)(x))$*, where* $\tilde{H}(e) = T(\tilde{F}(e), \tilde{G}(e)) = [T(\mu^-_{\tilde{F}(e)}(x), \mu^-_{\tilde{G}(e)}(x)), T(\mu^+_{\tilde{F}(e)}(x), \mu^+_{\tilde{G}(e)}(x))]$ *and* $\tilde{h}(e) = [T(\mu^-_{\tilde{f}(e)}(x), \mu^-_{\tilde{g}(e)}(x)), T(\mu^+_{\tilde{f}(e)}(x), \mu^+_{\tilde{g}(e)}(x))]$.

Definition 3.11 *The sum operation on the two PIVFSSs* $\tilde{F}_{\tilde{f}}$ *and* $\tilde{G}_{\tilde{g}}$*, denoted by* $\tilde{F}_{\tilde{f}} \oplus \tilde{G}_{\tilde{g}}$*, is defined by a mapping given by* $\tilde{H}_{\tilde{h}} : E \to IVF(U) \times IVF(U)$*, such that* $\tilde{H}_{\tilde{h}}(e) = (\tilde{H}(e)(x), \tilde{h}(e)(x))$*, where* $\tilde{H}(e) = \tilde{F}(e) \oplus \tilde{G}(e), \tilde{h}(e) = \tilde{f}(e) \oplus \tilde{g}(e)$.

Example 3.12 *Let us consider the PIVFSS* $\tilde{F}_{\tilde{f}}$ *in Example 3.3. Let* $\tilde{G}_{\tilde{g}}$ *be another PIVFSS over* (U, E) *defined as follows:*

$\tilde{G}_{\tilde{g}}(e_1) = \{(x_1, [0.4, 0.8], [0.1, 0.5]), (x_2, [0.5, 0.9], [0.1, 0.5]), (x_3, [0.1, 0.4], [0.3, 0.7])\}$,
$\tilde{G}_{\tilde{g}}(e_2) = \{(x_1, [0.4, 0.7], [0.4, 0.8]), (x_2, [0.3, 0.5], [0.6, 0.8]), (x_3, [0.7, 0.8], [0.3, 0.6])\}$,
$\tilde{G}_{\tilde{g}}(e_3) = \{(x_1, [0.3, 0.6], [0.2, 0.4]), (x_2, [0.1, 0.3], [0.5, 0.7]), (x_3, [0.4, 0.5], [0.7, 0.9])\}$.

If S=sup and T=inf, then we have

$$\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}} = \begin{pmatrix} [0.4,0.8],[0.1,0.8] & [0.6,0.9],[0.7,0.8] & [0.5,0.7],[0.8,0.9] \\ [0.7,0.8],[0.4,0.8] & [0.3,0.5],[0.6,0.8] & [0.7,0.8],[0.6,0.8] \\ [0.4,0.6],[0.5,0.7] & [0.4,0.8],[0.5,0.7] & [0.4,0.6],[0.7,0.9] \end{pmatrix}$$

$$\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}} = \begin{pmatrix} [0.2,0.6],[0.1,0.5] & [0.5,0.8],[0.1,0.5] & [0.1,0.4],[0.3,0.7] \\ [0.4,0.7],[0.1,0.3] & [0.2,0.4],[0.5,0.8] & [0.4,0.7],[0.3,0.6] \\ [0.3,0.6],[0.2,0.4] & [0.1,0.3],[0.5,0.7] & [0.1,0.5],[0.3,0.4] \end{pmatrix}$$

$$\tilde{F}_{\tilde{f}} \oplus \tilde{G}_{\tilde{g}} = \begin{pmatrix} [0.6,1.0],[0.2,1.0] & [1.0,1.0],[0.8,1.0] & [0.6,1.0],[1.0,1.0] \\ [1.0,1.0],[0.5,1.0] & [0.5,0.9],[1.0,1.0] & [1.0,1.0],[0.9,1.0] \\ [0.7,1.0],[0.7,1.0] & [0.5,1.0],[1.0,1.0] & [0.5,1.0],[1.0,1.0] \end{pmatrix}$$

Definition 3.13 *A PIVFSS is said to a possibility* $\tilde{F}-$*empty interval-valued fuzzy soft set, denoted by* $\tilde{F}_{\tilde{0}}$*, if* $\tilde{F}_{\tilde{0}} : E \to IVF(U) \times IVF(U)$*, such that* $\tilde{F}_{\tilde{0}}(e) = (\tilde{F}(e)(x), \tilde{0}(e)(x))$*, where* $\tilde{0}(e) = \varnothing, \forall e \in E$.

If $\tilde{F}(e) = \varnothing$*, then the possibility* $\tilde{F}-$*empty interval-valued fuzzy soft set is called a possibility empty interval-valued fuzzy soft set, denoted by* $\varnothing_{\tilde{0}}$.

Definition 3.14 *A PIVFSS is said to a possibility* $\tilde{F}-$*universal interval-valued fuzzy soft set, denoted by* $\tilde{F}_{\tilde{1}}$*, if* $\tilde{F}_{\tilde{1}} : E \to IVF(U) \times IVF(U)$*, such that* $\tilde{F}_{\tilde{1}}(e) = (\tilde{F}(e)(x), \tilde{1}(e)(x))$*, where* $\tilde{1}(e) = U, \forall e \in E$.

If $\tilde{F}(e) = U$, then the possibility $\tilde{F}-$universal interval-valued fuzzy soft set is called a possibility universal interval-valued fuzzy soft set, denoted by $\tilde{U}_{\tilde{1}}$.

From the above definitions , obviously we have
(1) $\tilde{\varnothing}_{\tilde{0}} \sqsubseteq \tilde{F}_{\tilde{0}} \sqsubseteq \tilde{F}_{\tilde{f}} \sqsubseteq \tilde{F}_{\tilde{1}} \sqsubseteq \tilde{U}_{\tilde{1}}$,
(2) $\tilde{\varnothing}^c_{\tilde{0}} = \tilde{U}_{\tilde{1}}$, $\tilde{U}^c_{\tilde{1}} = \tilde{\varnothing}_{\tilde{0}}$

Theorem 3.15 *Let $\tilde{F}_{\tilde{f}}$ be a PIVFSS over (U, E), then the following holds:*
(1) $\tilde{F}_{\tilde{f}} \sqcup \tilde{\varnothing}_{\tilde{0}} = \tilde{F}_{\tilde{f}}, \tilde{F}_{\tilde{f}} \sqcap \tilde{\varnothing}_{\tilde{0}} = \tilde{\varnothing}_{\tilde{0}}$,
(2) $\tilde{F}_{\tilde{f}} \sqcup \tilde{U}_{\tilde{1}} = \tilde{U}_{\tilde{1}}, \tilde{F}_{\tilde{f}} \sqcap \tilde{U}_{\tilde{1}} = \tilde{F}_{\tilde{f}}$.

Proof. By Definition 2.5, 2.6, 3.9, and 3.10, the above properties are straightforward. □

Remark 3.16 *Let $\tilde{F}_{\tilde{f}}$ be a PIVFSS over (U, E), if $\tilde{F}_{\tilde{f}} \neq \tilde{U}_{\tilde{1}}$ or $\tilde{F}_{\tilde{f}} \neq \tilde{\varnothing}_{\tilde{0}}$, then $\tilde{F}_{\tilde{f}} \sqcup \tilde{F}^c_{\tilde{f}} \neq \tilde{U}_{\tilde{1}}$, and $\tilde{F}_{\tilde{f}} \sqcap \tilde{F}^c_{\tilde{f}} \neq \tilde{\varnothing}_{\tilde{0}}$.*

Theorem 3.17 *Let $\tilde{F}_{\tilde{f}}, \tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three PIVFSSs over (U, E), then the following holds:*
(1) $\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}} = \tilde{G}_{\tilde{g}} \sqcup \tilde{F}_{\tilde{f}}$,
(2) $\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}} = \tilde{G}_{\tilde{g}} \sqcap \tilde{F}_{\tilde{f}}$,
(3) $\tilde{F}_{\tilde{f}} \sqcup (\tilde{G}_{\tilde{g}} \sqcup \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}) \sqcup \tilde{H}_{\tilde{h}}$,
(4) $\tilde{F}_{\tilde{f}} \sqcap (\tilde{G}_{\tilde{g}} \sqcap \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}) \sqcap \tilde{H}_{\tilde{h}}$.

Proof. The properties follow from Definition 2.5, 2.6, 3.9, and 3.10. □

Theorem 3.18 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSSs over (U, E). Then De-Morgan's laws are valid:*
(1) $(\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \sqcap \tilde{G}^c_{\tilde{g}}$,
(2) $(\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \sqcup \tilde{G}^c_{\tilde{g}}$.

Proof. Suppose that $\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}} = \tilde{H}_{\tilde{h}}$, then for all $e \in E$,
$\tilde{H}(e) = S(\tilde{F}(e), \tilde{G}(e)) = [S(\mu^-_{\tilde{F}(e)}(x), \mu^-_{\tilde{G}(e)}(x)), S(\mu^+_{\tilde{F}(e)}(x), \mu^+_{\tilde{G}(e)}(x))]$ and $\tilde{h}(e)) = [S(\mu^-_{\tilde{f}(e)}(x), \mu^-_{\tilde{g}(e)}(x)), S(\mu^+_{\tilde{f}(e)}(x), \mu^+_{\tilde{g}(e)}(x))]$.
Thus $\tilde{H}^c(e) = [1 - S(\mu^+_{\tilde{F}(e)}(x), \mu^+_{\tilde{G}(e)}(x)), 1 - S(\mu^-_{\tilde{F}(e)}(x), \mu^-_{\tilde{G}(e)}(x))]$ and $\tilde{h}^c(e)) = [1 - S(\mu^+_{\tilde{f}(e)}(x), \mu^+_{\tilde{g}(e)}(x)), 1 - S(\mu^-_{\tilde{f}(e)}(x), \mu^-_{\tilde{g}(e)}(x))]$.
Again suppose that $\tilde{F}^c_{\tilde{f}} \sqcap \tilde{G}^c_{\tilde{g}} = \tilde{I}_{\tilde{i}}$, then for all $e \in E$,

$$\begin{aligned}\tilde{I}(e) &= [T(1 - \mu^+_{\tilde{F}(e)}(x), 1 - \mu^+_{\tilde{G}(e)}(x)), T(1 - \mu^-_{\tilde{F}(e)}(x), 1 - \mu^-_{\tilde{G}(e)}(x))] \\ &= [1 - S(\mu^+_{\tilde{F}(e)}(x), \mu^+_{\tilde{G}(e)}(x)), 1 - S(\mu^-_{\tilde{F}(e)}(x), \mu^-_{\tilde{G}(e)}(x))]\end{aligned}$$

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and

$$\begin{aligned}\tilde{i}(e) &= [T(1-\mu^{+}_{\tilde{f}(e)}(x), 1-\mu^{+}_{\tilde{g}(e)}(x)), T(1-\mu^{-}_{\tilde{f}(e)}(x), 1-\mu^{-}_{\tilde{g}(e)}(x))] \\ &= [1-S(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x)), 1-S(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x))]\end{aligned}$$

We see that $\tilde{H}^{c}_{\tilde{h}} = \tilde{I}_{\tilde{i}}$.

Likewise, the proof of (2) can be made similarly. □

Remark 3.19 *Let $\tilde{F}_{\tilde{f}}$, $\tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three PIVFSSs over (U,E). Then the following does not hold here*

(1) $\tilde{F}_{\tilde{f}} \sqcup (\tilde{G}_{\tilde{g}} \sqcap \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}) \sqcap (\tilde{F}_{\tilde{f}} \sqcup \tilde{H}_{\tilde{h}})$,

(2) $\tilde{F}_{\tilde{f}} \sqcap (\tilde{G}_{\tilde{g}} \sqcup \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}) \sqcup (\tilde{F}_{\tilde{f}} \sqcap \tilde{H}_{\tilde{h}})$.

But if we take standard interval-valued fuzzy operations then distributive property holds.

Definition 3.20 *Let $(\tilde{F}_{\tilde{f}}, A)$ and $(\tilde{G}_{\tilde{g}}, B)$ be two PIVFSSs over (U,E). The "$(\tilde{F}_{\tilde{f}}, A)$ AND $(\tilde{G}_{\tilde{g}}, B)$", denoted by $(\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B)$, is defined by*

$$(\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B) = (\tilde{H}_{\tilde{h}}, A \times B),$$

where $\tilde{H}_{\tilde{h}}(\alpha,\beta) = (\tilde{H}(\alpha,\beta)(x), \tilde{h}(\alpha,\beta)(x))$, for all $(\alpha,\beta) \in A \times B$, such that $\tilde{H}(\alpha,\beta) = [T(\mu^{-}_{\tilde{F}(e)}(x), \mu^{-}_{\tilde{G}(e)}(x)), T(\mu^{+}_{\tilde{F}(e)}(x), \mu^{+}_{\tilde{G}(e)}(x))]$, and $\tilde{h}(\alpha,\beta) = [T(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x)), T(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x))]$.

Definition 3.21 *Let $(\tilde{F}_{\tilde{f}}, A)$ and $(\tilde{G}_{\tilde{g}}, B)$ be two PIVFSSs over (U,E). The "$(\tilde{F}_{\tilde{f}}, A)$ OR $(\tilde{G}_{\tilde{g}}, B)$", denoted by $(\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B)$, is defined by*

$$(\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B) = (\tilde{H}_{\tilde{h}}, A \times B),$$

where $\tilde{H}_{\tilde{h}}(\alpha,\beta) = (\tilde{H}(\alpha,\beta)(x), \tilde{h}(\alpha,\beta)(x))$, for all $(\alpha,\beta) \in A \times B$, such that $\tilde{H}(\alpha,\beta) = [S(\mu^{-}_{\tilde{F}(e)}(x), \mu^{-}_{\tilde{G}(e)}(x)), S(\mu^{+}_{\tilde{F}(e)}(x), \mu^{+}_{\tilde{G}(e)}(x))]$, and $\tilde{h}(\alpha,\beta) = [S(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x)), S(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x))]$.

Theorem 3.22 *Let $(\tilde{F}_{\tilde{f}}, A)$ and $(\tilde{G}_{\tilde{g}}, B)$ be two PIVFSSs over (U,E). Then*

(1) $((\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B))^{c} = (\tilde{F}_{\tilde{f}}, A)^{c} \vee (\tilde{G}_{\tilde{g}}, B)^{c}$,

(2) $((\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B))^{c} = (\tilde{F}_{\tilde{f}}, A)^{c} \wedge (\tilde{G}_{\tilde{g}}, B)^{c}$.

Proof. (1) Suppose that $(\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B) = (\tilde{H}_{\tilde{h}}, A \times B)$, where $\tilde{H}_{\tilde{h}}(\alpha,\beta) = (\tilde{H}(\alpha,\beta)(x), \tilde{h}(\alpha,\beta)(x))$, for all $(\alpha,\beta) \in A \times B$, such that $\tilde{H}(\alpha,\beta) = [T(\mu^{-}_{\tilde{F}(e)}(x), \mu^{-}_{\tilde{G}(e)}(x)), T(\mu^{+}_{\tilde{F}(e)}(x), \mu^{+}_{\tilde{G}(e)}(x))]$ and $\tilde{h}(\alpha,\beta) = [T(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x)), T(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x))]$.

Thus $\tilde{H}^{c}(\alpha,\beta) = [1-T(\mu^{+}_{\tilde{F}(e)}(x), \mu^{+}_{\tilde{G}(e)}(x)), 1-T(\mu^{-}_{\tilde{F}(e)}(x), \mu^{-}_{\tilde{G}(e)}(x))]$ and $\tilde{h}^{c}(\alpha,\beta) = [1-T(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x)), 1-T(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x))]$.

Again suppose that $(\tilde{F}_{\tilde{f}}, A)^c \vee (\tilde{G}_{\tilde{g}}, B)^c = (\tilde{I}_{\tilde{i}}, A \times B)$, where $\tilde{I}_{\tilde{i}}(\alpha,\beta) = (\tilde{I}(\alpha,\beta)(x), \tilde{i}(\alpha,\beta)(x))$, for all $(\alpha,\beta) \in A \times B$, such that

$$\begin{aligned}\tilde{I}(\alpha,\beta) &= [S(1-\mu^{+}_{\tilde{F}(e)}(x), 1-\mu^{+}_{\tilde{G}(e)}(x)), S(1-\mu^{-}_{\tilde{F}(e)}(x), 1-\mu^{-}_{\tilde{G}(e)}(x))]\\ &= [1-T(\mu^{+}_{\tilde{F}(e)}(x), \mu^{+}_{\tilde{G}(e)}(x)), 1-T(\mu^{-}_{\tilde{F}(e)}(x), \mu^{-}_{\tilde{G}(e)}(x))]\end{aligned}$$

and

$$\begin{aligned}\tilde{i}(\alpha,\beta) &= [S(1-\mu^{+}_{\tilde{f}(e)}(x), 1-\mu^{+}_{\tilde{g}(e)}(x)), S(1-\mu^{-}_{\tilde{f}(e)}(x), 1-\mu^{-}_{\tilde{g}(e)}(x))]\\ &= [1-T(\mu^{+}_{\tilde{f}(e)}(x), \mu^{+}_{\tilde{g}(e)}(x)), 1-T(\mu^{-}_{\tilde{f}(e)}(x), \mu^{-}_{\tilde{g}(e)}(x))]\end{aligned}$$

We see that $\tilde{H}^c_{\tilde{h}} = \tilde{I}_{\tilde{i}}$.

Likewise, the proof of (2) can be made similarly. □

Theorem 3.23 *Let $(\tilde{F}_{\tilde{f}}, A)$, $(\tilde{G}_{\tilde{g}}, B)$ and $(\tilde{H}_{\tilde{h}}, C)$ be any three PIVFSSs over (U, E). Then we have*

(1) $(\tilde{F}_{\tilde{f}}, A) \wedge ((\tilde{G}_{\tilde{g}}, B) \wedge (\tilde{H}_{\tilde{h}}, C)) = ((\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B)) \wedge (\tilde{H}_{\tilde{h}}, C)$,

(2) $(\tilde{F}_{\tilde{f}}, A) \vee ((\tilde{G}_{\tilde{g}}, B) \vee (\tilde{H}_{\tilde{h}}, C)) = ((\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B)) \vee (\tilde{H}_{\tilde{h}}, C)$.

Proof. The proof follows from Definition 2.5, 2.6, 3.20 and 3.21. □

Remark 3.24 *Let $(\tilde{F}_{\tilde{f}}, A)$, $(\tilde{G}_{\tilde{g}}, B)$ and $(\tilde{H}_{\tilde{h}}, C)$ be any three PIVFSSs over (U, E) . Then the following does not hold here*

(1) $(\tilde{F}_{\tilde{f}}, A) \wedge ((\tilde{G}_{\tilde{g}}, B) \vee (\tilde{H}_{\tilde{h}}, C)) = ((\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B)) \vee ((\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{H}_{\tilde{h}}, C))$,

(2) $(\tilde{F}_{\tilde{f}}, A) \vee ((\tilde{G}_{\tilde{g}}, B) \wedge (\tilde{H}_{\tilde{h}}, C)) = ((\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B)) \wedge ((\tilde{F}_{\tilde{f}}, A) \vee (\tilde{H}_{\tilde{h}}, C))$.

But if we take standard interval-valued fuzzy operations then distributive property holds.

Remark 3.25 *Let $(\tilde{F}_{\tilde{f}}, A)$ and $(\tilde{G}_{\tilde{g}}, B)$ be two PIVFSSs over (U, E). For all $(\alpha,\beta) \in A \times B$, if $\alpha \neq \beta$, then $(\tilde{G}_{\tilde{g}}, B) \wedge (\tilde{F}_{\tilde{f}}, A) \neq (\tilde{F}_{\tilde{f}}, A) \wedge (\tilde{G}_{\tilde{g}}, B)$, and $(\tilde{G}_{\tilde{g}}, B) \vee (\tilde{F}_{\tilde{f}}, A) \neq (\tilde{F}_{\tilde{f}}, A) \vee (\tilde{G}_{\tilde{g}}, B)$.*

4 Similarity between two possibility interval-valued fuzzy soft sets

In this section, a measure of similarity between two PIVFSSs has been given.

Definition 4.1 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSSs over (U, E). Similarity interval between two PIVFSSs $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$, denoted by $S(\tilde{F}_{\tilde{f}}, \tilde{G}_{\tilde{g}})$, is defined by*

$S(\tilde{F}_{\tilde{f}}, \tilde{G}_{\tilde{g}}) = [\varphi^{-}(\tilde{F}, \tilde{G}) \cdot \psi^{-}(\tilde{f}, \tilde{g}), \varphi^{+}(\tilde{F}, \tilde{G}) \cdot \psi^{+}(\tilde{f}, \tilde{g})]$,

such that

$\varphi^-(\tilde{F},\tilde{G}) = min(\varphi_1(\tilde{F},\tilde{G}),\varphi_2(\tilde{F},\tilde{G})),$
$\varphi^+(\tilde{F},\tilde{G}) = max(\varphi_1(\tilde{F},\tilde{G}),\varphi_2(\tilde{F},\tilde{G})),$
where

$$\varphi_1(\tilde{F},\tilde{G}) = \begin{cases} 0, & \mu^-_{\tilde{F}(e_i)}(x) = \mu^-_{\tilde{G}(e_i)}(x) = 0, \\ \frac{\sum_{i=1}^n \max_{x\in U}\{min(\mu^-_{\tilde{F}(e_i)}(x),\mu^-_{\tilde{G}(e_i)}(x))\}}{\sum_{i=1}^n \max_{x\in U}\{max(\mu^-_{\tilde{F}(e_i)}(x),\mu^-_{\tilde{G}(e_i)}(x))\}}, & otherwise. \end{cases}$$

$$\varphi_2(\tilde{F},\tilde{G}) = \frac{\sum_{i=1}^n \max_{x\in U}\{min(\mu^+_{\tilde{F}(e_i)}(x),\mu^+_{\tilde{G}(e_i)}(x))\}}{\sum_{i=1}^n \max_{x\in U}\{max(\mu^+_{\tilde{F}(e_i)}(x),\mu^+_{\tilde{G}(e_i)}(x))\}},$$

$$\psi^-(\tilde{f},\tilde{g}) = \begin{cases} 0, & \mu^-_{\tilde{f}(e_i)}(x) = \mu^-_{\tilde{g}(e_i)}(x) = 0, \\ \frac{\sum_{i=1}^n \max_{x\in U}\{min(\mu^-_{\tilde{f}(e_i)}(x),\mu^-_{\tilde{g}(e_i)}(x))\}}{\sum_{i=1}^n \max_{x\in U}\{max(\mu^-_{\tilde{f}(e_i)}(x),\mu^-_{\tilde{g}(e_i)}(x))\}}, & otherwise. \end{cases}$$

$$\psi^+(\tilde{f},\tilde{g}) = \frac{\sum_{i=1}^n \max_{x\in U}\{min(\mu^+_{\tilde{f}(e_i)}(x),\mu^+_{\tilde{g}(e_i)}(x))\}}{\sum_{i=1}^n \max_{x\in U}\{max(\mu^+_{\tilde{f}(e_i)}(x),\mu^+_{\tilde{g}(e_i)}(x))\}}.$$

Definition 4.2 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSSs over (U,E). We can write*

$$\bar{s} = \frac{\varphi^-(\tilde{F},\tilde{G})\cdot\psi^-(\tilde{f},\tilde{g}) + \varphi^+(\tilde{F},\tilde{G})\cdot\psi^+(\tilde{f},\tilde{g})}{2}$$

which is called the similarity measure of two PIVFSSs $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$. If $\bar{s} \geq \frac{1}{2}$, we say that two PIVFSS are significantly similar. Otherwise, we say that two PIVFSS are not significantly similar.

From the above definition, we can easily obtain the following theorem.

Theorem 4.3 *Let $\tilde{F}_{\tilde{f}}$, $\tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three PIVFSSs over (U,E). Then the following holds:*
(1) $S(\tilde{F}_{\tilde{f}},\tilde{G}_{\tilde{g}}) = S(\tilde{G}_{\tilde{g}},\tilde{F}_{\tilde{f}})$,
(2) $\tilde{F}_{\tilde{f}} = \tilde{G}_{\tilde{g}} \neq \emptyset \Rightarrow S(\tilde{F}_{\tilde{f}},\tilde{G}_{\tilde{g}}) = [1,1]$,
(3) $\tilde{F}_{\tilde{f}} \sqsubseteq \tilde{G}_{\tilde{g}} \sqsubseteq \tilde{H}_{\tilde{h}} \Rightarrow S(\tilde{F}_{\tilde{f}},\tilde{H}_{\tilde{h}}) \leq S(\tilde{G}_{\tilde{g}},\tilde{H}_{\tilde{h}})$.

Example 4.4 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two PIVFSSs over (U,E), respectively, defined as follows:*
$\tilde{F}_{\tilde{f}}(e_1) = \{(x_1,[0.4,0.7],[0.1,0.3]),(x_2,[0.1,0.5],[0.2,0.7]),(x_3,[0.5,0.6],[0.1,0.3])\}$,
$\tilde{F}_{\tilde{f}}(e_2) = \{(x_1,[0.1,0.5],[0.2,0.6]),(x_2,[0.3,0.4],[0.1,0.4]),(x_3,[0.8,0.9],[0.2,0.6])\}$,
$\tilde{F}_{\tilde{f}}(e_3) = \{(x_1,[0.3,0.6],[0.4,0.7]),(x_2,[0.5,0.9],[0.4,0.7]),(x_3,[0.3,0.7],[0.4,0.9])\}$,
$\tilde{G}_{\tilde{g}}(e_1) = \{(x_1,[0.5,0.7],[0.2,0.6]),(x_2,[0.0,0.4],[0.1,0.3]),(x_3,[0.3,0.5],[0.4,0.6])\}$,
$\tilde{G}_{\tilde{g}}(e_2) = \{(x_1,[0.4,0.8],[0.1,0.5]),(x_2,[0.8,0.9],[0.2,0.5]),(x_3,[0.4,0.7],[0.5,0.8])\}$,
$\tilde{G}_{\tilde{g}}(e_3) = \{(x_1,[0.4,0.7],[0.5,0.8]),(x_2,[0.4,0.7],[0.2,0.6]),(x_3,[0.6,0.9],[0.4,0.7])\}$.

By Definition 4.1, we have $\varphi_1(\tilde{F},\tilde{G}) = 0.63, \psi_1(\tilde{f},\tilde{g}) = 0.5, \varphi_2(\tilde{F},\tilde{G}) = 0.84$, *and* $\psi_2(\tilde{f},\tilde{g}) = 0.67$. *Then* $\varphi^-(\tilde{F},\tilde{G}) = 0.63, \psi^-(\tilde{f},\tilde{g}) = 0.5, \varphi^+(\tilde{F},\tilde{G}) = 0.84$, *and* $\psi^+(\tilde{f},\tilde{g}) = 0.67$.

Hence, the similarity interval between the two PIVFSSs $\tilde{F}_{\tilde{f}}$ *and* $\tilde{G}_{\tilde{g}}$ *will be*

$$S(\tilde{F}_{\tilde{f}},\tilde{G}_{\tilde{g}}) = [\varphi^-(\tilde{F},\tilde{G})\cdot\psi^-(\tilde{f},\tilde{g}), \varphi^+(\tilde{F},\tilde{G})\cdot\psi^+(\tilde{f},\tilde{g})] = [0.32, 0.56].$$

Here the similarity measure $\bar{s} = 0.44 < \frac{1}{2}$. *So* $\tilde{F}_{\tilde{f}}$ *and* $\tilde{G}_{\tilde{g}}$ *aren't significantly similar.*

5 Application of possibility interval-valued fuzzy soft set

In this section an application of AND operation of PIVFSS theory in a decision making problem is shown below.

Assume that a company want to fill a position. There are three candidates who form the set of alternatives, $U = \{x_1, x_2, x_3\}$. The hiring committee consider a set of parameters, $E = \{e_1, e_2, e_3\}$. The parameters $e_i (i = 1, 2, 3)$ stand for "experience", "computer knowledge" and "young age", respectively. Suppose the company wants to select one such candidate depending on any two of the parameters only. Let there be two observations $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ by two experts A and B, respectively, defined as follows:

$\tilde{F}_{\tilde{f}}(e_1) = \{(x_1, [0.2, 0.5], [0.6, 0.8]), (x_2, [0.2, 0.4], [0.1, 0.3]), (x_3, [0.4, 0.5], [0.5, 0.8])\}$,
$\tilde{F}_{\tilde{f}}(e_2) = \{(x_1, [0.6, 0.8], [0.1, 0.4]), (x_2, [0.3, 0.5], [0.4, 0.7]), (x_3, [0.6, 0.9], [0.1, 0.4])\}$,
$\tilde{F}_{\tilde{f}}(e_3) = \{(x_1, [0.6, 0.9], [0.6, 0.7]), (x_2, [0.4, 0.6], [0.3, 0.6]), (x_3, [0.3, 0.5], [0.4, 0.6])\}$.
$\tilde{G}_{\tilde{g}}(e_1) = \{(x_1, [0.2, 0.4], [0.5, 0.7]), (x_2, [0.3, 0.6], [0.0, 0.2]), (x_3, [0.2, 0.3], [0.6, 0.7])\}$,
$\tilde{G}_{\tilde{g}}(e_2) = \{(x_1, [0.5, 0.9], [0.2, 0.3]), (x_2, [0.4, 0.5], [0.5, 0.6]), (x_3, [0.7, 0.8], [0.2, 0.3])\}$,
$\tilde{G}_{\tilde{g}}(e_3) = \{(x_1, [0.7, 0.8], [0.5, 0.8]), (x_2, [0.5, 0.7], [0.4, 0.5]), (x_3, [0.4, 0.5], [0.5, 0.6])\}$.

Here, we use AND operation since both experts A and B's opinions have to be considered. By Definition 3.20, if T=inf, we have $(\tilde{F}_{\tilde{f}}, A)$ AND $(\tilde{G}_{\tilde{g}}, B) = (\tilde{H}_{\tilde{h}}, A\times B)$, where

$\tilde{H}_{\tilde{h}}(e_1, e_1) = \{(x_1, [0.2, 0.4], [0.5, 0.7]), (x_2, [0.2, 0.4], [0.0, 0.2]), (x_3, [0.2, 0.3], [0.5, 0.7])\}$,
$\tilde{H}_{\tilde{h}}(e_1, e_2) = \{(x_1, [0.2, 0.5], [0.2, 0.3]), (x_2, [0.2, 0.4], [0.1, 0.3]), (x_3, [0.4, 0.5], [0.2, 0.3])\}$,
$\tilde{H}_{\tilde{h}}(e_1, e_3) = \{(x_1, [0.2, 0.5], [0.5, 0.8]), (x_2, [0.2, 0.4], [0.1, 0.3]), (x_3, [0.4, 0.5], [0.5, 0.6])\}$.
$\tilde{H}_{\tilde{h}}(e_2, e_1) = \{(x_1, [0.2, 0.4], [0.1, 0.4]), (x_2, [0.3, 0.5], [0.0, 0.2]), (x_3, [0.2, 0.3], [0.1, 0.4])\}$,
$\tilde{H}_{\tilde{h}}(e_2, e_2) = \{(x_1, [0.5, 0.8], [0.1, 0.3]), (x_2, [0.3, 0.5], [0.4, 0.6]), (x_3, [0.6, 0.8], [0.1, 0.3])\}$,
$\tilde{H}_{\tilde{h}}(e_2, e_3) = \{(x_1, [0.6, 0.8], [0.1, 0.4]), (x_2, [0.3, 0.5], [0.4, 0.5]), (x_3, [0.4, 0.5], [0.1, 0.4])\}$,
$\tilde{H}_{\tilde{h}}(e_3, e_1) = \{(x_1, [0.2, 0.4], [0.5, 0.7]), (x_2, [0.3, 0.6], [0.0, 0.2]), (x_3, [0.2, 0.3], [0.4, 0.6])\}$,
$\tilde{H}_{\tilde{h}}(e_3, e_2) = \{(x_1, [0.5, 0.9], [0.2, 0.3]), (x_2, [0.4, 0.5], [0.3, 0.6]), (x_3, [0.3, 0.5], [0.2, 0.3])\}$,
$\tilde{H}_{\tilde{h}}(e_3, e_3) = \{(x_1, [0.6, 0.8], [0.5, 0.7]), (x_2, [0.4, 0.6], [0.3, 0.5]), (x_3, [0.3, 0.5], [0.4, 0.6])\}$.

Now, to determine the best candidate, we first compute the numerical grade $r_{ij}(x_k)$ and the corresponding possibility grade $\lambda_{ij}(x_k)$ for each (e_i, e_j) such that

Table 1: Numerical grade and possibility grade

	(e_1,e_1)	(e_1,e_2)	(e_1,e_3)	(e_2,e_1)	(e_2,e_2)	(e_2,e_3)	(e_3,e_1)	(e_3,e_2)	(e_3,e_3)
x_1	0.1,1.0	-0.1,0.1	-0.1,1.1	-0.1,0.3	0.4,-0.6	1.1,-0.4	-0.2,1.2	1.1,-0.4	1.0,0.6
x_2	0.1,-2.0	-0.4,-0.2	-0.4,-1.6	0.5,-0.6	-1.1,1.2	-0.7,0.8	0.7,-1.8	-0.4,0.8	-0.2,-0.6
x_3	-0.2,1.0	0.5,0.1	0.5,0.5	-0.4,0.3	0.7,-0.6	-0.4,-0.4	-0.5,0.6	-0.7,-0.4	-0.8,0.0

Table 2: Grade table

	(e_1,e_1)	(e_1,e_2)	(e_1,e_3)	(e_2,e_1)	(e_2,e_2)	(e_2,e_3)	(e_3,e_1)	(e_3,e_2)	(e_3,e_3)
x_i	x_1,x_2	x_3	x_3	x_2	x_3	x_1	x_2	x_1	x_1
Highest grade	$\times$	0.5	0.5	0.5	$\times$	1.1	0.7	1.1	$\times$
Possibility grade		0.1	0.5	-0.6		-0.4	-1.8	-0.4	

$$r_{ij}(x_k) = \sum_{x\in U}((c_k^- - \mu^-_{\tilde{H}(e_i,e_j)}(x)) + (c_k^+ - \mu^+_{\tilde{H}(e_i,e_j)}(x)),$$

$$\lambda_{ij}(x_k) = \sum_{x\in U}((c_k^- - \mu^-_{\tilde{h}(e_i,e_j)}(x)) + (c_k^+ - \mu^+_{\tilde{h}(e_i,e_j)}(x)).$$

The result is shown in Tables 1.

Now, we mark the highest numerical grade in each column excluding the columns which are the possibility grade of such belongingness of a candidate against each pair of parameters (see Table 2). Now, the score of each such candidate is calculated by taking the sum of the products of these numerical grades with the corresponding possibility λ_{ij}. The candidate with the highest score is the desired person. We do not consider the numerical grades of the candidate against the pairs $(e_i, e_i), i = 1, 2, 3$, as both the parameters are the same:

$Score(x_1) = 1.1 \times (-0.4) + 1.1 \times (-0.4) = -0.88$,
$Score(x_2) = 0.5 \times (-0.6) + 0.7 \times (-1.8) = -1.56$,
$Score(x_3) = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3$.

The firm will select the candidate with the highest score. Hence, they will select the candidate x_3.

6 Conclusion

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainty. However, it is difficult to be used to represent the fuzziness of problem. In order to handle these types of problem parameters, some fuzzy extensions of soft set theory are presented, yielding fuzzy soft set theory. In this paper, the notion of possibility interval-valued fuzzy soft set theory is proposed. Our possibility interval-valued fuzzy soft set theory is a combination of a possibility fuzzy soft set theory and an interval-

valued fuzzy set theory. In other words, our possibility interval-valued fuzzy soft set theory is an extension of generalized interval-valued fuzzy soft set theory and possibility fuzzy soft set theory. The basic properties of the possibility interval-valued fuzzy soft sets are also presented and discussed. Similarity measure of two possibility interval-valued fuzzy soft sets is discussed. Finally, an application of this theory has been applied to solve a decision making problem.

In further research, the parameterization reduction of possibility interval-valued fuzzy soft sets is an important and interesting issue to be addressed.

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On General System of Generalized Quasi-variational-like Inclusions with Maximal η-monotone Mappings in Hilbert Spaces

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Abstract. In this paper, we consider a new general system of generalized quasi-variational-like inclusions in Hilbert spaces. We suggest a new iterative algorithm for finding an approximate solution to the generalized quasi-variational-like inclusion systems, and prove the convergence of the iterative sequence generated by the algorithm. The presented results improve and extend some known results.

Key Words and Phrases. General system of generalized quasi-variational-like inclusion, maximal η-monotone mapping, iterative algorithm, existence, convergence criteria.

AMS Subject Classification. 47H15, 54E70, 47S40.

1 Introduction

Let H_i be a Hilbert spaces with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, and $CB(H_i)$ and 2^{H_i} denote the family of all nonempty closed bounded subsets of H_i and the family of all subsets of H_i for $i = 1, 2, \cdots, m$, respectively. Without loss of generality, we suggest that for all $i = 1, 2, \cdots, m$, $g_i : H_i \to H_i$, $\eta_i : H_i \times H_i \to H_i$, $F_i, T_i : H_1 \times H_2 \times H_3 \times \cdots \times H_m \to H_i$ are single -valued mappings, $A_{ij} : H_j \to CB(H_j)$ $(j = 1, 2, \cdots, m)$ is a multivalued mapping, and $M_i : H_i \to 2^{H_i}$ is a maximal η_i-monotone mapping. We consider the following general system of generalized quasi-variational-like inclusion problem: find $(x_1^*, x_2^*, \cdots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$ such that

$$\begin{aligned} 0 \in\ & g_i(x_i^*) - g_{i+1}(x_{i+1}^*) \\ & + \rho_i(T_i(x_1^*, x_2^*, \cdots, x_m^*) + F_i(u_{i1}, u_{i2}, \cdots, u_{im}) + M_i(g_i(x_i^*))), \end{aligned} \tag{1.1}$$

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where $i = 1, 2, \cdots, m$, $g_{m+1}(x^*_{m+1}) = g_1(x^*_1)$, ρ_i is a positive constant.

Some special cases of problem (1.1) had been studied by many authors. See, for example, [1–10, 12–20] and the references therein. Here, we mention some of them as follows:

Case 1. If $m = 3$, then problem (1.1) reduces to the following problem: find $(x^*, y^*, z^*) \in H_1 \times H_2 \times H_3$, $u_i \in A_i(x^*), v_i \in B_i(y^*), w_i \in C_i(z^*), i = 1, 2, 3,$ such that

$$\begin{array}{l} 0 \in g_1(x^*) - g_2(y^*) + \rho_1(T_1(x^*, y^*, z^*) + F_1(u_1, v_1, w_1) + M_1(g_1(x^*))), \\ 0 \in g_2(y^*) - g_3(z^*) + \rho_2(T_2(x^*, y^*, z^*) + F_2(u_2, v_2, w_2) + M_2(g_2(y^*))), \\ 0 \in g_3(z^*) - g_1(x^*) + \rho_3(T_3(x^*, y^*, z^*) + F_1(u_3, v_3, w_3) + M_3(g_3(z^*))), \end{array} \quad (1.2)$$

which is called a system of generalized quasi-variational-like inclusion considered by Qiu and Liu [14].

Case 2. When $m = 2$, and $H_i = H$, $F_i = O$, $T_i = T$, $g_i = g$, $M_i = M(i = 1, 2)$, where O is zero mapping on the H, then problem (1.1) becomes to the following problem: decide elements $x^*, y^* \in H$ such that

$$\begin{array}{l} 0 \in g(x^*) - g(y^*) + \rho_1(T(y^*) + M(g(x^*))), \\ 0 \in g(y^*) - g(x^*) + \rho_2(T(x^*) + M(g(y^*))), \end{array} \quad (1.3)$$

which was considered by Kazmi and Bhat [9].

Case 3. If $m = 2$, and $H_i = H$, $F_i = O$, $T_i = P_i + Q_i$, $g_i = I$ $(i = 1, 2)$, where I is identity mapping on the H, then problem (1.1) reduces to finding $x^*, y^* \in H$ such that

$$\begin{array}{l} 0 \in x^* - y^* + \rho_1(P_1(y^*) + Q_1(y^*) + M_1(x^*)), \\ 0 \in y^* - x^* + \rho_2(P_2(x^*) + Q_2(x^*) + M_2(y^*)). \end{array} \quad (1.4)$$

Problem (1.4) is called a system of (generalized) nonlinear mixed quasi-variational inclusions, which was considered by Peng and Zhu [12] in Banach spaces and Agarwal et al. [1] in Hilbert spaces, respectively.

Case 4. When $M_i = \partial\varphi_i (i = 1, 2)$, where $\varphi_i : H \to (-\infty, +\infty) \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals on H and $\partial\varphi_i$ denote the sub-differentials of the operators φ_i for $i = 1, 2$, then problem (1.4) reduces to the following system: find $x^*, y^* \in H$ such that

$$\begin{array}{l} \langle \rho_1(P_1(y^*) + Q_1(y^*)) + x^* - y^*, s - x^*\rangle \geq \rho_1(\varphi_1(x^*) - \varphi_1(s)), \forall s \in H, \\ \langle \rho_2(P_2(x^*) + Q_2(x^*)) + y^* - x^*, t - y^*\rangle \geq \rho_2(\varphi_2(y^*) - \varphi_2(t)), \forall t \in H, \end{array} \quad (1.5)$$

which is called a system of generalized nonlinear variational inequalities. If $\varphi_1 = \varphi_2 = \varphi$, then problem (1.5) changes into the system of generalized nonlinear mixed variational inequalities, which was dealt by Kim and Kim [10].

Case 5. If $Q_i = O(i = 1, 2)$, then problem (1.5) reduces to the following problem: find $x^*, y^* \in H$ such that

$$\begin{array}{l} \langle \rho_1 P_1(y^*) + x^* - y^*, s - x^*\rangle \geq \rho_1(\varphi(x^*) - \varphi(s)), \quad \forall s \in H, \\ \langle \rho_2 P_2(x^*) + y^* - x^*, t - y^*\rangle \geq \rho_2(\varphi(y^*) - \varphi(t)), \quad \forall t \in H. \end{array} \quad (1.6)$$

Problem (1.6) was introduced and studied by Verma [17]. Further, in problem (1.6), when φ is the indicator function of a nonempty closed convex set in H defined by

$$\varphi(y) = \begin{cases} 0, & y \in K, \\ \infty, & y \notin K, \end{cases}$$

then the system (1.6) reduces to the following system: find $x^*, y^* \in H$ such that

$$\begin{aligned} &\langle \rho_1 P_1(y^*) + x^* - y^*, s - x^* \rangle \geq 0, \quad \forall s \in H, \\ &\langle \rho_2 P_2(x^*) + y^* - x^*, t - y^* \rangle \geq 0, \quad \forall t \in H, \end{aligned} \tag{1.7}$$

which was introduced and studied by Verma [16, 18, 19].

We remark that for appropriate and suitable choices of the mappings g_i, η_i, F_i, T_i, A_{ij}, M_i $(i, j = 1, 2, \cdots, m)$ and positive integer m, problem (1.1) includes a number of known class of problems of variational inequality, variational inclusion and system of variational inclusion, which were studied previously by many authors. For more details, see [1, 3–10, 12, 14–20] and the references therein. Moreover, Cao [2], Peng et al. [12, 13], and other researchers constructed some N-step iterative algorithms for dealing the related problems of variational inclusion systems.

Inspired and motivated by recent works, the purpose of this paper is to study a new general system of generalized quasi-variational-like inclusion in Hilbert spaces. Further, we construct a new iterative algorithm for finding an approximate solution to this system and discuss the convergence analysis of this algorithm.

2 Preliminaries

In the sequel, we give some concepts and lemmas needed later.

Definition 2.1. Let $\eta : H \times H \to H$ be a single-valued operator. Then the multivalued mapping $M : H \to CB(H)$ is said to be

(i) η-monotone, if for any $x, y \in H$, $u \in Mx$, $v \in My$, $\langle u - v, \eta(x, y) \rangle \geq 0$;

(ii) maximal η-monotone, if M is η-monotone and $(I + \rho M)H = H$ for any $\rho > 0$;

(iii) l-$\hat{\mathbf{H}}$- Lipschitz continuous, if there exists a constant $l > 0$ such that $\hat{\mathbf{H}}(Mx, My) \leq l\|x - y\|$ for all $x, y \in H$, where $\hat{\mathbf{H}}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Definition 2.2. A single-valued operator $g : H \to H$ is said to be

(i) monotone, if for all $x, y \in H$, $\langle g(x) - g(y), x - y \rangle \geq 0$;

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that $\langle g(x) - g(y), x - y \rangle \geq \alpha\|x - y\|^2$ for all $x, y \in H$;

(iii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that $\|g(x) - g(y)\| \leq \beta\|x - y\|$ for all $x, y \in H$.

Definition 2.3. Let $T_i : H_1 \times H_2 \times H_3 \times \cdots \times H_m \to H_i$ and $g_i : H_i \to H_i$ be single -valued mappings for $i = 1, 2, \cdots, m$. Then T_i is said to be

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(i) $(\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{im})$-Lipschitz continuous, if there exists constant $\zeta_{ij} > 0$ such that $\|T_i(x_1, x_2, \cdots, x_m) - T_i(y_1, y_2, \cdots, y_m)\| \leq \sum_{j=1}^{m} \zeta_{ij} \|x_j - y_j\|$ for all $x_j, y_j \in H_j$, $j = 1, 2, \cdots, m$;

(ii) monotone with respect to g_j in the j-th argument, if for $x_j, y_j \in H_j$, $j = 1, 2, \cdots, m$,

$$\begin{aligned}&\langle T_i(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_m)\\ &\quad -T_i(x_1, \cdots, x_{j-1}, y_j, x_{j+1}, \cdots, x_m), g_j(x_j) - g_j(y_j)\rangle \geq 0;\end{aligned}$$

(iii) k_j-strongly monotone with respect to g_j in the j-th argument, if there exists constant $k_j > 0$ such that for $x_j, y_j \in H_j$, $j = 1, 2, \cdots, m$,

$$\begin{aligned}\langle T_i(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_m) - T_i(x_1, \cdots, x_{j-1}, y_j, x_{j+1}, \cdots, x_m),\\ g_j(x_j) - g_j(y_j)\rangle \geq k_j \|x_j - y_j\|^2.\end{aligned}$$

Definition 2.4. Let $\eta : H \times H \to H$ be a single-valued operator. Then η is said to be

(i) monotone, if for all $x, y \in H$, $\langle \eta(x, y), x - y\rangle \geq 0$;

(ii) δ-strongly monotone, if there exists a constant $\delta > 0$ such that $\langle \eta(x, y), x - y\rangle \geq \delta \|x - y\|^2$ for all $x, y \in H$;

(iii) τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau \|x - y\|$ for all $x, y \in H$.

Definition 2.5. Let $\eta : H \times H \to H$ be a single-valued operator, $M : H \to 2^H$ be a maximal η -monotone mapping. Then the resolvent operator $J_M^\rho : H \to H$ is defined by

$$J_M^\rho(x) = (I + \rho M)^{-1}(x), \quad \forall x \in H,$$

where $\rho > 0$ is a constant.

Lemma 2.1. ([4]) Let $\eta : H \times H \to H$ be a single-valued δ-strongly monotone and τ-Lipschitz continuous operator, $M : H \to 2^H$ be a maximal η-monotone mapping. Then the resolvent operator $J_M^\rho : H \to H$ is $\frac{\tau}{\delta}$-Lipschitz continuous.

Lemma 2.2. Let $\eta_i : H_i \times H_i \to H_i$ be a single-valued operator, $M_i : H_i \to 2^{H_i}$ be a maximal η_i-monotone mapping, $F_i, T_i : H_1 \times H_2 \times H_3 \times \cdots \times H_m \to H_i$ are also single-valued mappings, $A_{ij} : H_j \to CB(H_j)$ be a multivalued mapping for $i, j = 1, 2, \cdots, m$. Then $(x_1^*, x_2^*, \cdots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$ is a solution of problem (1.1) if and only if

$$g_i(x_i^*) = J_{M_i}^{\rho_i}(g_{i+1}(x_{i+1}^*) - \rho_i(F_i(u_{i1}, u_{i2}, \cdots, u_{im}) + T_i(x_1^*, x_2^*, \cdots, x_m^*))), \tag{2.1}$$

where $g_{m+1}(x_{m+1}^*) = g_1(x_1^*)$, $J_{M_i}^{\rho_i} = (I + \rho_i M_i)^{-1}$ and $\rho_i > 0 (i = 1, 2, \cdots, m)$.

Proof. The proof directly follows from the definition of $J_{M_i}^{\rho_i}$ for all $i = 1, 2, \cdots, m$ and so it is omitted. □

Lemma 2.3. ([12]) Let H be a real Hilbert space. Then, for any $x, y \in H$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

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3 Iterative Algorithm and Convergence

Algorithm 3.1. For any given $(x_1^0, x_2^0, \cdots, x_m^0) \in H_1 \times H_2 \times \cdots \times H_m$, take $u_{ij}^0 \in A_{ij}(x_j^0)$, $(i, j = 1, 2, \cdots, m)$. let $p_i^0 = J_{M_i}^{\rho_i}(g_{i+1}(x_{i+1}^0) - \rho_i(F_i(u_{i1}^0, u_{i2}^0, \cdots, u_{im}^0) + T_i(x_1^0, x_2^0, \cdots, x_m^0)))$, $i = 1, 2, \cdots, m$. Since $p_i^0 \in H_i, (i = 1, 2, \cdots, m)$, there exists $x_i^1 \in H_i$, such that $p_i^0 = g_i(x_i^1)(i = 1, 2, \cdots, m)$. By the results of Nadler [11], there exists $u_{ij}^1 \in A_{ij}(x_j^1)$, such that $\|u_{ij}^1 - u_{ij}^0\| \leq (1+1)\hat{\mathbf{H}}(A_{ij}(x_j^1), A_{ij}(x_j^0))$ for all $i, j = 1, 2, \cdots, m$. Let $p_i^1 = J_{M_i}^{\rho_i}(g_{i+1}(x_{i+1}^1) - \rho_i(F_i(u_{i1}^1, u_{i2}^1, \cdots, u_{im}^1) + T_i(x_1^1, x_2^1, \cdots, x_m^1))) \in H_i$ for all $i = 1, 2, \cdots, m$. Thus, there exists $x_i^2 \in H_i$, such that $p_i^1 = g_i(x_i^2)$. By induction, we can define iterative sequences $\{x_i^n\}(i = 1, 2, \cdots, m)$, and $\{u_{ij}^n\}(i, j = 1, 2, \cdots, m)$ satisfying

$$\begin{aligned} g_i(x_i^{n+1}) \quad &= J_{M_i}^{\rho_i}(g_{i+1}(x_{i+1}^n) \\ &\quad -\rho_i(F_i(u_{i1}^n, u_{i2}^n, \cdots, u_{im}^n) + T_i(x_1^n, x_2^n, \cdots, x_m^n))) \end{aligned} \tag{3.1}$$

and

$$\|u_{ij}^{n+1} - u_{ij}^n\| \leq (1 + (n+1)^{-1})\hat{\mathbf{H}}(A_{ij}(x_j^{n+1}), A_{ij}(x_j^n)), \tag{3.2}$$

where $u_{ij}^n \in A_{ij}(x_j^n), i, j = 1, 2, \cdots, m$, and $n = 0, 1, 2, \cdots$.

Theorem 3.1. For $i = 1, 2, \cdots, m$, let $\eta_i : H_i \times H_i \to H_i$ be δ_i-strongly monotone and τ_i -Lipschitz continuous mapping, $g_i : H_i \to H_i$ be α_i-strongly monotone and β_i-Lipschitz continuous mapping, $A_{ij} : H_j \to CB(H_j)$ be l_{ij}-$\hat{\mathbf{H}}$-Lipschitz continuous mapping for $j = 1, 2, \cdots, m$, $M_i : H_i \to 2^{H_i}$ is maximal η_i-monotone mapping. Let mapping $T_i : H_1 \times H_2 \times H_3 \times \cdots \times H_m \to H_i$ be $(\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{im})$-Lipschitz continuous and k_i-strongly monotone with respect to g_{i+1} in the $(i+1)$-th argument. Suppose that $F_i : H_1 \times H_2 \times H_3 \times \cdots \times H_m \to H_i$ is $(\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{im})$-Lipschitz continous mapping, remark to where $g_{m+1} = g_1$. If $\alpha_j > 1$ and there exists constant $\rho_j > 0 (j = 1, 2, \cdots, m)$ such that

$$a \cdot \max_{1 \leq j \leq m}(\sum_{i=1, i \neq j-1}^{m} \rho_i \zeta_{ij} + \sum_{i=1}^{m} \rho_i \lambda_{ij} l_{ij} + (\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2 \zeta_{j-1j}^2)^{\frac{1}{2}}) < 1, \tag{3.3}$$

where $a = \max_{1 \leq i \leq m} \frac{\tau_i}{\delta_i (2\alpha_i - 1)^{\frac{1}{2}}}$, remark to where if $j = 1$, then

$$(\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2 \zeta_{j-1j}^2)^{\frac{1}{2}} = (\beta_1^2 - 2k_m \rho_m + \rho_m^2 \zeta_{m1}^2)^{\frac{1}{2}}.$$

Then problem (1.1) admits a solution $(x_1^*, x_2^*, \cdots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$ and sequences $\{x_i^n\}$ and $\{u_{ij}^n\}$ generated by Algorithm 3.1 strongly converges to $x_i^*(i = 1, 2, \cdots, m)$ and $u_{ij}(i, j = 1, 2, \cdots, m)$, respectively.

Proof. By Lemma 2.3 and strong monotonicity of $g_i(i = 1, 2, \cdots, m)$, we have

$$\begin{aligned} &\|x_i^{n+2} - x_i^{n+1}\|^2 \\ &= \|g_i(x_i^{n+2}) - g_i(x_i^{n+1}) + (x_i^{n+2} - x_i^{n+1} - g_i(x_i^{n+2}) + g_i(x_i^{n+1}))\|^2 \\ &\leq \|g_i(x_i^{n+2}) - g_i(x_i^{n+1})\|^2 \\ &\quad -2\langle (g_i - I)(x_i^{n+2}) - (g_i - I)(x_i^{n+1}), x_i^{n+2} - x_i^{n+1} \rangle \\ &\leq \|g_i(x_i^{n+2}) - g_i(x_i^{n+1})\|^2 - 2(\alpha_i - 1)\|x_i^{n+2} - x_i^{n+1}\|^2, \end{aligned}$$

which implies

$$\|x_i^{n+2} - x_i^{n+1}\| \leq \frac{1}{(2\alpha_i - 1)^{\frac{1}{2}}}\|g_i(x_i^{n+2}) - g_i(x_i^{n+1})\|. \tag{3.4}$$

It follows from Algorithm 3.1 and Lemma 2,1 that

$$\begin{aligned}
&\|g_i(x_i^{n+2}) - g_i(x_i^{n+1})\| \\
&\leq \frac{\rho_i\tau_i}{\delta_i}\|F_i(u_{i1}^{n+1}, u_{i2}^{n+1}, \cdots, u_{im}^{n+1}) - F_i(u_{i1}^n, u_{i2}^n, \cdots, u_{im}^n)\| \\
&\quad + \frac{\tau_i}{\delta_i}\|g_{i+1}(x_{i+1}^{n+1}) - g_{i+1}(x_{i+1}^n) \\
&\qquad -\rho_i(T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^{n+1}, x_{i+2}^{n+1}, \cdots, x_m^{n+1}) \\
&\qquad\qquad -T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^n, x_{i+2}^{n+1}, \cdots, x_m^{n+1}))\| \\
&\quad + \frac{\rho_i\tau_i}{\delta_i}\|T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^n, x_{i+2}^{n+1}, \cdots, x_m^{n+1}) \\
&\qquad\qquad -T_i(x_1^n, \cdots, x_i^n, x_{i+1}^n, x_{i+2}^n, \cdots, x_m^n)\|.
\end{aligned} \tag{3.5}$$

Since T_i is $(\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{im})$-Lipschitz continuous and k_i-strongly monotone with respect to g_{i+1} in the $(i+1)-th$ argument, and g_i is β_i-Lipschitz contionuous, we have

$$\begin{aligned}
&\|g_{i+1}(x_{i+1}^{n+1}) - g_{i+1}(x_{i+1}^n) \\
&\quad -\rho_i(T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^{n+1}, x_{i+2}^{n+1}, \cdots, x_m^{n+1}) \\
&\qquad -T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^n, x_{i+2}^{n+1}, \cdots, x_m^{n+1}))\| \\
&\leq (\beta_{i+1}^2 - 2k_i\rho_i + \rho_i^2\zeta_{i(i+1)}^2)^{\frac{1}{2}}\|x_{i+1}^{n+1} - x_{i+1}^n\|,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\|T_i(x_1^{n+1}, \cdots, x_i^{n+1}, x_{i+1}^n, x_{i+2}^{n+1}, \cdots, x_m^{n+1}) \\
&\quad -T_i(x_1^n, \cdots, x_i^n, x_{i+1}^n, x_{i+2}^n, \cdots, x_m^n)\| \\
&\leq \zeta_{i1}\|x_1^{n+1} - x_1^n\| + \cdots + \zeta_{ii}\|x_i^{n+1} - x_i^n\| \\
&\quad +\zeta_{i(i+2)}\|x_{i+2}^{n+1} - x_{i+2}^n\| + \cdots + \zeta_{im}\|x_m^{n+1} - x_m^n\| \\
&= \sum_{j=1, j\neq i+1}^{m} \zeta_{ij}\|x_j^{n+1} - x_j^n\|.
\end{aligned} \tag{3.7}$$

It follows from the $(\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{im})$-Lipschitz continuity of F_i, the l_{ij}-$\hat{\mathbf{H}}$-Lipschitz continuity of A_{ij} and (3.2) that

$$\begin{aligned}
&\|F_i(u_{i1}^{n+1}, u_{i2}^{n+1}, \cdots, u_{im}^{n+1}) - F_i(u_{i1}^n, u_{i2}^n, \cdots, u_{im}^n)\| \\
&\leq \lambda_{i1}\|u_{i1}^{n+1} - u_{i1}^n\| + \lambda_{i2}\|u_{i2}^{n+1} - u_{i2}^n\| + \cdots + \lambda_{im}\|u_{im}^{n+1} - u_{im}^n\| \\
&\leq \sum_{j=1}^{m} \lambda_{ij}(1 + (n+1)^{-1})\hat{\mathbf{H}}(A_{ij}x_j^{n+1}, A_{ij}x_j^n) \\
&\leq \sum_{j=1}^{m} \lambda_{ij}l_{ij}(1 + (n+1)^{-1})\|x_j^{n+1} - x_j^n\|.
\end{aligned} \tag{3.8}$$

It follows from (3.4)-(3.8) that

$$\begin{aligned}
&\|x_i^{n+2} - x_i^{n+1}\| \\
&\leq \frac{\tau_i}{\delta_i(2\alpha_i - 1)^{\frac{1}{2}}}[(\beta_{i+1}^2 - 2k_i\rho_i + \rho_i^2\zeta_{i(i+1)}^2)^{\frac{1}{2}}\|x_{i+1}^{n+1} - x_{i+1}^n\| \\
&\quad +\rho_i \sum_{j=1, j\neq i+1}^m \zeta_{ij}\|x_j^{n+1} - x_j^n\| \\
&\quad +\rho_i \sum_{j=1}^m \lambda_{ij} l_{ij}(1 + (n+1)^{-1})\|x_j^{n+1} - x_j^n\|]. \qquad (3.9)
\end{aligned}$$

Pay attention to $\beta_m = \beta_1$ and $\zeta_{m(m+1)} = \zeta_{m1}$. We have

$$\begin{aligned}
&\sum_{j=1}^m \|x_j^{n+2} - x_j^{n+1}\| = \sum_{i=1}^m \|x_i^{n+2} - x_i^{n+1}\| \\
&\leq a \cdot \sum_{i=1}^m [(\beta_{i+1}^2 - 2k_i\rho_i + \rho_i^2\zeta_{i(i+1)}^2)^{\frac{1}{2}}\|x_{i+1}^{n+1} - x_{i+1}^n\| \\
&\quad +\rho_i \sum_{j=1, j\neq i+1}^m \zeta_{ij}\|x_j^{n+1} - x_j^n\| \\
&\quad +\rho_i \sum_{j=1}^m \lambda_{ij} l_{ij}(1 + (n+1)^{-1})\|x_j^{n+1} - x_j^n\|] \\
&\leq a \cdot \sum_{j=1}^m [\sum_{i=1, i\neq j-1}^m \rho_i\zeta_{ij} + (1 + (n+1)^{-1})\sum_{i=1}^m \rho_i\lambda_{ij} l_{ij} \\
&\quad +(\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2\zeta_{(j-1)j}^2)^{\frac{1}{2}}]\|x_j^{n+1} - x_j^n\| \\
&\leq \theta_n \sum_{j=1}^m \|x_j^{n+1} - x_j^n\|, \qquad (3.10)
\end{aligned}$$

where

$$\begin{aligned}
\theta_n \quad &= a \cdot \max_{1\leq j\leq m}[\sum_{i=1, i\neq j-1}^m \rho_i\zeta_{ij} + (1 + (n+1)^{-1})\sum_{i=1}^m \rho_i\lambda_{ij} l_{ij} \\
&\quad +(\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2\zeta_{(j-1)j}^2)^{\frac{1}{2}}].
\end{aligned}$$

If $j = 1$, then we get

$$(\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2\zeta_{(j-1)j}^2)^{\frac{1}{2}} = (\beta_1^2 - 2k_m\rho_m + \rho_m^2\zeta_{m1}^2)^{\frac{1}{2}}.$$

Let

$$\theta = a \cdot \max_{1\leq j\leq m}[\sum_{i=1, i\neq j-1}^m \rho_i\zeta_{ij} + \sum_{i=1}^m \rho_i\lambda_{ij} l_{ij} + (\beta_j^2 - 2k_{j-1}\rho_{j-1} + \rho_{j-1}^2\zeta_{(j-1)j}^2)^{\frac{1}{2}}].$$

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Then $\theta_n \downarrow \theta$ as $n \to \infty$. By (3.3), we know that $0 < \theta < 1$ and so (3.10) implies that $\{x_i^n\}(i = 1, 2, \cdots, m)$ are both Cauchy sequences. Thus, there exists $x_i^* \in H_i$ such that $x_i^n \to x_i^*$ as $n \to \infty$ for $i = 1, 2, \cdots, m$. Now, we prove that $u_{ij}^n \to u_{ij} \in A_{ij}(x_j^*)$. In fact, it follow from the l_{ij}-$\hat{\mathbf{H}}$-Lipschitz continuity of $A_{ij}(i, j = 1, 2, \cdots, m)$ and (3.2) that $\|u_{ij}^{n+1} - u_{ij}^n\| \le (1+(n+1)^{-1})l_{ij}\|x_j^{n+1} - x_j^n\|$ for all $i, j = 1, 2, \cdots, m$ and $n = 0, 1, 2, \cdots$. Hence, $\{u_{ij}^n\}$ are also both Cauchy sequences for $i, j = 1, 2, \cdots, m$. Therefore, for any $i, j = 1, 2, \cdots, m$, there exists $u_{ij} \in H_j$, such that $u_{ij}^n \to u_{ij}(n \to \infty)$. Further

$$\begin{aligned} d(u_{ij}, A_{ij}x_j^*) &\le \|u_{ij}^n - u_{ij}\| + d(u_{ij}^n, A_{ij}x_j^*) \\ &\le |u_{ij}^n - u_{ij}\| + \hat{\mathbf{H}}(A_{ij}x_j^n, A_{ij}x_j^*) \\ &\le \|u_{ij}^n - u_{ij}\| + l_{ij}\|x_j^n - x_j^*\| \to 0 \quad (n \to \infty). \end{aligned}$$

Since $A_{ij}(x_j^*)$ is closed, $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$. By continuity of $g_i, F_i, T_i, A_{ij}, J_{M_i}^{\rho_i}$ and Algorithm 3.1, we know that $(x_1^*, x_2^*, \cdots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$ satisfy the relation (2.1). By Lemma 2.2, we claim $(x_1^*, x_2^*, \cdots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij} \in A_{ij}(x_j^*)(i, j = 1, 2, \cdots, m)$ is a solution of the problm (1.1). This completes the proof. □

Remak 3.1. If $m = 3$, Algorithm 3.1 and Theorem 3.1 reduce to Algorithm 2.1 and Theorem 3.1 in [14], respectively. Our results improve and extend the corresponding results in the literature.

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Stability and Convergence of Fourier Pseudospectral Method for Generalized Zakharov Equations*

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Abstract

In this article a Fourier pseudospectral method for the generalized Zakharov equations is applied. Fully discrete pseudospectral scheme is developed. Convergence of the pseudospectral scheme is proved by energy estimation method. By using convergence theorem stability of the fully discrete scheme is proved. Numerical results of the fully discrete scheme are compare with the results already available in the literature to check the efficiency of the proposed method.

Key words: Generalized Zakharov equations, Fourier pseudospectral method, convergence, stability.

1 Introduction

Generalized Zakharov equations have the following partial differential equations form [1]:

$$i\partial_t u + \partial_x^2 u + \alpha(|u|^2)u + \beta uw = 0, \qquad x \in \Omega, t \geq 0, \tag{1.1}$$

$$\partial_t v + \partial_x w - \gamma\partial_x(|u|^2) = 0, \qquad x \in \Omega, t \geq 0, \tag{1.2}$$

$$\partial_t w + \partial_x v = 0, \qquad x \in \Omega, t \geq 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \qquad x \in \Omega, \tag{1.4}$$

$$u(-L,t) = u(L,t), \quad v(-L,t) = v(L,t), \quad w(-L,t) = w(L,t), \qquad t \geq 0, \tag{1.5}$$

where $u(x,t)$, $v(x,t)$ and $w(x,t)$ to represent the envelop of the high frequency electric field, the plasma density from its equilibrium value and the deviation of the ion density from its equilibrium value respectively. $i^2 = -1$, the parameters α, β and γ are real constants. Here $\partial_x = \frac{\partial}{\partial x}$, $\partial_x^2 = \frac{\partial^2}{\partial x^2}$, $\partial_t = \frac{\partial}{\partial t}$ and $\Omega = [-L, L]$.

The derivation of the generalized Zakharov equations can take place from hydrodynamics description of the plasma [2, 3]. The above equations (1.1)–(1.5) represent a universal model for the study of the interaction between non-dispersive and dispersive waves. The system reduced to the classical Zakharov equations of plasma physics when $\alpha = 0$, $\beta = 2$ and $\gamma = 1$.

Numerical methods for the generalized Zakharov equations were studied in the last two decades. For example, an energy-preserving implicit finite difference scheme for the generalized Zakharov equations presented by Glassey and proved the convergence of the scheme [4, 5]. An implicit or semi explicit conservative finite difference scheme for the Zakharov equations were developed by Chang and Jiang [6], while they extended their method for the generalized Zakharov equations in [7]. The various powerful methods have been study for the Zakharov equations or the generalized Zakharov equations, such as Homotopy perturbation method [9, 30], Adomain decomposition method [8], Variational iteration methods [10, 11, 12, 13, 14, 15, 16, 17].

An exact and approximate analytical solution for nonlinear problems have been developed by some authors in [18, 19, 20]. A time splitting spectral scheme to solve the generalized Zakharov system (GZS) was proposed by Bao et al. [26], while the vector GZS for multi-component plasmas was solved by [27]. Ma Shuqing and Chang Qianshun [24] has studied the dissipative Zakharov equations, in which they apply pseudospectral method and proved the convergence by priori estimates.

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A spectral method for one dimensional Zakharov system (ZS) was designed by Payne et al. [21]. An error estimation of semi-discrete and fully discrete of pseudospectral schemes for (1.1)-(1.5) proved by Guangye in [22]. The time splitting spectral methods for the generalized Zakharov system was studied by Shi Jin et al. [25]. The Chebyshev cardinal functions for Klein-Gordon-Zakharov Equations were used by Ghoreishi et al. [31].

The initial periodic boundary-value problem of generalized Zakharov equations (1.1)-(1.5) is considered in this article. A second order finite difference approximation is used in time direction, while the pseudospectral method is applied in space for generalized Zakharov equations. An energy estimation method is used for analysis of error estimates of fully discrete pseudospectral scheme. $O(\tau^2+N^{-S})$ is the rate of convergence of the resulting pseudospectral scheme, where τ is the step length in time direction, S is depending on the smoothness of the exact solution and N is the number of spatial Fourier modes. The stability of the fully discrete pseudospectral scheme is proved by using convergence theorem and numerical results are presented.

2 Notations and Lemmas

Suppose $H^S(\Omega)$ represent the Sobolev space having norm $\|\Psi\|_S^2=\sum\limits_{\ell=0}^{S}\left\|\frac{\partial^\ell\Psi}{\partial x^\ell}\right\|^2$. Let $L^\infty(\Omega)$ indicate the space of Lebesgue measure with norm $\|\Psi\|_\infty=\operatorname{ess\,sup}_{x\in\Omega}|\Psi(x)|$. Assume $L^2(\Omega)$ is the L^2 space with norm $\|\Psi\|^2=(\Psi,\Psi)$ and the associated inner product is defined as $(\Psi,\Phi)=\int_{-L}^{L}\Psi(x)\Phi(x)dx$. The discrete inner product and norm are defined as follows:

$$(\Psi,\Phi)_N=h\sum_{\ell=0}^{N}\sum_{\ell=0}^{N}\Psi(x_\ell)\Phi(x_\ell),\qquad \|\Psi\|_N=(\Psi,\Psi)_N^{1/2},\quad h=\frac{1}{N+1}.$$

Space of trigonometric polynomials of degree N is denoted by S_N and is defined as:

$$S_N=\operatorname{span}\left\{\frac{1}{\sqrt{L}}\exp\left(\frac{i\pi x_j}{L}\right):j=0,1,\cdots,N\right\},$$

$$-L=x_0<x_1<x_2,\cdots,<x_N=L,\quad \text{with}\quad x_j=L\left(\frac{2j}{N}-1\right),\quad j=0,1,\cdots,N,$$

where N is an even number. The orthogonal projection operator $P_N:L^2(\Omega)\longrightarrow S_N$ is defined as

$$(P_N\Psi,\Phi)=(\Psi,\Phi),\quad \forall\Phi\in S_N.$$

The interpolation operator $P_c:C(\Omega)\longrightarrow S_N$ is also defined as

$$P_c\Psi(x_\ell)=\Psi(x_\ell),\qquad 0\le\ell\le N.$$

Suppose $R_\tau=\left\{t=k\tau:0\le k\le\left[\frac{T}{\tau}\right]\right\}$, where τ is the step length along t direction. Assume $\Psi^k(x)=\Psi(x,k\tau)=\Psi^k$ for simplicity. Define

$$\Psi^k_{\hat{t}}=\frac{1}{2\tau}(\Psi^{k+1}-\Psi^{k-1}),$$
$$\widehat{\Psi}^k=\frac{1}{2}(\Psi^{k+1}+\Psi^{k-1}).$$

Lemma 2.1. *[23] Suppose $\Psi\in H^S(\Omega)$,$0\le\mu\le S$, $\exists$ a constant $\digamma$ not depend on Ψ and N*

$$\|\Psi-P_N\Psi\|_\mu\le\digamma N^{\mu-S}\|\Psi\|_S.$$

Lemma 2.2. *[23] Suppose $\Psi\in H^S(\Omega)$,$0\le\mu\le S$, $\exists$ a constant $\digamma$ not depend on Ψ and N.*

$$\|\Psi-P_c\Psi\|_\mu\le\digamma N^{\mu-S}\|\Psi\|_S.$$

Lemma 2.3. *[28] If $\Psi, \Phi \in F(\Omega)$, then*

$$(P_c\Psi, P_c\Phi)_N = (P_c\Psi, P_c\Phi) = (\Psi, \Phi)_N.$$

Lemma 2.4. *[28] If $S \geq 1$, and $\Psi, \Phi \in H^S(\Omega)$, $\exists$ a constant F not depend on Ψ, Φ and N, such that*

$$\|\Psi\Phi\|_S \leq F \|\Psi\|_S \|\Phi\|_S .$$

3 Error Estimates of Fully Discrete Scheme

Consider the fully discrete Fourier pseudospectral scheme for finding: $u_c^k, v_c^k, w_c^k \in S_N$, such that for $k = 1, \ldots, \left[\frac{T}{\tau}\right]$, the equations

$$u_{c\hat{t}}^k + \partial_x^2 \widehat{u}_c^k + \alpha P_c(|u_c^k|^2 \widehat{u}_c^k) + \beta P_c(v_c^k \widehat{w}_c^k) = 0, \tag{3.1}$$

$$v_{c\hat{t}}^k + \partial_x \widehat{w}_c^k - \gamma \partial_x P_c(|u_c^k|^2) = 0, \tag{3.2}$$

$$w_{c\hat{t}}^k + \partial_x \widehat{v}_c^k = 0, \tag{3.3}$$

$$u_c^0(x) = P_N u_0(x), \quad v_c^0(x) = P_N v_0(x), \quad w_c^0(x) = P_N w_0(x), \tag{3.4}$$

$$u_c^1(x) = P_N\left[u_0 + \tau\partial_t u(0)\right], v_c^1(x) = P_N\left[v_0 + \tau\partial_t v(0)\right], w_c^1(x) = P_N\left[w_0 + \tau\partial_t w(0)\right], \tag{3.5}$$

are satisfied at $x = x_\ell, \ell = 0, \ldots, N$. Let

$$\begin{aligned}
e_1^k &= u^k - u_c^k = (u^k - P_N u^k) + (P_N u^k - u_c^k) = \xi_1^k + \eta_1^k, \\
e_2^k &= v^k - v_c^k = (v^k - P_N v^k) + (P_N v^k - v_c^k) = \xi_2^k + \eta_2^k, \\
e_3^k &= w^k - w_c^k = (w^k - P_N w^k) + (P_N w^k - w_c^k) = \xi_3^k + \eta_3^k.
\end{aligned}$$

Note that $(\xi_\ell^k, \psi) = 0, \ell = 1, 2, 3, \forall \psi \in S_N$, subtracting (3.1) from (1.1), (3.2) from (1.2), and (3.3) from (1.3) then η_1^k , η_2^k and η_3^k satisfy the system

$$(e_{1\hat{t}}^k, \psi) + (\partial_x \widehat{e}_1^k, \partial_x \psi) + \alpha(P_c(|u^k|^2 \widehat{u}^k - |u_c^k|^2 \widehat{u}_c^k), \psi) + \beta(P_c(w^k \widehat{u}^k - w_c^k \widehat{u}_c^k), \psi) = (\tau_1^k, \psi), \tag{3.6}$$

$$(e_{2\hat{t}}^k, \psi) + (\widehat{e}_3^k, \partial_x \psi) + \gamma(P_c(|u^k|^2 - |u_c^k|^2), \partial_x \psi) = (\tau_2^k, \psi), \tag{3.7}$$

$$(e_{3\hat{t}}^k, \psi) + (\widehat{e}_2^k, \partial_x \psi) = (\tau_3^k, \psi), \tag{3.8}$$

where τ_1^k, τ_1^k and τ_3^k are truncation errors

$$\begin{aligned}
\tau_1^k &= \left(u_{\hat{t}}^k - \partial_t u^k\right) + \partial_x^2\left(\widehat{u}^k - u^k\right) + \alpha\left(|u^k|^2 u^k - P_c(|u^k|^2 \widehat{u}^k)\right) + \beta\left(w^k u^k - P_c\left(w^k \widehat{u}^k\right)\right), \\
\tau_2^k &= \left(v_{\hat{t}}^k - \partial_t v^k\right) + \partial_x\left(\widehat{w}^k - w^k\right) + \gamma \partial_x\left(|u^k|^2 - P_c\left(|u^k|^2\right)\right), \\
\tau_3^k &= \left(w_{\hat{t}}^k - \partial_t w^k\right) + \partial_x\left(\widehat{v}^k - v^k\right).
\end{aligned}$$

By using Lemma 2.3 and Taylor's theorem, we get

$$\begin{aligned}
\tau_1^k =& \frac{\tau^2}{12}\left(\partial_t^3 u(t_1^k) + \partial_t^3 u(t_2^k)\right) + \frac{\tau^2}{4}\left(\partial_t^2(\partial_x^2 u(t_3^k)) + \partial_t^2(\partial_x^2 u(t_4^k))\right) + \alpha\left(|u^k|^2 u^k - P_c(|u^k|^2 \widehat{u}^k)\right) \\
&+ \beta\left(w^k u^k - P_c\left(w^k \widehat{u}^k\right)\right), \\
\tau_2^k =& \frac{\tau^2}{12}\left(\partial_t^3 v(t_5^k) + \partial_t^3 v(t_6^k)\right) + \frac{\tau^2}{6}\left(\partial_t^2(\partial_x^2 v(t_7^k)) + \partial_t^2(\partial_x^2 v(t_8^k))\right) + \partial_x\left(|u^k|^2 - P_c\left(|u^k|^2\right)\right), \\
\tau_3^k =& \frac{\tau^2}{12}\left(\partial_t^3 w(t_9^k) + \partial_t^3 w(t_{10}^k)\right) + \frac{\tau^2}{4}\left(\partial_t^2 w(t_{11}^k)) + \partial_t^2 w(t_{12}^k))\right),
\end{aligned}$$

where $t^{k-1} \leq t_\ell^k \leq t^{k+1}$, $\ell = 1, 2, ..., 12$. The imaginary part and real part of complex function is denoted by $\Im$ and $\Re$ respectively. Setting $\psi = \widehat{\eta}_1^k$ in (3.6) and taking the imaginary parts, we get

$$\frac{1}{4\tau}\left(\|\eta_1^{k+1}\|^2 - \|\eta_1^{k-1}\|^2\right) + \|\partial_x \widehat{\eta}_1^k\|^2 + F_1^k + F_2^k = \Im(\tau_1^k, \widehat{\eta}_1^k), \tag{3.9}$$

where

$$F_1^k = \alpha\Im(P_c(|u^k|^2\widehat{u}^k - |u_c^k|^2\widehat{u}_c^k), \widehat{\eta}_1^k),$$
$$F_2^k = \beta\Im(P_c(w^k\widehat{u}^k - w_c^k\widehat{u}_c^k), \widehat{\eta}_1^k).$$

A general positive constant C not depend on τ and N will be used for errors estimation and this constant can be considered different in different cases. By using algebraic inequality and Cauchy-Schwartz inequality, we can estimate F_1^k, F_2^k and the right hand term of (3.9).

$$|\alpha\Im(P_c(|u^k|^2\widehat{u}^k - |u_c^k|^2\widehat{u}_c^k), \widehat{\eta}_1^k)| \le C(\|P_c(|u^k|^2\widehat{u}^k - (|u_c^k|^2\widehat{u}_c^k))\|^2 + \|\widehat{\eta}_1^k\|^2).$$

By applying Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}\|P_c(|u^k|^2\widehat{u}^k - (|u_c^k|^2\widehat{u}_c^k))\| &= \|P_c(|u^k|^2(\widehat{u}^k - \widehat{u}_c^k)) + P_c(\widehat{u}_c^k(|u^k|^2 - |u_c^k|^2))\| \\ &\le \|P_c(|u^k|^2(\widehat{u}^k - \widehat{u}_c^k))\| + \|P_c(\widehat{u}_c^k(|u^k|^2 - |u_c^k|^2))\| \\ &\le \|u^k\|_\infty\|P_c(\widehat{u}^k - \widehat{u}_c^k)\| + \|u_c^k\|_\infty\|P_c(|u^k|^2 - |u_c^k|^2)\| \\ &\le C(N^{-S} + \|\eta_1^k\| + \|\widehat{\eta}_1^k\|).\end{aligned}$$

Hence

$$|F_1^k| \le C(N^{-2S} + \|\eta_1^k\|^2 + \|\widehat{\eta}_1^k\|^2). \tag{3.10}$$

Similarly by using algebraic inequality and Cauchy-Schwartz inequality, we have

$$|\beta\Im(P_c(w^k\widehat{u}^k - w_c^k\widehat{u}_c^k), \widehat{\eta}_1^k)| \le C(\|P_c(w^k\widehat{u}^k - w_c^k\widehat{u}_c^k)\|^2 + \|\widehat{\eta}_1^k\|^2).$$

By applying Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}\|P_c(w^k\widehat{u}^k - w_c^k\widehat{u}_c^k)\| &= \|P_c(w^k(\widehat{u}^k - \widehat{u}_c^k)) + P_c(\widehat{u}_c^k(w^k - w_c^k))\| \\ &\le \|w^k\|_\infty(CN^{-S}\|u^k\|_s + \|\widehat{\eta}_1^k\|) + \|\widehat{u}_c^k\|_\infty(CN^{-S}\|w^k\|_S + \|\eta_3^k\|) \\ &\le (CN^{-S} + \|\widehat{\eta}_1^k\| + \|\eta_3^k\|).\end{aligned}$$

Therefore

$$|F_2^k| \le C(N^{-2S} + \|\widehat{\eta}_1^k\|^2 + \|\eta_3^k\|^2). \tag{3.11}$$

The right hand term can be estimated by using the Cauchy-Schwartz inequality and Lemma 2.2, we get

$$\begin{aligned}|\Im(\tau_1^k, \widehat{\eta}_1^k)| &\le C(\|\tau_1^k\|^2 + \|\widehat{\eta}_1^k\|^2) \\ &\le C(\|\widehat{\eta}_1^k\|^2 + \tau^4 + |\alpha|\| \left(|u^k|^2u^k - P_c(|u^k|^2\widehat{u}^k)\right)\|^2 \\ &+ |\beta|\| \left(w^ku^k - P_c\left(w^k\widehat{u}^k\right)\right)\|^2 \\ &\le C(\|\widehat{\eta}_1^k\|^2 + \|\eta_3^k\|^2 + \tau^4 + N^{-2S}).\end{aligned} \tag{3.12}$$

Substituting (3.10), (3.11) and (3.12) into (3.9), we obtain

$$\frac{1}{4\tau}\left(\|\eta_1^{k+1}\|^2 - \|\eta_1^{k-1}\|^2\right) + \|\partial_x\widehat{\eta}_1^k\|^2 \le C(\|\eta_1^k\|^2 + \|\widehat{\eta}_1^k\|^2 + \|\eta_3^k\|^2 + \tau^4 + N^{-2S}). \tag{3.13}$$

Now setting $\psi = \widehat{\eta}_2^k$ in (3.7), we get

$$\frac{1}{4\tau}(\|\eta_2^{k+1}\|^2 - \|\eta_2^{k-1}\|^2) + (\widehat{e}_3^k, \partial_x\widehat{\eta}_2^k) + F_3^k = (\tau_2^k, \widehat{\eta}_2^k), \tag{3.14}$$

where

$$F_3^k = \gamma(P_c(|u^k|^2 - |u_c^k|^2), \partial_x\widehat{\eta}_2^k).$$

We are going to estimate F_3^k and the right hand term by using the Taylor's theorem and Cauchy-Schwartz inequality, we get

$$|F_3^k| = |(P_c(|u^k|^2 - |u_c^k|^2), \partial_x \widehat{\eta}_2^k)| \leq (\|P_c(|u^k|^2 - |u_c^k|^2)\|^2 + \|\partial_x \widehat{\eta}_2^k\|^2).$$

Again by using Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned}\| P_c(|u^k|^2 - |u_c^k|^2) \| = \|P_c(u^k\overline{u}^k - u_c^k\overline{u}_c^k)\| &= \|P_c(u^k(\overline{u}^k - \overline{u}_c^k) + \overline{u}_c^k(u^k - u_c^k))\| \\ &\leq \|u^k\|_\infty \|P_c(\overline{u}^k - \overline{u}_c^k)\| + \|\overline{u}_c^k\|_\infty \|P_c(u^k - u_c^k)\| \\ &\leq C(N^{-S} + \|\eta_1^k\|).\end{aligned}$$

Hence

$$|F_3^k| \leq C(N^{-2S} + \|\eta_1^k\|^2 + \|\partial_x \widehat{\eta}_2^k\|^2). \tag{3.15}$$

Similarly the right hand term of (3.14) can be estimated by using the Cauchy-Schwartz inequality and Lemma 2.2, we get

$$\begin{aligned}|(\tau_2^k, \widehat{\eta}_2^k)| &\leq C(\tau^4 + \|\partial_x \left(|u^k|^2 - P_c\left(|u^k|^2\right)\right)\|^2 + \|\widehat{\eta}_2^k\|^2) \\ &\leq C(N^{-2S} + \tau^4 + \|\widehat{\eta}_2^k\|^2).\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we obtain

$$\frac{1}{4\tau}(\|\eta_2^{k+1}\|^2 - \|\eta_2^{k-1}\|^2) \leq C(\tau^4 + N^{-2S} + \|\eta_1^k\|^2 + \|\partial_x \widehat{\eta}_2^k\|^2). \tag{3.17}$$

Now setting $\psi = \widehat{\eta}_3^k$ in (3.8), we get

$$\frac{1}{4\tau}(\|\eta_3^{k+1}\|^2 - \|\eta_3^{k-1}\|^2) + (\widehat{e}_2^k, \partial_x \widehat{\eta}_3^k) = (\tau_3^k, \widehat{\eta}_3^k), \tag{3.18}$$

The second term and right hand term of (3.18) can be estimated by the same procedure as applied in (3.15) and (3.16), we get

$$\frac{1}{4\tau}\left(\|\eta_3^{k+1}\|^2 - \|\eta_3^{k-1}\|^2\right) + \|\partial_x \widehat{\eta}_3^k\|^2 \leq C(\tau^4 + N^{-2S} + \|\widehat{\eta}_2^k\|^2 + \|\widehat{\eta}_2^k\|^2 + \|\partial_x \widehat{\eta}_3^k\|^2). \tag{3.19}$$

Combining (3.13), (3.17) and (3.19), we get

$$\begin{aligned}&\frac{1}{4\tau}(\|\eta_1^{k+1}\|^2 - \|\eta_1^{k-1}\|^2) + \frac{1}{4\tau}(\|\eta_2^{k+1}\|^2 - \|\eta_2^{k-1}\|^2) + \frac{1}{4\tau}(\|\eta_3^{k+1}\|^2 - \|\eta_3^{k-1}\|^2) \\ &\quad + \leq C(\tau^4 + N^{-2S} + \|\eta_1^k\|^2 + \|\eta_2^k\|^2 + \|\eta_3^k\|^2 + \|\widehat{\eta}_1^k\|^2 + \|\widehat{\eta}_2^k\|^2 + \|\partial_x \widehat{\eta}_3^k\|^2 + \|\partial_x \widehat{\eta}_2^k\|^2).\end{aligned} \tag{3.20}$$

In fact

$$\|u\|_1^2 = \|u\|^2 + \|\partial_x u\|^2 \quad \text{and} \quad \|\widehat{u}^k\|^2 \leq \frac{1}{2}\left(\|u^{k+1}\|^2 + \|u^{k-1}\|^2\right).$$

Let

$$E^k = \|\eta_1^{k+1}\|^2 + \|\eta_1^k\|^2 + \|\eta_2^{k+1}\|^2 + \|\eta_2^k\|^2 + \|\eta_3^{k+1}\|^2 + \|\eta_3^k\|^2,$$

Summing up (3.20) for $k = 1, \cdots, n-1$, we obtain

$$E^n \leq C(E^0 + N^{-2S} + \tau^4) + C\tau \sum_{k=1}^{n-1} E^k.$$

Note that

$$\|\eta_1^0\|^2 = \|\eta_2^0\|^2 = \|\eta_3^0\|^2 = 0, \quad \text{and} \quad \|\eta_1^1\|^2 = \|\eta_2^1\|^2 = \|\eta_3^1\|^2 \leq C(\tau^4 + N^{-2S}).$$

By applying Grönwall's Lemma, we get

$$C(N^{-2S} + \tau^4) \leq \Xi e^{-CT}.$$

$$E^n \leq C(N^{-2S} + \tau^4)e^{c(n+1)\tau}, \qquad \forall (n+1)\tau \leq T,$$

where Ξ is positive constant and we get

Theorem 1. *Suppose τ is small enough, the solutions $u(x,t)$, $v(x,t)$ and $w(x,t)$ of (1.1)–(1.5) satisfy $\partial_t^3 u \in L^\infty(0,T;H^0(\Omega))$, $\partial_t^2 u \in L^\infty(0,T;H^2(\Omega))$, $\partial_t^3 v \in L^\infty(0,T;H^0(\Omega))$, $\partial_t^2 v \in L^\infty(0,T;H^2(\Omega))$, $\partial_t^3 w \in L^\infty(0,T;H^0(\Omega))$, $\partial_t^2 u \in L^\infty(0,T;H^1(\Omega))$, are the solutions of pseudospectral scheme (3.1)–(3.5). Then $\exists$ constant Ξ, not depend on τ and N, such that for $k = 0,1,\cdots,n-1$*

$$\|u^{k+1} - u_c^{k+1}\| + \|v^{k+1} - v_c^{k+1}\| + \|w^{k+1} - w_c^{k+1}\| \le \Xi(\tau^2 + N^{-S}).$$

Theorem 2. *Assume that conditions of Theorem 1 are satisfied, when $\tau \to 0$, $N \to \infty$, the solution of Fourier pseudospectral method (3.1)–(3.5) converge to the solution of (1.1)–(1.5). The convergence rate is $(\tau^2 + N^{-S})$*

Theorem 3. *Suppose that u_c^k, v_c^k, w_c^k are solutions for (3.1)–(3.5). when time step τ is small enough and T is bigger. Then $\|u_c^k\| \le M_1$, $\|v_c^k\| \le M_2$, $\|w_c^k\| \le M_3$. The proof may be found in [29].*

Theorem 4. *Suppose that the conditions of Theorem 3 are satisfied. Then the Fourier pseudospectral scheme (3.1)–(3.5) is stable for the given initial values.*

4 Numerical Results

We present some numerical results to show the computational complexity of the Fourier pseudospectral method for (1.1)-(1.5). All computations were done by using Matlab 7.3 on personal Laptop Inspiron 6400. The generalized Zakharov equations have the family of one soliton [25]:

$$\begin{aligned} u(x,t) = \Re\left\{\left[\lambda + \mu\left(1 - \frac{\nu^2}{c^2}\right)^{-1}\right] 2i\mathrm{sech}(2\eta(x-\nu t)) \right. \\ \left. \times \exp i\left[\frac{\nu}{2}x - \left(\frac{\nu^2}{4} - 4\eta^2\right)t + \phi_0\right]\right\} \end{aligned} \tag{4.1}$$

$$\begin{aligned} v(x,t) = \Im\left\{\left[\lambda + \mu\left(1 - \frac{\nu^2}{c^2}\right)^{-1}\right] 2i\mathrm{sech}(2\eta(x-\nu t)) \right. \\ \left. \times \exp i\left[\frac{\nu}{2}x - \left(\frac{\nu^2}{4} - 4\eta^2\right)t + \phi_0\right]\right\} \end{aligned} \tag{4.2}$$

$$w(x,t) = \mu\left(1 - \frac{\nu^2}{c^2}\right)^{-1} |u|^2. \tag{4.3}$$

The relative discrete L^2-norm error is defined as follows:

$$E_2(\mathbf{u}(t)) = \left[\frac{\sum_{x\in\Omega} |\mathbf{u}(x,t) - \mathbf{u}_N(x,t)|^2}{\sum_{x\in\Omega|\mathbf{u}(x,t)|^2}}\right]^{1/2}, \tag{4.4}$$

where $\mathbf{u} = (u,v,w)$ is the exact solution of equation (1.1)-(1.3) and $\mathbf{u}_N = (u_N, v_N, w_N)$ is the solution of the Fourier pseudospectral scheme (3.1)-(3.3).

The calculation is carried out with $\lambda = 0.0$, $\mu = 1.0$, $c = 1.0$, $\eta = 0.5$, $\nu = 0.5$ and $\phi_0 = 0.0$ through out the computation. For comparison, we consider time splitting of spectral scheme of [25]. In Table 1 the numerical results show that present scheme (3.1)-(3.3) gives much better results than scheme [25]. Present scheme provide the numerical solution with high accuracy even if N is small. In order to check the rate of convergence of the present scheme. Table 2 shows the numerical results of the present scheme. We obtained that if N increases and τ decreases proportionally, then the errors become smaller quickly, which shows the convergence of the present scheme. The relative error for w is given in Table 3.

At time $t = 1.0$, the single soliton is plotted in Figure 1, where $L = 10$. The surface graph of the approximate solution at $t = 1$, $L = 20$ are given in Figure 2. We examine the behavior of the approximate solution.

It can be seen that Fourier pseudospectral scheme presents better solution for nonlinear partial differential equations. The advantage of spectral methodology provides fast convergence by using Fast Fourier Transform.

Table 1
Comparison of errors at $N = 8$, $\tau = 0.001$

	$E_2(u(t))$		$E_2(v(t))$	
Time	Present Scheme	Scheme[25]	Present Scheme	Scheme[25]
0.2	0.8212E-5	0.1508E-4	0.2231E-5	0.6214E-4
0.4	0.6916E-5	0.7527E-4	0.3346E-5	0.4565E-4
0.6	0.2529E-4	0.6395E-3	0.6666E-4	0.6689E-3
0.8	0.4829E-4	0.4646E-3	0.2364E-4	0.3478E-3
1.0	0.5612E-4	0.5310E-3	0.6883E-4	0.4985E-3

Table 2
The Relative Errors for u and v at t=1.0

	$E_2(u(t))$			$E_2(v(t))$		
N	$\tau = 0.005$	$\tau = 0.001$	$\tau = 0.0005$	$\tau = 0.005$	$\tau = 0.001$	$\tau = 0.0005$
4	0.4832E-3	0.5309E-3	0.76667E-3	0.1163E-3	0.4667E-3	0.7263E-3
8	0.4785E-5	0.3218E-6	0.4258E-7	0.4491E-5	0.4168E-6	0.4454E-7
16	0.4808E-5	0.5363E-6	0.3987E-7	0.4573E-5	0.4545E-6	0.4578E-7
32	0.4979E-5	0.4364E-6	0.2153E-7	0.4684E-5	0.3667E-6	0.3581E-7
64	0.4989E-5	0.4365E-6	0.1265E-7	0.4795E-5	0.6788E-6	0.3591E-7
128	0.4999E-5	0.4367E-6	0.1376E-7	0.4856E-5	0.3897E-6	0.3681E-7

Table 3: The Relative Errors for w at t=1.0

N	$\tau = 0.005$	$\tau = 0.001$
4	0.6701E-4	0.3998E-5
8	0.4637E-4	0.8166E-5
16	0.311E-3	0.5775E-4
32	0.444E-3	0.4322E-4
64	0.5211E-3	0.2475E-4
128	0.454E-3	0.3422E-4

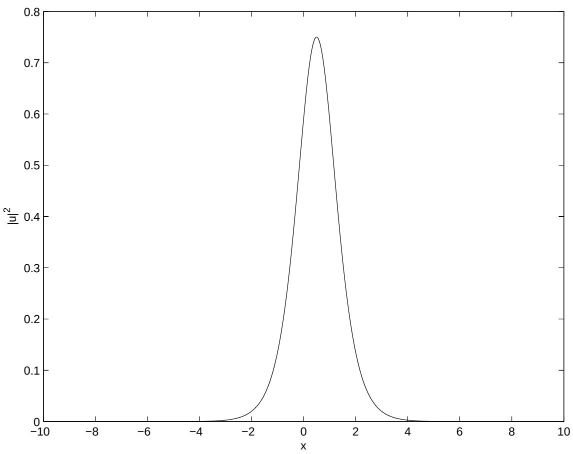
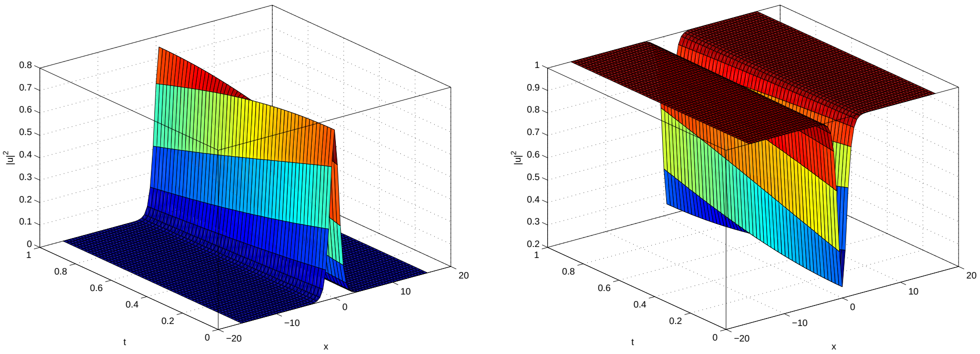


Figure 1: plot of soliton at t=1.0



(a) Approximate solution of u   (b) Approximate solution of v

Figure 2: Approximate solution of u and v

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Mathematical analysis of a humoral immunity virus infection model with Crowley-Martin functional response and distributed delays

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Abstract

In this paper, we investigate the basic and global properties of a virus infection model with humoral immunity and distributed intracellular delays. The incidence rate of the infection is given by Crowley-Martin functional response. Two types of distributed time delays have been incorporated into the model to describe the time needed for infection of uninfected cell and virus replication. Using the method of Lyapunov functional, we have established that the global stability of the model is completely determined by two threshold numbers, the basic reproduction number R_0 and the humoral immunity reproduction number R_1. We have proven that if $R_0 \leq 1$ then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without immune response is GAS, and if $R_1 > 1$, then the infected steady state with humoral immunity is GAS.
Keywords: Global stability; humoral immunity; Distributed delay; Crowley-Martin functional response.
AMS subject classifications. 92D25, 34D20, 34D23

1 Introduction

In the last decade, several mathematical models have been developed to describe the dynamics of several viruses, such as human immunodeficiency virus (HIV) [1]-[18], hepatitis B virus (HBV) [19]-[21], hepatitis C virus (HCV) [22], [23], human T-cell leukemia virus (HTLV) [24], and human cytomegalo virus (HCMV) [25], etc. Mathematical modeling and model analysis of the viral infection process can help for estimating key parameter values and guiding development efficient anti-viral treatments. Some of these models take into account the main role of immune system of human body. The immune system is described as having two "arms": the cellular arm, which depends on T cells to mediate attacks on viral infected or cancerous cells; and the humoral arm, which depends on B cells to make antibodies to clear antigens circulating in blood and lymph. The humoral immunity is more effective than the cell-mediated immune in some diseases like in malaria infection [26]. Mathematical models for virus dynamics with humoral immunity have been developed in [27]-[34].

The basic model with humoral immunity was introduced by Murase et. al. [28] as:

$$\dot{x}(t) = \lambda - dx(t) - \beta x(t)v(t), \tag{1}$$
$$\dot{y}(t) = \beta x(t)v(t) - \delta y(t), \tag{2}$$
$$\dot{v}(t) = N\delta y(t) - cv(t) - qv(t)z(t), \tag{3}$$
$$\dot{z}(t) = rv(t)z(t) - \mu z(t), \tag{4}$$

where $x(t)$, $y(t)$, $v(t)$ and $z(t)$ represent the populations of uninfected cells, infected cells, viruses and B cells at time t, respectively; λ and d are the birth rate and death rate constants of uninfected cells, respectively; β is the infection rate constant; N is the number of free virus produced during the average infected cell life span; δ is the death rate constant of infected cells; c is the death rate constant of the virus. The viruses are cleared by antibodies with rate $qv(t)z(t)$. The B cells are proliferated at a rate $rv(t)z(t)$ and die at rate $\mu z(t)$. Model (1)-(4) is based on the assumption that the infection could occur and the viruses are produced from infected cells instantaneously, once the uninfected cells are contacted by the virus particles. Other accurate models incorporate the delay between the time the viral entry into the uninfected cell and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations (see e.g. [9]-[17]). In these papers, the viral infection models are presented without taking into consideration the humoral immunity. In [32] and [34], the global stability of viral infection models with humoral immunity and with discrete-time delays has been studied.

In model (1)-(4), the infection rate is assumed to be bilinear in x and v. However, the actual incidence rate is probably not linear over the entire range of x and v [36], [37]. In [33] and [34], a virus infection model with humoral immunity and with saturated infection rate of the form $\frac{\beta xv}{1+\alpha v}$, was suggested where α is a positive constant. However, the time delay was not considered in [33] and [34]. Huang and Takeuchi [38] investigated a viral infection model with Beddington-DeAngelis functional response, $\frac{\beta xv}{1+\gamma x+\alpha v}$, where α and γ are positive constants. Crowley-Martin functional response in the form $\frac{\beta xv}{(1+\gamma x)(1+\alpha v)}$ has been introduced in HIV model in [39]. However, the humoral immunity was not included in [33] and [39].

In this paper, we assume that the infection rate is given by Crowley-Martin functional response. We incorporate two types of distributed delays into the model to account the time delay between the time that target cells are contacted by the virus particle and the time the emission of infectious (matures) virus particles. The global stability of the model is established using Lyapunov functionals, which are similar in nature to those used in [40]. We prove that the global dynamics of the model is determined by the basic reproduction number R_0 and humoral immunity reproduction number R_1. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immunity exists and it is GAS, if $R_1 > 1$ then the infected steady state with humoral immunity exists and it is GAS.

2 The Model

In this section we propose a mathematical model of viral infection with Crowley-Martin functional response which describes the interaction of the virus with uninfected cells, taking into account the humoral immunity.

$$\dot{x}(t) = \lambda - dx(t) - \frac{\beta x(t)v(t)}{(1+\gamma x(t))\,(1+\alpha v(t))}, \tag{5}$$

$$\dot{y}(t) = \int_0^h f(\tau)e^{-m\tau}\frac{\beta x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}d\tau - \delta y(t), \tag{6}$$

$$\dot{v}(t) = N\delta\int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv(t) - qv(t)z(t), \tag{7}$$

$$\dot{z}(t) = rv(t)z(t) - \mu z(t), \tag{8}$$

where γ and α are a positive constants, and all the variables and parameters of the model have the same meanings as given in (1)-(4). To account for the time lag between viral contacting the uninfected cell and the production of new virus particles, two types of distributed intracellular delays are introduced. It assumed that the uninfected cells are contacted by the virus particles at time $t-\tau$ becomes infected cells at time t, where τ is a random variable with a probability distribution $f(\tau)$ over the interval $[0,h]$ and h is limit superior of this delay. The factor $e^{-m\tau}$ account for the probability of surviving the time period of delay, where m is the death rate constant of infected cells but not yet virus producer cells. On the other hand, it is assumed that, a cell infected at time $t-\tau$ starts to yield new infectious virus at time t where τ is distributed according to a probability distribution $g(\tau)$ over the interval $[0,\omega]$ and ω is limit superior of this delay. The factor $e^{-n\tau}$ account for the probability of surviving the time period of delay, where n is a constant.

The probability distribution functions $f(\tau)$ and $g(\tau)$ are assumed to satisfy $f(\tau) > 0$ and $g(\tau) > 0$, and

$$\int_0^h f(\tau)d\tau = \int_0^\omega g(\tau)d\tau = 1, \qquad f(r)e^{sr}dr < \infty, \quad \int_0^\omega g(r)e^{sr}dr < \infty,$$

where s is a positive number. Then

$$0 < \int_0^h f(\tau)e^{-m\tau}d\tau \leq 1,\ 0 < \int_0^\omega g(\tau)e^{-n\tau}d\tau \leq 1, \text{for } m,n \geq 0,$$

let us denote:

$$F = \int_0^h f(\tau)e^{-m\tau}d\tau, \qquad G = \int_0^\omega g(\tau)e^{-n\tau}d\tau.$$

The initial conditions for system (5)-(8) take the form

$$\begin{aligned} x(\theta) &= \varphi_1(\theta),\ y(\theta) = \varphi_2(\theta) \\ v(\theta) &= \varphi_3(\theta), z(\theta) = \varphi_4(\theta) \\ \varphi_j(\theta) &\geq 0,\ \theta \in [-\rho, 0),\quad j = 1, ..., 4, \\ \varphi_j(0) &> 0,\ j = 1, ..., 4, \end{aligned} \tag{9}$$

where $\rho = \max\{h, \omega\}$, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_4(\theta)) \in C([-\rho, 0], \mathbb{R}^4_+)$, where $C([-\rho, 0], \mathbb{R}^4_+)$ is the Banach space of continuous functions mapping the interval $[-\rho, 0]$ into $\mathbb{R}^4_+$. By the fundamental theory of functional differential equations [35], system (5)-(8) has a unique solution satisfying the initial conditions (9).

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (5)-(8) with initial conditions (9).

Proposition 1. Let $(x(t), y(t), v(t), z(t))$ be any solution of system (5)-(8) satisfying the initial conditions (9), then $x(t), y(t), v(t)$ and $z(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x(t) > 0$, for all $t \geq 0$. Assume that $x(t)$ lose its non-negativity on some local existence interval $[0, \ell]$ for some constant ℓ and let $t_1 \in [0, \ell]$ be such that $x(t_1) = 0$. From Eq. (5) we have $\dot{x}(t_1) = \lambda > 0$. Hence $x(t) > 0$ for some $t \in (t_1, t_1 + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. This leads to contradiction and hence $x(t) > 0$, for all $t \geq 0$. Now from Eqs. (6)-(8) we have

$$\begin{aligned} y(t) &= y(0)e^{-\delta t} + \beta \int_0^t e^{-\delta(t-\eta)} \int_0^h f(\tau)e^{-m\tau} \frac{x(\eta-\tau)v(\eta-\tau)}{(1+\gamma x(\eta-\tau))(1+\alpha v(\eta-\tau))} d\tau d\eta, \\ v(t) &= v(0)e^{-\int_0^t (c+qz(\xi))d\xi} + N\delta \int_0^t e^{-\int_\eta^t (c+qz(\xi))d\xi} \int_0^\omega g(\tau)e^{-n\tau} y(\eta-\tau) d\tau d\eta, \\ z(t) &= z(0)e^{-\int_0^t (\mu - rv(\xi))d\xi}, \end{aligned}$$

confirming that $y(t) \geq 0$, $v(t) \geq 0$ and $z(t) \geq 0$ for all $t \in [0, \rho]$. By a recursive argument, we obtain $y(t) \geq 0, v(t) \geq 0$ and $z(t) \geq 0$ for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (5) we have $\dot{x}(t) \leq \lambda - dx(t)$. This implies $\limsup_{t\to\infty} x(t) \leq \frac{\lambda}{d}$.

Let $X_1(t) = \int_0^h f(\tau)e^{-m\tau} x(t-\tau)d\tau + y(t)$, then

$$\begin{aligned} \dot{X}_1(t) &= \int_0^h f(\tau)e^{-m\tau} \left(\lambda - dx(t-\tau) - \frac{\beta x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))} \right) d\tau \\ &+ \int_0^h f(\tau)e^{-m\tau} \frac{\beta x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))} d\tau - \delta y(t), \\ &= \lambda \int_0^h f(\tau)e^{-m\tau} d\tau - d \int_0^h f(\tau)e^{-m\tau} x(t-\tau) d\tau - \delta y(t) \\ &\leq \lambda \int_0^h f(\tau)e^{-m\tau} d\tau - \sigma_1 \left[\int_0^h f(\tau)e^{-m\tau} x(t-\tau) d\tau + y(t) \right] \\ &= \lambda \int_0^h f(\tau)e^{-m\tau} d\tau - \sigma_1 X_1(t) \leq \lambda - \sigma_1 X_1(t), \end{aligned}$$

where $\sigma_1 = \min\{d, \delta\}$. Hence $\limsup_{t\to\infty} X_1(t) \leq L_1$, where $L_1 = \dfrac{\lambda}{\sigma_1}$. Since $\int_0^h f(\tau)e^{-m\tau} x(t-\tau)d\tau > 0$ then

$\limsup_{t\to\infty} y(t) \le L_1$. On the other hand, let $X_2(t) = v(t) + \frac{q}{r}z(t)$, then

$$\dot{X}_2(t) = N\delta \int_0^{\omega} g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv(t) - \frac{q\mu}{r}z(t) \le N\delta L_1 - \sigma_2 X_2(t),$$

where $\sigma_2 = \min\{c,\mu\}$. Hence $\limsup_{t\to\infty} X_2(t) \le L_2$, where $L_2 = \dfrac{N\delta L_1}{\sigma_2}$. Since $v(t) \ge 0$ and $y(t) \ge 0$ then $\limsup_{t\to\infty} v(t) \le L_2$ and $\limsup_{t\to\infty} z(t) \le L_2$. Therefore, $x(t), y(t), v(t)$ and $z(t)$ are ultimately bounded.

2.2 Steady states

We define the basic reproduction number for system (5)-(8) as:

$$R_0 = \frac{NFG\beta x_0}{c(1+\gamma x_0)}.$$

Lemma 1.

(a) If $R_0 \le 1$, then there exists only an uninfected steady state $E_0(x_0, 0, 0, 0)$.

(b) If $R_0 > 1$, then there exists an infected steady state without humoral immunity $E_1(x_1, y_1, v_1, 0)$.

(c) If $R_1 > 1$, then there exists an infected steady state with humoral immunity $E_2(x_2, y_2, v_2, z_2)$.

Proof.

Let the right-hand side of Eqs. (5)-(8) be zero,

$$\lambda - dx(t) - \frac{\beta xv}{(1+\gamma x)(1+\alpha v)} = 0, \tag{10}$$

$$F\frac{\beta xv}{(1+\gamma x)(1+\alpha v)} - \delta y = 0, \tag{11}$$

$$N\delta G y - cv - qvz = 0, \tag{12}$$

$$rvz - \mu z = 0. \tag{13}$$

Eq. (13) has two possible solutions, $z = 0$ or $v = \mu/r$. If $z = 0$, then from (10) we obtain x as

$$x^+ = \frac{1}{2\gamma(1+\alpha v)}\left(\gamma x_0(1+\alpha v) - (1+\zeta v) + \sqrt{[(1+\zeta v) - \gamma x_0(1+\alpha v)]^2 + 4\gamma x_0(1+\alpha v)^2}\right),$$

$$x^- = \frac{1}{2\gamma(1+\alpha v)}\left(\gamma x_0(1+\alpha v) - (1+\zeta v) - \sqrt{[(1+\zeta v) - \gamma x_0(1+\alpha v)]^2 + 4\gamma x_0(1+\alpha v)^2}\right),$$

where $\zeta = \alpha + \frac{\beta}{d}$. It is clear if $v > 0$ then $x^+ > 0$ and $x^- < 0$. Let us choose $x = x^+$. From Eqs. (11) and (12) we have

$$y = \frac{F\beta xv}{\delta(1+\gamma x)(1+\alpha v)},$$

$$\frac{GNF\beta xv}{(1+\gamma x)(1+\alpha v)} - cv = 0. \tag{14}$$

Eq. (14) has two possible solutions $v = 0$ and $v \ne 0$. If $v = 0$, then we get the uninfected steady state $E_0(\frac{\lambda}{d}, 0, 0, 0)$. If $v \ne 0$ then let

$$M(v) = \frac{GNF\beta xv}{(1+\gamma x)(1+\alpha v)} - cv = FGN(\lambda - dx) - cv.$$

It is clear that when $v = 0$, then $x = x_0$ and $M(0) = 0$ and when $v = \bar{v} = \dfrac{FGN\lambda}{c}$, then substituting it in x we obtain $\overline{x} > 0$, and so $M(\bar{v}) = -GNFd\overline{x} < 0$. Since $M(v)$ is continuous for all $v \ge 0$, we have that

$$M'(0) = \frac{FNG\beta x_0}{(1+\gamma x_0)} - c = c(R_0 - 1).$$

Therefore, if $R_0 > 1$, then $M'(0) > 0$. It follows that there exists $v_1 \in (0, \bar{v})$ such that $M(v_1) = 0$. Then there exists an infected steady state without immune response $E_1(x_1, y_1, v_1, 0)$ where

$$
\begin{aligned}
x_1 &= \frac{1}{2\gamma\,(1+\alpha v_1)}\left(\gamma x_0(1+\alpha v_1) - (1+\zeta v_1) + \sqrt{[(1+\zeta v_1) - \gamma x_0(1+\alpha v_1)]^2 + 4\gamma x_0(1+\alpha v_1)^2}\right), \\
y_1 &= \frac{F\beta x_1 v_1}{\delta\,(1+\gamma x_1)\,(1+\alpha v_1)}, \\
v_1 &= \frac{1}{2c\alpha\gamma}\left(dFGN\alpha + FGN\beta - c\gamma + FGN\alpha\gamma\lambda - \sqrt{(c\gamma - FGN(d\alpha+\beta-\alpha\gamma\lambda))^2 + 4dFGN\alpha\gamma(c+FGN\alpha\lambda)}\right).
\end{aligned}
$$

If $z \neq 0$, we obtain the infected steady state with immune response $E_2(x_2, y_2, v_2, z_2)$ where

$$
\begin{aligned}
x_2 &= \frac{1}{2\gamma(1+\alpha v_2)}\left(\gamma x_0(1+\alpha v_2) - (1+\zeta v_2) + \sqrt{[(1+\zeta v_2) - \gamma x_0(1+\zeta v_2)]^2 + 4\gamma x_0(1+\alpha v_2)^2}\right), \\
y_2 &= \frac{F\beta x_2 v_2}{\delta\,(1+\gamma x_2)\,(1+\alpha v_2)}, \quad v_2 = \frac{\mu}{r}, \quad z_2 = \frac{c}{q}\,(R_1 - 1)\,,
\end{aligned}
$$

where R_1 is an humoral immunity reproduction number given by:

$$
R_1 = \frac{NFG\beta x_2}{c\,(1+\gamma x_2)\,(1+\alpha v_2)}.
$$

It is clear that x_2, y_2 and v_2 are positive and if $R_1 > 1$, then z_2 is positive. Since $0 < x_2 \leq x_0$ and $v_2 > 0$, then

$$
R_1 \leq \frac{NFG\beta x_2}{c(1+\gamma x_2)} \leq \frac{NFG\beta x_0}{c(1+\gamma x_0)} = R_0.
$$

2.3 Global stability

In this section, we prove the global stability of the steady states of system (5)-(8) employing the method of Lyapunov functional which is used in [40] for SIR epidemic model with distributed delay. Next we shall use the following notation: $u = u(t)$, for any $u \in \{x, y, v, z\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as $H(u) = u - 1 - \ln u$. It is clear that $H(u) \geq 0$ for any $u > 0$ and H has the global minimum $H(1) = 0$.

Theorem 1. If $R_0 \leq 1$, then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$
\begin{aligned}
W_0 = NFG &\left[\frac{x_0}{(1+\gamma x_0)} H\left(\frac{x}{x_0}\right) + \frac{1}{F} y + \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\int_0^\tau \frac{x(t-\theta)v(t-\theta)}{(1+\gamma x(t-\theta))\,(1+\alpha v(t-\theta))} d\theta d\tau \right. \\
&\left. + \frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}\int_0^\tau y(t-\theta) d\theta d\tau\right] + v + \frac{q}{r} z. \qquad (15)
\end{aligned}
$$

The time derivative of W_0 along the trajectories of (5)-(8) satisfies

$$
\begin{aligned}
\frac{dW_0}{dt} = NFG &\left[\frac{1}{(1+\gamma x_0)}\left(1 - \frac{x_0}{x}\right)\left(\lambda - dx - \frac{\beta x v}{(1+\gamma x)(1+\alpha v)}\right)\right. \\
&+ \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))} d\tau - \frac{\delta}{F} y \\
&+ \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\left(\frac{xv}{(1+\gamma x)(1+\alpha v)} - \frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right) d\tau \\
&\left. + \frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}(y - y(t-\tau)) d\tau\right] + N\delta\int_0^\omega g(\tau)e^{-n\tau} y(t-\tau) d\tau - cv - qvz + qvz - \frac{q\mu}{r} z. \qquad (16)
\end{aligned}
$$

Collecting terms of (16) we get

$$\begin{aligned}
\frac{dW_0}{dt} &= NFG\left[-\frac{d(x-x_0)^2}{(1+\gamma x_0)\,x} - \frac{\beta xv}{(1+\gamma x_0)\,(1+\gamma x)\,(1+\alpha v)} + \frac{\beta x_0 v}{(1+\gamma x_0)\,(1+\gamma x)\,(1+\alpha v)}\right.\\
&\left.\quad + \frac{\beta xv}{(1+\gamma x)\,(1+\alpha v)}\right] - cv - \frac{q\mu}{r}z\\
&= NFG\left[\frac{-d(x-x_0)^2}{(1+\gamma x_0)\,x} + \frac{\beta x_0 v}{(1+\gamma x_0)(1+\alpha v)}\right] - cv - \frac{q\mu}{r}z\\
&= -NFGd\frac{(x-x_0)^2}{(1+\gamma x_0)\,x} + \frac{cvR_0}{(1+\alpha v)} - cv - \frac{q\mu}{r}z\\
&= -NFGd\frac{(x-x_0)^2}{(1+\gamma x_0)\,x} + cv(R_0-1) - \frac{c\alpha v^2 R_0}{(1+\alpha v)} - \frac{q\mu}{r}z. \qquad (17)
\end{aligned}$$

From Eq. (17) we can see that if $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x, v, z > 0$. By Theorem 5.3.1 in [35], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (17) that $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $v = 0$ and $z = 0$. Noting that M is invariant, for each element of M we have $v = 0$ and $z = 0$, then $\dot{v} = 0$. From Eq. (7) we drive that $0 = \dot{v} = N\delta \int\limits_0^{\omega} g(\tau)e^{-n\tau}y(t-\tau)d\tau$. This yields $y = 0$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $y = 0$, $v = 0$ and $z = 0$. From LaSalle's Invariance Principle, E_0 is GAS.

Theorem 2. If $R_1 \le 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$\begin{aligned}
W_1 &= NFG\left[x - x_1 - \int\limits_{x_1}^{x}\frac{x_1(1+\gamma\eta)}{\eta(1+\gamma x_1)}d\eta + \frac{1}{F}y_1 H\left(\frac{y}{y_1}\right)\right.\\
&\quad + \frac{1}{F}\frac{\beta x_1 v_1}{(1+\gamma x_1)\,(1+\alpha v_1)}\int\limits_0^h f(\tau)e^{-m\tau}\int\limits_0^{\tau} H\left(\frac{x(t-\theta)v(t-\theta)(1+\gamma x_1)\,(1+\alpha v_1)}{x_1 v_1(1+\gamma x(t-\theta))\,(1+\alpha v(t-\theta)}\right)d\theta d\tau\\
&\left.\quad + \frac{\delta y_1}{FG}\int\limits_0^{\omega} g(\tau)e^{-n\tau}\int\limits_0^{\tau} H\left(\frac{y(t-\theta)}{y_1}\right)d\theta d\tau\right] + v_1 H\left(\frac{v}{v_1}\right) + \frac{q}{r}z. \qquad (18)
\end{aligned}$$

The time derivative of W_1 along the trajectories of (5)-(8) is given by

$$\begin{aligned}
\frac{dW_1}{dt} &= NFG\left[\left(1 - \frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right)\left(\lambda - dx - \frac{\beta xv}{(1+\gamma x)(1+\alpha v)}\right)\right.\\
&\quad + \frac{1}{F}\left(1 - \frac{y_1}{y}\right)\left(\beta\int\limits_0^h f(\tau)e^{-m\tau}\frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}d\tau - \delta y\right)\\
&\quad + \frac{\beta}{F}\int\limits_0^h f(\tau)e^{-m\tau}\left(\frac{xv}{(1+\gamma x)\,(1+\alpha v)} - \frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right.\\
&\quad \left. + \frac{x_1 v_1}{(1+\gamma x_1)\,(1+\alpha v_1)}\ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)\,(1+\alpha v)}{xv(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right)\right)d\tau\\
&\left.\quad + \frac{\delta}{FG}\int\limits_0^{\omega} g(\tau)e^{-n\tau}\left(y - y(t-\tau) + y_1\ln\left(\frac{y(t-\tau)}{y}\right)\right)d\tau\right]\\
&\quad + \left(1 - \frac{v_1}{v}\right)\left(N\delta\int\limits_0^{\omega} g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv - qzv\right) + qvz - \frac{q\mu}{r}z. \qquad (19)
\end{aligned}$$

Using the steady state conditions for E_1:

$$\lambda = dx_1 + \frac{\beta x_1 v_1}{(1+\gamma x_1)\,(1+\alpha v_1)},\ \delta y_1 = F\frac{\beta x_1 v_1}{(1+\gamma x_1)\,(1+\alpha v_1)},\ cv_1 = N\delta G y_1,$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= NFG\left[-d(x-x_1)\left(1-\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right)+\frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\right.\\ &\quad -\frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}+\frac{\beta x_1 v}{(1+\gamma x_1)(1+\alpha v)}\\ &\quad -\frac{1}{F}\frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\int_0^h f(\tau)e^{-m\tau}\frac{y_1 x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{yx_1v_1(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}d\tau+\frac{\delta}{F}y_1\\ &\quad +\frac{1}{F}\frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\int_0^h f(\tau)e^{-m\tau}\ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)(1+\alpha v)}{xv(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right)d\tau\\ &\quad \left.+\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\ln\left(\frac{y(t-\tau)}{y}\right)d\tau-\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\frac{v_1 y(t-\tau)}{vy_1}d\tau\right]\\ &\quad -cv+cv_1+qv_1z-\frac{q\mu}{r}z. \end{aligned} \tag{20}$$

Using the following equalities:

$$cv = cv_1\frac{v}{v_1} = NFG\left(\frac{\delta}{F}y_1\frac{v}{v_1}\right),$$

$$\begin{aligned}\ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)(1+\alpha v)}{xv(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right) &= \ln\left(\frac{y_1x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{yx_1v_1(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right)\\ &\quad +\ln\left(\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right)+\ln\left(\frac{v_1y}{vy_1}\right)+\ln\left(\frac{1+\alpha v}{1+\alpha v_1}\right),\\ \ln\left(\frac{y(t-\tau)}{y}\right) &= \ln\left(\frac{vy_1}{v_1y}\right)+\ln\left(\frac{v_1y(t-\tau)}{vy_1}\right).\end{aligned}$$

Then collecting terms of (20), we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= NFG\left[-d\frac{(x-x_1)^2}{x(1+\gamma x_1)}-\frac{\delta y_1}{F}\left(\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}-1-\ln\left(\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right)\right)\right.\\ &\quad +\frac{\delta y_1}{F}\frac{v(1+\alpha v_1)}{v_1(1+\alpha v)}-\frac{\delta y_1}{F^2}\int_0^h f(\tau)e^{-m\tau}\left(\frac{y_1x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{yx_1v_1(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}-1\right.\\ &\quad \left.-\ln\left(\frac{y_1x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{yx_1v_1(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right)\right)d\tau\\ &\quad -\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\left(\frac{v_1y(t-\tau)}{vy_1}-1-\ln\left(\frac{v_1y(t-\tau)}{vy_1}\right)\right)d\tau\\ &\quad \left.+\frac{\delta y_1}{F}\left(\ln\left(\frac{1+\alpha v}{1+\alpha v_1}\right)-\frac{v}{v_1}\right)\right]+q\left(v_1-\frac{\mu}{r}\right)z\end{aligned}$$

$$\begin{aligned}
&= NFG\left[-d\frac{(x-x_1)^2}{x(1+\gamma x_1)} + \frac{\delta}{F}y_1\left(\frac{v\,(1+\alpha v_1)}{v_1\,(1+\alpha v)} - \frac{v}{v_1} + \frac{1+\alpha v}{1+\alpha v_1} - 1\right)\right.\\
&- \frac{\delta y_1}{F}H\left(\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right) - \frac{\delta y_1}{F}H\left(\frac{1+\alpha v}{1+\alpha v_1}\right)\\
&- \frac{\delta y_1}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_1 x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{y x_1 v_1(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right)d\tau\\
&\left.-\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_1 y(t-\tau)}{v y_1}\right)d\tau\right] + q\left(v_1 - \frac{\mu}{r}\right)z\\
&= NFG\left[-d\frac{(x-x_1)^2}{x(1+\gamma x_1)} - \frac{\delta y_1}{F}\frac{\alpha(v-v_1)^2}{v_1\,(1+\alpha v)\,(1+\alpha v_1)} - \frac{\delta y_1}{F}H\left(\frac{x_1(1+\gamma x)}{x(1+\gamma x_1)}\right)\right.\\
&- \frac{\delta y_1}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_1 x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{y x_1 v_1(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right)d\tau\\
&\left.-\frac{\delta y_1}{F}H\left(\frac{1+\alpha v}{1+\alpha v_1}\right) - \frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_1 y(t-\tau)}{v y_1}\right)d\tau\right] + q\left(v_1 - \frac{\mu}{r}\right)z. \qquad (21)
\end{aligned}$$

Now we show that if $R_1 \le 1$ then $v_1 \le \frac{\mu}{r} = v_2$. Let $R_0 > 1$, then we want to show that

$$sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2) = sgn\,(R_1 - 1)\,.$$

For $x_1, x_2, v_1, v_2 > 0$, we have

$$\left(\frac{\beta x_2 v_1}{(1+\gamma x_2)(1+\alpha v_1)} - \frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\right)(x_2 - x_1) = \frac{\beta v_1\,(x_2-x_1)^2}{(1+\gamma x_2)(1+\alpha v_1)\,(1+\gamma x_1)} > 0, \qquad (22)$$

$$\left(\frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_2 v_1}{(1+\gamma x_2)(1+\alpha v_1)}\right)(v_2 - v_1) = \frac{\beta x_2\,(v_2-v_1)^2}{(1+\gamma x_2)(1+\alpha v_2)\,(1+\alpha v_1)} > 0, \qquad (23)$$

$$((\lambda - dx_2) - (\lambda - dx_1))\,(x_2 - x_1) = -d\,(x_2 - x_1)^2 < 0, \qquad (24)$$

$$\left(\frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_1)}\right)(v_1 - v_2) = \frac{\beta\alpha x_2\,(v_2-v_1)^2}{(1+\alpha v_2)(1+\alpha v_1)(1+\gamma x_2)} > 0. \qquad (25)$$

Suppose that, $sgn\,(x_2 - x_1) = sgn\,(v_2 - v_1)$. Using the conditions of the steady states E_1 and E_2 we have

$$\begin{aligned}
(\lambda - dx_2) - (\lambda - dx_1) &= \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\\
&= \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_2 v_1}{(1+\gamma x_2)(1+\alpha v_1)} + \frac{\beta x_2 v_1}{(1+\gamma x_2)(1+\alpha v_1)} - \frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)},
\end{aligned}$$

and from (22) and (23) we get $sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)$, which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the steady stae conditions for E_1 we have $\frac{NFG\beta x_1}{c(1+\gamma x_1)(1+\alpha v_1)} = 1$, then

$$\begin{aligned}
R_1 - 1 &= \frac{NFG}{c}\left(\frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_1}{(1+\gamma x_1)(1+\alpha v_1)}\right)\\
&= \frac{NFG}{c}\left(\frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_1)} + \frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_1)} - \frac{\beta x_1}{(1+\gamma x_1)(1+\alpha v_1)}\right)\\
&= \frac{NFG}{c}\left[\frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_2)} - \frac{\beta x_2}{(1+\gamma x_2)(1+\alpha v_1)} + \frac{1}{v_1}\left(\frac{\beta x_2 v_1}{(1+\gamma x_2)(1+\alpha v_1)} - \frac{\beta x_1 v_1}{(1+\gamma x_1)(1+\alpha v_1)}\right)\right].
\end{aligned}$$

From (22) and (25) , we get $sgn\,(R_1 - 1) = sgn\,(v_1 - v_2)$ .Hence, if $R_0 > 1$, then $x_1, y_1, v_1 > 0$, and if $R_1 \le 1$, then $v_1 \le v_2 = \frac{\mu}{r}$ and $\frac{dW_1}{dt} \le 0$ for all $x, y, v, z > 0$. By Theorem 5.3.1 in [35], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_1}{dt} = 0\right\}$. It can be seen that $\frac{dW_1}{dt} = 0$ if and only if $x = x_1, v = v_1$, $z = 0$ and $H = 0$ i.e.

$$\frac{y_1 x(t-\tau)v(t-\tau)(1+\gamma x_1)(1+\alpha v_1)}{y x_1 v_1(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))} = \frac{v_1 y(t-\tau)}{v y_1} = 1 \text{ for almost all } \tau \in [0, \rho]. \qquad (26)$$

From Eq. (26), if $v=v_1$ then $y=y_1$ and hence $\frac{dW_1}{dt}$ equal to zero at E_1. LaSalle's Invariance Principle implies global stability of E_1.

Theorem 2. If $R_1>1$, then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$\begin{aligned} W_2 = NFG \Bigg[& x - x_2 - \int_{x_2}^{x} \frac{x_2(1+\gamma\eta)}{\eta(1+\gamma x_2)} d\eta + \frac{1}{F} y_2 H\left(\frac{y}{y_2}\right) \\ & + \frac{1}{F} \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \int_0^h f(\tau) e^{-m\tau} \int_0^\tau H\left(\frac{x(t-\theta)v(t-\theta)(1+\gamma x_2)(1+\alpha v_2)}{x_2 v_2 (1+\gamma x(t-\theta))(1+\alpha v(t-\theta)}\right) d\theta d\tau \\ & + \frac{\delta y_2}{FG} \int_0^\omega g(\tau) e^{-n\tau} \int_0^\tau H\left(\frac{y(t-\theta)}{y_2}\right) d\theta d\tau \Bigg] + v_2 H\left(\frac{v}{v_2}\right) + \frac{q}{r} z_2 H\left(\frac{z}{z_2}\right). \end{aligned} \tag{27}$$

The time derivative of W_2 along the trajectories of (5)-(8) is given by

$$\begin{aligned} \frac{dW_2}{dt} = NFG \Bigg[& \left(1 - \frac{x_2(1+\gamma x)}{x(1+\gamma x_2)}\right)\left(\lambda - dx - \frac{\beta x v}{(1+\gamma x)(1+\alpha v)}\right) \\ & + \frac{1}{F}\left(1 - \frac{y_2}{y}\right)\left(\beta \int_0^h f(\tau) e^{-m\tau} \frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))} d\tau - \delta y\right) \\ & + \frac{\beta}{F} \int_0^h f(\tau) e^{-m\tau} \left(\frac{xv}{(1+\gamma x)(1+\alpha v)} - \frac{x(t-\tau)v(t-\tau)}{(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right. \\ & \left. + \frac{x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)(1+\alpha v)}{xv(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right)\right) d\tau \\ & + \frac{\delta}{FG} \int_0^\omega g(\tau) e^{-n\tau} \left(y - y(t-\tau) + y_2 \ln\left(\frac{y(t-\tau)}{y}\right)\right) d\tau \Bigg] \\ & + \left(1 - \frac{v_2}{v}\right)\left(N\delta \int_0^\omega g(\tau) e^{-n\tau} y(t-\tau) d\tau - cv - qzv\right) + \left(1 - \frac{z_2}{z}\right)\left(qvz - \frac{q\mu}{r} z\right). \end{aligned} \tag{28}$$

Using the steady state conditions for E_2:

$$\lambda = dx_2 + \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)}, \; cv_2 = \; NFG\left(\frac{\delta}{F} y_2\right) - qv_2 z_2, \; \mu = rv_2,$$

we obtain

$$\begin{aligned} \frac{dW_2}{dt} = NFG \Bigg[& -d(x-x_2)\left(1 - \frac{x_2(1+\gamma x)}{x(1+\gamma x_2)}\right) + \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \\ & - \frac{\beta x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \frac{x_2(1+\gamma x)}{x(1+\gamma x_2)} + \frac{\beta x_2 v}{(1+\gamma x_2)(1+\alpha v)} \\ & - \frac{\beta}{F} \frac{x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \int_0^h f(\tau) e^{-m\tau} \frac{y_2 x(t-\tau)v(t-\tau)(1+\gamma x_2)(1+\alpha v_2)}{y x_2 v_2 (1+\gamma x(t-\tau))(1+\alpha v(t-\tau))} d\tau + \frac{\delta}{F} y_2 \\ & + \frac{\beta}{F} \frac{x_2 v_2}{(1+\gamma x_2)(1+\alpha v_2)} \int_0^h f(\tau) e^{-m\tau} \ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)(1+\alpha v)}{xv(1+\gamma x(t-\tau))(1+\alpha v(t-\tau))}\right) d\tau \\ & + \frac{\delta y_2}{FG} \int_0^\omega g(\tau) e^{-n\tau} \ln\left(\frac{y(t-\tau)}{y}\right) d\tau - \frac{\delta y_2}{FG} \int_0^\omega g(\tau) e^{-n\tau} \frac{v_2 y(t-\tau)}{v y_2} d\tau \Bigg] \\ & - cv + cv_2 + qv_2 z - qvz_2 - \frac{q\mu}{r} z + \frac{q\mu}{r} z_2. \end{aligned} \tag{29}$$

Using the following equalities:

$$cv = cv_2\frac{v}{v_2} = NFG\left(\frac{\delta}{F}y_2\frac{v}{v_2}\right) - qvz_2,$$

$$\ln\left(\frac{x(t-\tau)v(t-\tau)(1+\gamma x)\,(1+\alpha v)}{xv(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right) = \ln\left(\frac{y_2x(t-\tau)v(t-\tau)(1+\gamma x_2)(1+\alpha v_2)}{yx_2v_2(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right) + \ln\left(\frac{x_2(1+\gamma x)}{x(1+\gamma x_2)}\right) + \ln\left(\frac{v_2y}{vy_2}\right) + \ln\left(\frac{1+\alpha v}{1+\alpha v_2}\right),$$

$$\ln\left(\frac{y(t-\tau)}{y}\right) = \ln\left(\frac{vy_2}{v_2y}\right) + \ln\left(\frac{v_2y(t-\tau)}{vy_2}\right).$$

Then collecting terms of (29), we obtain

$$\begin{aligned}\frac{dW_2}{dt} = NFG\Bigg[&-d\frac{(x-x_2)^2}{x(1+\gamma x_2)} + \frac{\delta}{F}y_2\left(\frac{v\,(1+\alpha v_2)}{v_2\,(1+\alpha v)} - \frac{v}{v_2} + \frac{1+\alpha v}{1+\alpha v_2} - 1\right)\\ &- \frac{\delta y_2}{F}H\left(\frac{x_2(1+\gamma x)}{x(1+\gamma x_2)}\right) - \frac{\delta y_2}{F}H\left(\frac{1+\alpha v}{1+\alpha v_2}\right)\\ &- \frac{\delta y_2}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_2x(t-\tau)v(t-\tau)(1+\gamma x_2)(1+\alpha v_2)}{yx_2v_2(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right)d\tau - \frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_2y(t-\tau)}{vy_2}\right)d\tau\Bigg].\end{aligned} \tag{30}$$

Eq. (30) can be rewritten as

$$\begin{aligned}\frac{dW_2}{dt} = NFG\Bigg[&-d\frac{(x-x_2)^2}{x(1+\gamma x_2)} - \frac{\delta y_2}{F}\frac{\alpha(v-v_2)^2}{v_2\,(1+\alpha v)\,(1+\alpha v_2)} - \frac{\delta y_2}{F}H\left(\frac{x_2(1+\gamma x)}{x(1+\gamma x_2)}\right)\\ &- \frac{\delta y_2}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_2x(t-\tau)v(t-\tau)(1+\gamma x_2)(1+\alpha v_2)}{yx_2v_2(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))}\right)d\tau\\ &- \frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_2y(t-\tau)}{vy_2}\right)d\tau - \frac{\delta y_2}{F}H\left(\frac{1+\alpha v}{1+\alpha v_2}\right)\Bigg].\end{aligned}$$

Hence, it is easy to see that if $x, y, v > 0$, then $\frac{dW_2}{dt} \leq 0$. By Theorem 5.3.1 in [35], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_2}{dt} = 0\right\}$. It can be seen that $\frac{dW_2}{dt} = 0$ if and only if $x = x_2, v = v_2$ and $H = 0$ i.e.

$$\frac{y_2x(t-\tau)v(t-\tau)(1+\gamma x_2)(1+\alpha v_2)}{yx_2v_2(1+\gamma x(t-\tau))\,(1+\alpha v(t-\tau))} = \frac{v_2y(t-\tau)}{vy_2} = 1 \text{ for almost all } \tau \in [0, \rho]. \tag{31}$$

From Eq. (31), if $v = v_2$ then $y = y_2$ and from Eq. (12), then we get $z = z_2$. Thus, $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's Invariance Principle implies global stability of E_2.

3 Conclusion

In this paper, we have proposed a virus infection model which describes the interaction of the virus with uninfected cells taking into account the humoral immunity. The infection rate is given by Crowley-Martin functional response. Two types of distributed time delays describing the time needed for infection of uninfected cell and virus replication have been incorporated into the model. Using the method of Lyapunov functional, we have established that the global dynamics of the model is determined by two threshold parameters R_0 and R_1. The basic reproduction number viral infection R_0 determines whether a chronic infection can be established. The humoral immunity reproduction number R_1 determines whether a persistent B cells response can be established. We have proven that if $R_0 \leq 1$, then the uninfected steady state E_0 is GAS, and the viruses are cleared. If $R_1 \leq 1 < R_0$, then the infected steady state without immune response E_1 is GAS, and the infection becomes chronic but with no persistent B cells response. If $R_1 > 1$, then the infected steady state with immune response E_2 is GAS, and the infection is chronic with persistent B cells response.

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Multi-Poly-Cauchy polynomials

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Abstract

We investigate the properties of the multi-poly-Cauchy numbers $c_n^{(k_1,\dots,k_r)}$ and polynomials $c_n^{(k_1,\dots,k_r)}(z)$, which are generalizations of poly-Cauchy numbers $c_n^{(k)}$ and polynomials $c_n^{(k)}(z)$, introduced by the third author. We also study a complex variable function $Z_{k_1,\dots,k_r}(s,z)$ interpolating the multi-poly-Cauchy polynomial $c_n^{(k_1,\dots,k_r)}(z)$.

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1

1 Introduction

For positive integers $k_1, k_2, \dots, k_r$, define *multi-poly-factorial function* $\mathrm{Lif}_{k_1,k_2,\dots,k_r}(z)$ by

$$\mathrm{Lif}_{k_1,k_2,\dots,k_r}(z) = \sum_{0\le m_1<m_2<\cdots<m_r} \frac{z^{m_r}}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\,.$$

If $r=1$, $\mathrm{Lif}_k(z)$ ([8]) is called *polylogarithm factorial function* (or simply, *polyfactorial function*) and defined by

$$\mathrm{Lif}_k(z) = \sum_{m=0}^\infty \frac{z^m}{m!(m+1)^k}\,.$$

Define *multi-poly-Cauchy polynomials of the first kind* $c_n^{(k_1,k_2,\dots,k_r)}(z)$ $(n=0,1,2,\dots)$ by the generating function

$$(1+t)^z \mathrm{Lif}_{k_1,k_2,\dots,k_r}\bigl(\ln(1+t)\bigr) = \sum_{n=0}^\infty c_n^{(k_1,k_2,\dots,k_r)}(z)\frac{t^n}{n!}\,.$$

When $z=0$, define *multi-poly-Cauchy numbers of the first kind* $c_n^{(k_1,k_2,\dots,k_r)} = c_n^{(k_1,k_2,\dots,k_r)}(0)$ by

$$\mathrm{Lif}_{k_1,k_2,\dots,k_r}\bigl(\ln(1+t)\bigr) = \sum_{n=0}^\infty c_n^{(k_1,k_2,\dots,k_r)}\frac{t^n}{n!}\,.$$

If $r=1$, then $c_n^{(k)}(z)$ and $c_n^{(k)}$ are the poly-Cauchy polynomials and numbers of the first kind ([5, 8, 9]), defined by

$$(1+t)^z \mathrm{Lif}_k\bigl(\ln(1+t)\bigr) = \sum_{n=0}^\infty c_n^{(k)}(z)\frac{t^n}{n!}$$

and

$$\mathrm{Lif}_k\bigl(\ln(1+t)\bigr) = \sum_{n=0}^\infty c_n^{(k)}\frac{t^n}{n!}\,,$$

respectively. These numbers and polynomials are introduced in [8] and in [5], respectively. Their characteristic and combinatorial properties have been investigated in [9, 10, 12] and so on. If $k=1$, then $c_n^{(1)} = c_n$ are the classical Cauchy numbers ([3, 15]). A different generalization of Cauchy numbers can be found in [11].

The concept of poly-Cauchy numbers is an analogue of that of poly-Bernoulli numbers $B_n^{(k)}$, introduced by Kaneko ([6]) as follows:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^\infty B_n^{(k)}\frac{t^n}{n!}\,,$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

2

denotes the k-th polylogarithm if $k \ge 1$ and a rational function if $k \le 0$. When $k = 1$, $B_n^{(1)}$ are the Bernoulli numbers with $B_1^{(1)} = 1/2$, defined by

$$\frac{t}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{t^n}{n!}\,.$$

Note that the classical Bernoulli numbers B_n with $B_1 = -1/2$ are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\,,$$

satisfying $B_n = B_n^{(1)}$ except $n = 1$. One generalization of poly-Bernoulli numbers is called *multi-poly-Bernoulli number* $B_n^{(k_1,\dots,k_r)}$ ($k_i \ge 1$ for $i = 1,\dots,r$), defined by the generating function

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}\bigl(1-e^{-t}\bigr)}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)} \frac{t^n}{n!}\,,$$

where $\mathrm{Li}_{k_1,\dots,k_r}(z)$ is the multiple polylogarithm function defined by

$$\mathrm{Li}_{k_1,\dots,k_r}(z) = \sum_{0<m_1<\dots<m_r} \frac{z^{m_r}}{m_1^{k_1}\dots m_r^{k_r}} \quad (|z|<1)\,.$$

The properties of $B_n^{(k_1,\dots,k_r)}$ and/or $\mathrm{Li}_{k_1,\dots,k_r}(z)$ have been investigated in [13, 14, 7]. We can say that $c_n^{(k_1,\dots,k_r)}$ and/or $\mathrm{Lif}_{k_1,\dots,k_r}(z)$ are their analogous concepts.

In [5] we give some duality relations between $Z_k(s,z)$, defined by

$$Z_k(s,z) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))dt \quad (\Re(s)>0 \text{ and } z>-1)$$

and $\xi_k(s,z)$ ([2, 4]), defined by

$$\xi_k(s,z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-zt} t^{s-1} dt \quad (\Re(s)>0 \text{ and } z>0)\,.$$

When $z = 1$, the function $\xi_k(s,1) = \xi_k(s)$ is the Arakawa-Kaneko zeta function ([1]). It is easy to see that $\xi_1(s,z) = s\zeta(s+1,z)$ where $\zeta(s,z)$ is the Hurwitz zeta function defined by

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} \quad (\Re(s)>1,\ z>0).$$

It is known that the function $\xi_k(s,z)$ can be analytically continued to the whole complex s-plane and its values at non-positive integers are given by $\xi_k(-n,z) = (-1)^n B_n^{(k)}(z)$ ($n \ge 0$) ([4, Theorem 2]). We show that the function $Z_k(s,z)$ can be extended to an entire function, and its values at non-positive integers are given as

$$Z_k(-n,z) = c_n^{(k)}(z) \quad (n = 0,1,2,\dots).$$

In this paper, we also study a complex variable function $Z_{k_1,\dots,k_r}(s,z)$, which is a generalization of $Z_k(s,z)$, interpolating the multi-poly-Cauchy polynomial $c_n^{k_1,\dots,k_r}(z)$.

3

2 Multi-poly-Cauchy polynomials

Let $s(n,m)$ be the (signed) Stirling numbers of the first kind, defined by

$$\underbrace{x(x-1)\cdots(x-n+1)}_{n}=\sum_{m=0}^{n}s(n,m)x^m\,.$$

Poly-Cauchy numbers of the first kind $c_n^{(k)}$ can be written explicitly in terms of the Stirling numbers of the first kind:

$$c_n^{(k)}=\sum_{m=0}^{n}\frac{s(n,m)}{(m+1)^k}$$

([8, Theorem 1]). Similarly, multi-poly-Cauchy numbers of the first kind can be written in terms of the Stirling numbers of the first kind.

Theorem 1 *For a nonnegative integer n and positive integers $k_1,k_2,\dots,k_r$, we have*

$$c_n^{(k_1,k_2,\dots,k_r)}=\sum_{0\le m_1<m_2<\cdots<m_r\le n}\frac{s(n,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\,.$$

Proof. By

$$\frac{\bigl(\ln(1+t)\bigr)^m}{m!}=\sum_{n=m}^{\infty}s(n,m)\frac{t^n}{n!}\,,$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty}c_n^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\\
&=\sum_{0\le m_1<m_2<\cdots<m_r}\frac{\bigl(\ln(1+t)\bigr)^{m_r}}{m_1!\cdots m_{r-1}!m_r!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\\
&=\sum_{0\le m_1<m_2<\cdots<m_r}\frac{1}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{n=m_r}^{\infty}s(n,m_r)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{0\le m_1<m_2<\cdots<m_r\le n}\frac{s(n,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\right)\frac{t^n}{n!}\,.\end{aligned}$$

By comparing the coefficients of the both sides, we get the desired result. ∎

The multi-poly-Cauchy polynomials of the first kind can be also written in terms of the Stirling numbers of the first kind.

Theorem 2 *For a nonnegative integer n and positive integers $k_1, k_2, \dots, k_r$, we have*

$$
\begin{aligned}
&c_n^{(k_1,\dots,k_r)}(z) \\
&= n! \sum_{0\le m_1<m_2<\cdots<m_r\le n} \frac{1}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \sum_{\nu=m_r}^{n} \binom{z}{n-\nu} \frac{s(\nu,m_r)}{\nu!} .
\end{aligned}
$$

Proof. By Theorem 1, we obtain that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} c_n^{(k_1,\dots,k_r)}(z)\frac{t^n}{n!} = (1+t)^z \mathrm{Lif}_{k_1,\dots,k_r}\big(\ln(1+x)\big) \\
&= \sum_{\mu=0}^{\infty}\binom{z}{\mu}t^{\mu}\sum_{\nu=0}^{\infty}\left(\sum_{0\le m_1<m_2<\cdots<m_r\le\nu}\frac{s(\nu,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\right)\frac{t^{\nu}}{\nu!} \\
&= \sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\binom{z}{n-\nu}\sum_{0\le m_1<m_2<\cdots<m_r\le\nu}\frac{s(\nu,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\frac{n!}{\nu!}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} n! \sum_{0\le m_1<m_2<\cdots<m_r\le n}\frac{1}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{\nu=m_r}^{n}\binom{z}{n-\nu}\frac{s(\nu,m_r)}{\nu!}\frac{t^n}{n!} ,
\end{aligned}
$$

where for a complex number z and a nonnegative integer μ

$$
\binom{z}{\mu} = \frac{\overbrace{z(z-1)\cdots(z-\mu+1)}^{\mu}}{\mu!} .
$$

By comparing the coefficients of the both sides, we get the desired result. ∎

Remark. If $r=1$ in Theorem 2, then

$$
c_n^{(k)}(z) = n!\sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{\nu=m}^{n}\binom{z}{n-\nu}\frac{s(\nu,m)}{\nu!} .
$$

On the other hand, by Theorem 2.1 in [5]

$$
\begin{aligned}
c_n^{(k)}(z) &= \sum_{m=0}^{n} s(n,m)\sum_{i=0}^{m}\binom{m}{i}\frac{z^i}{(m-i+1)^k} \\
&= \sum_{i=0}^{n} z^i \sum_{m=i}^{n}\binom{m}{i}\frac{s(n,m)}{(m-i+1)^k} .
\end{aligned}
$$

The following fundamental result holds. The proof is done by a simple calculation.

5

Lemma 1 *If $k_r > 1$, then*

$$\frac{d}{dz}\big(z\mathrm{Lif}_{k_1,k_2,\dots,k_r}(z)\big) = \mathrm{Lif}_{k_1,k_2,\dots,k_{r-1},k_r-1}(z)\,.$$

If $k_r = 1$, then

$$\frac{d}{dz}\big(z\mathrm{Lif}_{k_1,k_2,\dots,k_{r-1},1}(z)\big) = \sum_{0\le m_1<\cdots<m_r} \frac{z^{m_r}}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}}\,.$$

By using Lemma 1 we can write the generating function of the multi-poly-Cauchy polynomials $c_n^{(k_1,\dots,k_r)}(z)$ in the form of iterated integrals.

Proposition 1 *For positive integers $k_1,\dots,k_r$*

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n^{(k_1,\dots,k_r)}(z)\frac{t^n}{n!}\\
&= \frac{(1+t)^z}{\ln(1+t)} \underbrace{\int_0^t \frac{1}{(1+t)\ln(1+t)}\cdots\int_0^t}_{k_r-1} \frac{\mathrm{Lif}_{k_1,\dots,k_{r-1},1}\big(\ln(1+t)\big)}{1+t} \underbrace{dt\cdots dt}_{k_r-1}\\
&= \sum_{0\le m_1<\cdots<m_r} \frac{1}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}} \frac{(1+t)^z}{\ln(1+t)}\\
&\quad\times \underbrace{\int_0^t \frac{1}{(1+t)\ln(1+t)}\cdots\int_0^t \frac{1}{(1+t)\ln(1+t)}\int_0^t}_{k_r} \frac{\big(\ln(1+t)\big)^{m_r}}{1+t} \underbrace{dt\cdots dt}_{k_r}\,.
\end{aligned}$$

Proof. By Lemma 1 we have

$$\begin{aligned}
&\mathrm{Lif}_{k_1,\dots,k_r}(z) = \frac{1}{z}\int_0^z \mathrm{Lif}_{k_1,\dots,k_{r-1},k_r-1}(z)dz\\
&= \frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \mathrm{Lif}_{k_1,\dots,k_{r-1},k_r-2}(z)dzdz\\
&= \cdots\\
&= \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\cdots\frac{1}{z}\int_0^z}_{k_r-1} \mathrm{Lif}_{k_1,\dots,k_{r-1},1}(z) \underbrace{dz\cdots dz}_{k_r-1}\\
&= \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\cdots\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z}_{k_r} \sum_{0\le m_1<\cdots<m_r} \frac{z^{m_r}}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}} \underbrace{dz\cdots dz}_{k_r}\,.
\end{aligned}$$

Putting $z = \ln(1+t)$, we get the result. ∎

6

Remark. If $r=1$ in Proposition 1, then by $\mathrm{Lif}_1(z)=(e^z-1)/z$ we have

$$\frac{(1+t)^z}{\ln(1+t)}\underbrace{\int_0^t\frac{1}{(1+t)\ln(1+x)}\cdots\int_0^t\frac{1}{(1+t)\ln(1+t)}}_{k-1}\times t\,\underbrace{dtdt\dots dt}_{k-1}$$
$$=\sum_{n=0}^\infty c_n^{(k)}(z)\frac{t^n}{n!}\,,$$

which is Corollary 2.3 in [5].

3 Functions interpolating the multi-poly-Cauchy polynomials

Let $k_1,\dots,k_r$ be positive integers. Define the function $Z_{k_1,\dots,k_r}(s,z)$ for $s\in\mathbb{C}$ with $\Re(s)>0$ and $z>-1$ by

$$Z_{k_1,\dots,k_r}(s,z):=\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\bigl(\ln(1-t)\bigr)dt\,. \tag{1}$$

By the change of the variables $t=1-e^{-u}$, this can be written as

$$Z_{k_1,\dots,k_r}(s,z)=\frac{1}{\Gamma(s)}\int_0^\infty(1-e^{-u})^{s-1}e^{-(z+1)u}\mathrm{Lif}_{k_1,\dots,k_r}(-u)du\,. \tag{2}$$

Theorem 3 *For $n=0,1,2,\dots,$ we have*

$$Z_{k_1,\dots,k_r}(-n,z)=c_n^{(k_1,\dots,k_r)}(z)\,.$$

Remark. If $r=1$, then Theorem 3 is reduced to Proposition 6.2 in [5].

Proof. Let b be an arbitrary non-negative integer and γ an arbitrary real number with $0<\gamma<1$. Then by the equation (1), we have

$$\begin{aligned}&\int_0^1 t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\bigl(\ln(1-t)\bigr)dt\\&=\int_0^\gamma t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\bigl(\ln(1-t)\bigr)dt+\int_\gamma^1 t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\bigl(\ln(1-t)\bigr)dt\\&=\int_0^\gamma t^{s-1}\sum_{m=0}^b\frac{(-1)^m c_m^{(k_1,\dots,k_r)}(z)}{m!}t^m dt+\int_0^\gamma t^{s-1}\sum_{m=b+1}^\infty\frac{(-1)^m c_m^{(k_1,\dots,k_r)}(z)}{m!}t^m dt\\&\quad+\int_\gamma^1 t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\bigl(\ln(1-t)\bigr)dt.\end{aligned}$$

The first term is equal to

$$\sum_{m=0}^{b}\frac{(-1)^m c_m^{(k_1,\dots,k_r)}(z)}{m!}\left[\frac{t^{s+m}}{s+m}\right]_0^{\gamma}=\sum_{m=0}^{b}\frac{(-1)^m c_m^{(k_1,\dots,k_r)}(z)}{m!}\frac{\gamma^{s+m}}{(s+m)}.$$

Since

$$\frac{1}{\Gamma(s)}\frac{1}{(s+m)}=\frac{s^{(m)}}{\Gamma(s+m+1)},$$

where $s^{(m)}=s(s+1)\cdots(s+m-1)$ $(m>0)$ is the rising factorial with $s^{(0)}=1$, we have for $\Re(s)>0$

$$\begin{aligned}
&Z_{k_1,\dots,k_r}(s,z)\\
&=\sum_{m=0}^{b}\frac{s^{(m)}}{m!\Gamma(s+m+1)}c_m^{(k_1,\dots,k_r)}(z)(-1)^m\gamma^{s+m}\\
&\quad+\frac{1}{\Gamma(s)}\Bigg(\int_0^{\gamma}t^{s-1}\left((1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\big(\ln(1-t)\big)-\sum_{m=0}^{b}\frac{(-1)^m c_m^{(k_1,\dots,k_r)}(z)}{m!}t^m\right)dt\\
&\qquad+\int_{\gamma}^{1}t^{s-1}(1-t)^z\mathrm{Lif}_{k_1,\dots,k_r}\big(\ln(1-t)\big)dt\Bigg).
\end{aligned}\tag{3}$$

The first integration converges if $\Re(s)>-1-b$ and the second integration converges for an arbitrary $s\in\mathbb{C}$. Hence the right-hand side of (3) defines holomorphic function for $\Re(s)>-1-b$. Since b is arbitrary, we obtain the holomorphic continuation of $Z_{k_1,\dots,k_r}(s,z)$ to the whole s-plane. Finally, when $s=-n$, only the term for $m=n$ in the first part remains and $Z_{k_1,\dots,k_r}(-n,z)=c_n^{(k_1,\dots,k_r)}(z)$ is obtained. ∎

Theorem 3 gives the values of $Z_{k_1,\dots,k_r}(s,z)$ at negative integers. The values at positive integers are expressed by using values of polylogarithm functions $\mathrm{Li}_k(z)$ and generalized harmonic functions $H_n^{(k)}(z)$, defined by

$$H_n^{(k)}(z)=\frac{z}{1^k}+\frac{z^2}{2^k}+\cdots+\frac{z^n}{n^k}\,.$$

Theorem 4 *Let n and $k_1,\dots,k_r$ be positive integers. For $z\ge 0$, we have*

$$\begin{aligned}
&Z_{k_1,\dots,k_r}(n,z)\\
&=\frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l+1}\sum_{0\le m_1<\cdots<m_{r-1}}\frac{1}{m_1!\dots m_{r-1}!(m_1+1)^{k_1}\dots(m_{r-1}+1)^{k_{r-1}}}\\
&\qquad\left(\mathrm{Li}_{k_r}\left(-\frac{1}{l+1+z}\right)-H_{m_{r-1}+1}^{(k_r)}\left(-\frac{1}{l+1+z}\right)\right)\,.
\end{aligned}$$

Remark. If $r=1$, then Theorem 4 is reduced to Proposition 6.3 in [5].

Proof. By the expression (2), we have

$$Z_{k_1,\dots,k_r}(n,z)=\frac{1}{(n-1)!}\int_0^\infty\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l e^{-(l+1+z)u}$$
$$\sum_{0\le m_1<\cdots<m_r}\frac{(-u)^{m_r}du}{m_1!\dots m_r!(m_1+1)^{k_1}\dots(m_r+1)^{k_r}}\,.$$

By changing the variables $u=v/(l+1+z)$, we have

$$Z_{k_1,\dots,k_r}(n,z)=\frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\int_0^\infty e^{-v}$$
$$\sum_{0\le m_1<\cdots<m_r}\frac{(-1)^{m_r}v^{m_r}dv}{m_1!\dots m_r!(m_1+1)^{k_1}\dots(m_r+1)^{k_r}(l+1+z)^{m_r+1}}\,.$$

Since $m!=\int_0^\infty e^{-v}v^m dv$, we obtain that

$$\begin{aligned}
&Z_{k_1,\dots,k_r}(n,z)\\
&=\frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\\
&\quad\sum_{0\le m_1<\cdots<m_r}\frac{(-1)^{m_r}}{m_1!\dots m_{r-1}!(m_1+1)^{k_1}\dots(m_r+1)^{k_r}(l+1+z)^{m_r+1}}\\
&=\frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l+1}\sum_{0\le m_1<\cdots<m_{r-1}}\frac{1}{m_1!\dots m_{r-1}!(m_1+1)^{k_1}\dots(m_{r-1}+1)^{k_{r-1}}}\\
&\quad\sum_{m_r=m_{r-1}+1}^{\infty}\left(\frac{-1}{l+1+z}\right)^{m_r+1}\frac{1}{(m_r+1)^{k_r}}\\
&=\frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l+1}\sum_{0\le m_1<\cdots<m_{r-1}}\frac{1}{m_1!\dots m_{r-1}!(m_1+1)^{k_1}\dots(m_{r-1}+1)^{k_{r-1}}}\\
&\quad\left(\mathrm{Li}_{k_r}\left(-\frac{1}{l+1+z}\right)-H_{m_{r-1}+1}^{(k_r)}\left(-\frac{1}{l+1+z}\right)\right)\,.
\end{aligned}$$

∎

We shall show the following duality formula between $Z_{k_1,\dots,k_r}(s,z)$ and $\xi_k(s,z)$.

Theorem 5 *For positive integers $k_1,\dots,k_r$ and $\rho\ge 2$ and a real number z with $-1<z\le 0$, we have*

$$\sum_{n=1}^{\infty}\frac{\Gamma(n)}{n^\rho}Z_{k_1,\dots,k_r}(n,z)$$
$$=\sum_{0\le m_1<\cdots<m_{r-1}<m_r-1}\frac{(-1)^{m_r-1}}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}}\frac{\xi_\rho(m_r,z+1)}{m_r^{k_r}}\,.$$

Remark. If $r=1$, then Theorem 5 is reduced to Corollary 6.6 in [5].

Proof. We shall calculate

$$\int_0^\infty \frac{e^{-uz}\mathrm{Li}_\rho(1-e^{-u})\mathrm{Lif}_{k_1,\dots,k_r}(-u)}{e^u-1}du \tag{4}$$

in two ways. Firstly, (4) is equal to

$$\begin{aligned}&\sum_{n=1}^\infty \frac{1}{n^\rho}\int_0^\infty \frac{e^{-uz}(1-e^{-u})^n}{e^u-1}\mathrm{Lif}_{k_1,\dots,k_r}(-u)du\\
&=\sum_{n=1}^\infty \frac{\Gamma(n)}{n^\rho}\frac{1}{\Gamma(n)}\int_0^\infty e^{-(z+1)u}(1-e^{-u})^{n-1}\mathrm{Lif}_{k_1,\dots,k_r}(-u)du\\
&=\sum_{n=1}^\infty \frac{\Gamma(n)}{n^\rho}Z_{k_1,\dots,k_r}(n,z)\,.\end{aligned}$$

On the other hand, (4) is equal to

$$\begin{aligned}&\sum_{0\le m_1<\dots<m_r}\frac{(-u)^{m_r}}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\int_0^\infty \frac{e^{-uz}\mathrm{Li}_\rho(1-e^{-u})}{e^u-1}du\\
&=\sum_{0\le m_1<\dots<m_{r-1}<m_r-1}\frac{(-1)^{m_r-1}}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}m_r^{k_r}}\\
&\qquad\times\frac{1}{\Gamma(m_r)}\int_0^\infty \frac{u^{m_r-1}e^{-uz}\mathrm{Li}_\rho(1-e^{-u})}{e^u-1}du\\
&=\sum_{0\le m_1<\dots<m_{r-1}<m_r-1}\frac{(-1)^{m_r-1}}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}}\frac{\xi_\rho(m_r,z+1)}{m_r^{k_r}}\,.\end{aligned}$$

Combining two expressions, we get the result. ∎

4 The second case

Let $k_1,\dots,k_r$ be positive integers. Define *multi-poly-Cauchy polynomials of the second kind* $\widehat{c}_n^{(k_1,k_2,\dots,k_r)}(z)$ $(n=0,1,2,\dots)$ by the generating function

$$\frac{\mathrm{Lif}_{k_1,k_2,\dots,k_r}\bigl(-\ln(1+t)\bigr)}{(1+t)^z}=\sum_{n=0}^\infty \widehat{c}_n^{(k_1,k_2,\dots,k_r)}(z)\frac{t^n}{n!}\,.$$

When $z=0$, define *multi-poly-Cauchy numbers of the second kind* $\widehat{c}_n^{(k_1,k_2,\dots,k_r)}=\widehat{c}_n^{(k_1,k_2,\dots,k_r)}(0)$ by

$$\mathrm{Lif}_{k_1,k_2,\dots,k_r}\bigl(-\ln(1+t)\bigr)=\sum_{n=0}^\infty \widehat{c}_n^{(k_1,k_2,\dots,k_r)}\frac{t^n}{n!}\,.$$

If $r=1$, then $\widehat{c}_n^{(k)}(z)$ and $\widehat{c}_n^{(k)}$ are the poly-Cauchy polynomials and numbers of the second kind ([5, 8, 9]), defined by

$$\frac{\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)}{(1+t)^z}=\sum_{n=0}^\infty \widehat{c}_n^{(k)}(z)\frac{t^n}{n!}$$

and

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)=\sum_{n=0}^\infty \widehat{c}_n^{(k)}\frac{t^n}{n!}\,,$$

respectively.

The multi-poly-Cauchy numbers of the second kind can be written in terms of the Stirling numbers of the first kind.

Theorem 6 *For a nonnegative integer n and positive integers $k_1,k_2,\dots,k_r$, we have*

$$\widehat{c}_n^{(k_1,k_2,\dots,k_r)}=\sum_{0\le m_1<m_2<\cdots<m_r\le n}\frac{(-1)^{m_r}s(n,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\,.$$

Proof. By

$$\frac{\bigl(\ln(1+t)\bigr)^m}{m!}=\sum_{n=m}^\infty s(n,m)\frac{t^n}{n!}\,,$$

we have

$$\begin{aligned}
&\sum_{n=0}^\infty \widehat{c}_n^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\\
&=\sum_{0\le m_1<m_2<\cdots<m_r}\frac{\bigl(-\ln(1+t)\bigr)^{m_r}}{m_1!\cdots m_{r-1}!m_r!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\\
&=\sum_{0\le m_1<m_2<\cdots<m_r}\frac{(-1)^{m_r}}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{n=m_r}^\infty s(n,m_r)\frac{t^n}{n!}\\
&=\sum_{n=0}^\infty\left(\sum_{0\le m_1<m_2<\cdots<m_r\le n}\frac{(-1)^{m_r}s(n,m_r)}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\right)\frac{t^n}{n!}\,.
\end{aligned}$$

By comparing the coefficients of the both sides, we get the desired result. ∎

The multi-poly-Cauchy polynomials of the second kind can be also written in terms of the Stirling numbers of the first kind.

Theorem 7 *For a nonnegative integer n and positive integers $k_1, k_2, \ldots, k_r$, we have*

$$\widehat{c}_n^{(k_1,\ldots,k_r)}(z) = n! \sum_{0 \le m_1 < m_2 < \cdots < m_r \le n} \frac{1}{m_1! \cdots m_{r-1}!(m_1+1)^{k_1} \cdots (m_r+1)^{k_r}} \sum_{\nu=m_r}^{n} \binom{z+n-\nu-1}{n-\nu} \frac{(-1)^{n+m_r-\nu} s(\nu, m_r)}{\nu!}.$$

Proof. By Theorem 6, observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{c}_n^{(k_1,\ldots,k_r)}(z) \frac{t^n}{n!} = \frac{1}{(1+t)^z} \mathrm{Lif}_{k_1,\ldots,k_r}\big(-\ln(1+t)\big) \\
&= \sum_{\mu=0}^{\infty} \binom{z+\mu-1}{\mu} (-1)^{\mu} t^{\mu} \\
&\qquad \sum_{\nu=0}^{\infty} \left(\sum_{0 \le m_1 < m_2 < \cdots < m_r \le \nu} \frac{(-1)^{m_r} s(\nu, m_r)}{m_1! \cdots m_{r-1}!(m_1+1)^{k_1} \cdots (m_r+1)^{k_r}} \right) \frac{t^{\nu}}{\nu!} \\
&= \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \binom{z+n-\nu-1}{n-\nu} (-1)^{n-\nu} \\
&\qquad \sum_{0 \le m_1 < m_2 < \cdots < m_r \le \nu} \frac{(-1)^{m_r} s(\nu, m_r)}{m_1! \cdots m_{r-1}!(m_1+1)^{k_1} \cdots (m_r+1)^{k_r}} \frac{n!}{\nu!} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} n! \sum_{0 \le m_1 < m_2 < \cdots < m_r \le n} \frac{1}{m_1! \cdots m_{r-1}!(m_1+1)^{k_1} \cdots (m_r+1)^{k_r}} \\
&\qquad \sum_{\nu=m_r}^{n} \binom{z+n-\nu-1}{n-\nu} \frac{(-1)^{n+m_r-\nu} s(\nu, m_r)}{\nu!} \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of the both sides, we get the desired result. ∎

Remark. If $r = 1$ in Theorem 7, then

$$\widehat{c}_n^{(k)}(z) = n! \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{\nu=m}^{n} \binom{z+n-\nu-1}{n-\nu} \frac{(-1)^{n+m-\nu} s(\nu, m)}{\nu!}.$$

On the other hand, by Theorem 3.1 in [5]

$$\begin{aligned}
\widehat{c}_n^{(k)}(z) &= \sum_{m=0}^{n} (-1)^m s(n, m) \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} \\
&= \sum_{i=0}^{n} z^i \sum_{m=i}^{n} \binom{m}{i} \frac{(-1)^m s(n, m)}{(m-i+1)^k}.
\end{aligned}$$

By using Lemma 1 we can write the generating function of the multi-poly-Cauchy polynomials of the second kind $\widehat{c}_n^{(k_1,\dots,k_r)}(z)$ in the form of iterated integrals. The proof is similar to that of Prop 1 and is omitted.

Proposition 2 *For positive integers $k_1,\dots,k_r$*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{c}_n^{(k_1,\dots,k_r)}(z)\frac{t^n}{n!}\\
&=\frac{1}{(1+t)^z\ln(1+t)}\underbrace{\int_0^t\frac{1}{(1+t)\ln(1+t)}\cdots\int_0^t}_{k_r-1}\frac{\mathrm{Lif}_{k_1,\dots,k_{r-1},1}\bigl(-\ln(1+x)\bigr)}{1+t}\underbrace{dt\cdots dt}_{k_r-1}\\
&=\sum_{0\le m_1<\cdots<m_r}\frac{1}{m_1!\cdots m_r!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}}\frac{1}{(1+t)^z\ln(1+t)}\\
&\qquad\underbrace{\int_0^t\frac{1}{(1+t)\ln(1+t)}\cdots\int_0^t\frac{1}{(1+t)\ln(1+t)}\int_0^t}_{k_r}\frac{\bigl(-\ln(1+t)\bigr)^{m_r}}{1+t}\underbrace{dt\cdots dt}_{k_r}\,.
\end{aligned}
$$

Remark. If $r=1$ in Proposition 2, then it is reduced to Corollary 3.3 in [5].

Define the function $\widehat{Z}_{k_1,\dots,k_r}(s,z)$ for $s\in\mathbb{C}$ with $\Re(s)>0$ and $z>-1$ by

$$\widehat{Z}_{k_1,\dots,k_r}(s,z):=\frac{1}{\Gamma(s)}\int_0^1\frac{t^{s-1}}{(1-t)^z}\mathrm{Lif}_{k_1,\dots,k_r}\bigl(-\ln(1-t)\bigr)dt \tag{5}$$

or equivalently,

$$\widehat{Z}_{k_1,\dots,k_r}(s,z)=\frac{1}{\Gamma(s)}\int_0^\infty(1-e^{-u})^{s-1}e^{(z-1)u}\mathrm{Lif}_{k_1,\dots,k_r}(u)du\,. \tag{6}$$

Theorem 8 *For $n=0,1,2,\dots,$ we have*

$$\widehat{Z}_{k_1,\dots,k_r}(-n,z)=\hat{c}_n^{k_1,\dots,k_r}(z)\,.$$

Proof. The analytic continuation of $\widehat{Z}_{k_1,\dots,k_r}(-n,z)$ can be shown in a similar way to that of $Z_{k_1,\dots,k_r}(-n,z)$ in Theorem 3. By the equation (5), for $n=0,1,2,\dots,$ we have $\widehat{Z}_{k_1,\dots,k_r}(-n,z)=\widehat{c}_n^{(k_1,\dots,k_r)}(z)$. ∎

Remark. If $r=1$, then Theorem 8 is reduced to Proposition 7.2 in [5].

The function $\hat{Z}_{k_1,\dots,k_r}(s,z)$ satisfies similar properties to those of $Z_{k_1,\dots,k_r}(s,z)$. They are proven in the same manner, so we state here only the results and omit their proofs.

Theorem 9 *Let n and $k_1, \dots, k_r$ be positive integers. For $z \geq 0$, we have*

$$\begin{aligned}&\widehat{Z}_{k_1,\dots,k_r}(n,z)\\&= \frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l \sum_{0\le m_1<\dots<m_{r-1}} \frac{1}{m_1!\dots m_{r-1}!(m_1+1)^{k_1}\dots(m_{r-1}+1)^{k_{r-1}}}\\&\qquad\left(\mathrm{Li}_{k_r}\left(\frac{1}{l+1-z}\right) - H^{(k_r)}_{m_{r-1}+1}\left(\frac{1}{l+1-z}\right)\right).\end{aligned}$$

Remark. If $r=1$, then Theorem 9 is reduced to Proposition 7.3 in [5].

Theorem 10 *For positive integers $k_1, \dots, k_r$ and $\rho \geq 2$ and a real number $-1 < z \leq 0$, we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\Gamma(n)}{n^\rho}\widehat{Z}_{k_1,\dots,k_r}(n,z)\\&\qquad= \sum_{0\le m_1<\dots<m_{r-1}<m_r-1} \frac{(-1)^{m_r-1}}{m_1!\cdots m_{r-1}!(m_1+1)^{k_1}\cdots(m_{r-1}+1)^{k_{r-1}}}\frac{\xi_\rho(m_r,1-z)}{m_r^{k_r}}.\end{aligned}$$

Remark. If $r=1$, then Theorem 10 is reduced to Corollary 7.5 in [5].

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RATES OF APPROXIMATION BY NEURAL NETWORKS WITH FOUR LAYERS

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Abstract. We invesigate the rates of approximation by a type of feed-forward neural networks with four layers, for some important function classes including functions of bounded $\phi-$variation, functions in Sobolev space $W_{p,[a,b]}$ and functions in L^p spaces.

1. Introduction

A bounded function $\sigma : R \to R$ is called a sigmoidal function if $\sigma(x) \to 1$ as $x \to +\infty$ and $\sigma(x) \to 0$ as $x \to -\infty$. Denote by $G_{n,\sigma}$ the set of all feed-forward neural networks (FNN) with activation function σ, that is,

$$G_{n,\sigma} := \left\{ N(x) := \sum_{j=0}^{n} c_j \sigma(a_j x + b_j),\ x, a_j, b_j, c_j \in \mathbb{R} \right\}.$$

As we know, FNNS are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons. A lot of results concerning the existence of an approximation and the determination of the number of neurons required to guarantee that all functions (belong to a certain class) can be approximated to the prescribed degree of accuracy, have been achieved by many mathematicians. When σ is a sigmoidal function, by a result of Gao and Xu ([5]), each continuous function of bounded variation f can be approximated, with respect to the uniform norm on the interval $[a, b]$, by feed-forward neural networks in $G_{n,\sigma}$ with the error $O(n^{-1})$. Later, Lewicki and Marino [6] generalized the result of Gao and Xu ([5]) by considering the approximation rate for functions satisfying a property (P). Their result can be read as follows:

Theorem LM *Let $\phi : R^+ \to R^+$ be a continuous, strictly increasing function such that $\phi(0) = 0$. Let the function $f \in C[a, b]$ satisfy the property*

(P) *There exists a constant $C > 0$ such that for every $n \in N$ we can select a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for every $i = 1, 2, \cdots, n$, if $x, y \in I_i = [x_{i-1}, x_i]$, then*

$$|f(x) - f(y)| \le \phi^{-1}\left(\frac{C}{n}\right), \tag{1.1}$$

and let $\sigma \in L_\infty(R)$ be a fixed sigmoidal function. Then

$$dist(f, G_{n,\sigma}) \le (1 + 8\|\sigma\|_\infty)\, \phi^{-1}\left(\frac{C}{n}\right),$$

where the distance is taken with respect to the supremum norm denoted by $\|\cdot\|_{[a,b]}$ on $[a, b]$.

Key words and phrases. feed-forward neural networks; functions of bounded variation; Sobolev space; approximation rate.

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Let ϕ be as in Theorem LM, $f \in C[a,b]$. Set

$$V_\phi(f)_{[a,b]} := \sup\left\{\sum_{j=0}^{n-1}\phi\left(|f(x_{j+1}) - f(x_j)|\right) : a = x_0 < x_1 < \cdots < x_n = b\right\}.$$

We say that f is of bounded $\phi-$variation if $V_\phi(f)_{[a,b]} < \infty$. It is shown that (see [6]), if f is of bounded $\phi-$variation, then f satisfies property (P) with $C = V_\phi(f)_{[a,b]}$.

Recently, some authors investigated the FNN with activation function $g_j : \mathbb{R}^d \to \mathbb{R}$, defined by

$$g_j(x) := \frac{e^{-A\rho(x,x_j)}}{\sum_{i=0}^n e^{-A\rho(x,x_i)}}, j = 0, 1, 2, \cdots,$$

where $x_0, x_1, \cdots, x_n$ are the points in $\mathbb{R}^d$, $\rho(a,b)$ denotes the Euclidean distance between the points a and b in $\mathbb{R}^d$, and $A > 0$ is a parameter. In this case, $N(x)$ can be regarded as a FNN with four layers: the first layer is the input layer, the input is x; the second layer is processing layer for computing the distances of $\rho(x, x_i), i = 0, 1, \cdots, n$, between the input x and the prototypical input points x_i; The values $\rho(x, x_i)$ from the second layer are the inputs of the third layer that contains $n+1$ neurons, each output $g_j(x)$ is activation function of the j-th neuron; the fourth layer is output layer, the output is $N(x)$. Although $g_j(x)$ are not sigmoidal, they have some better properties than the usual sigmoidal functions. Indeed, we have: (i) $0 < g_j(x) \le 1, j = 0, 1, \cdots n$; (ii) $\sum_{j=0}^n g_j(x) = 1,\ x \in \mathbb{R}^d$.

In [4], Cao, Zhang and Xu constructed a class of neural networks $N_{n,a}(f,x)$ with activation functions $g_j(x)$ on a finite interval $[a,b]$ as follows:

$$N_{n,a}(f,x) := \sum_{j=0}^n f(x_j)\frac{e^{-A(n)|x-x_j|}}{\sum_{i=0}^n e^{-A(n)|x-x_i|}}, \tag{1.2}$$

where $\{x_j\}_{j=0}^n$ are nodes on $[a,b]$ and $A(n)$ is a parameter depending on n. They also gives the rate of approximation by operator $N_{n,a}(f,x)$ for continuous functions on $[a,b]$.

Set

$$N_n := \left\{N_{n,a}(f,x) = \sum_{j=0}^n f(x_j)\frac{e^{-A(n)|x-x_j|}}{\sum_{i=0}^n e^{-A(n)|x-x_i|}} : x_i \in [a,b]\right\}.$$

The approximation properties of neural networks with four layers were also studied by Anastassiou (see [1]-[3]).

The first purpose of this note is to estimate the rate of approximation by elements in N_n for functions satisfying the property (P), and show that similar result of Theorem LM holds for N_n. In fact, we have

Theorem 1. *Let ϕ be defined as in Theorem LM, $f \in C[a,b]$ satisfy property (P). Then*

$$dist(f, N_n) \le \frac{2e^2+4e}{(e+1)^2}\phi^{-1}\left(\frac{C}{n}\right).$$

Define the Sobolev space $W_{p,[a,b]}$ as follows:

$$W_{p,[a,b]} := \{f : f \in AC([a,b]), f' \in L^p[a,b]\},\ 1 \le p < \infty.$$

For $f \in W_{p,[a,b]}$, we have

Theorem 2. *For any $f \in W_{p,[a,b]}$, it holds that*

$$dist(f, N_n)_p \le C_p(n+1)^{-1}\|f'\|_p,\ 1 \le p < \infty,$$

where $\|\cdot\| p$ is the norm in $L^p[a,b]$ space, the distance $dist(f,N_n)_p$ is taken with respect to the L^p-norm, and C_p is a constant only depending on p.

Theorem 2 above generalzies the result given by Petrushev and Popov in [7] for approximation by rational functions, where they only gave the result for $p>1$.

Finally, for $f\in L^p_{[a,b]}$, we have

Theorem 3. *For any $f\in L^p_{[a,b]}$, it holds that*

$$dist\,(f,N_n)_p\le C_p\omega\left(f,\frac{1}{n+1}\right)_p,\ 1\le p<\infty,$$

where $\omega\,(f,t)_p$ is the modulus of continuity of f under the L^p-norm, that is,

$$\omega\,(f,t)_p:=\sup_{0<h\le t}\left(\int_a^{b-h}|f(x+h)-f(x)|^p dx\right)^{1/p}.$$

Throughout the paper, C_p denotes a positive constant only depending on p, which may be different in different occurrence.

2. Proofs

2.1. **Proof of Theorem 1.** Since f and ϕ satisfy property (P), there exits a partition $a=x_0<x_1<\cdots<x_n=b$ such that (1.1) holds for $x,y\in I_i=[x_{i-1},x_i]\,,i=1,2,\cdots,n$. Taking

$$A\,(n):=\frac{1}{\min_{0\le i\le n-1}\{x_{i+1}-x_i\}}.$$

We only need to prove the following

$$\|N_{n,a}\,(f)-f\|\le\frac{2e^2+4e}{(e+1)^2}\phi^{-1}\left(\frac{C}{n}\right),\tag{2.1}$$

where $N_{n,a}\,(f,x)$ is defined by (1.2). For convenience, we write

$$r_j\,(x):=\frac{e^{-A(n)|x-x_j|}}{\sum_{i=0}^n e^{-A(n)|x-x_i|}},\ j=0,1,\cdots,n.$$

Assume that $\min_{0\le i\le n-1}\{x_{i+1}-x_i\}=x_{j_0+1}-x_{j_0},0\le j_0\le n-1$ and $x\in[x_k,x_{k+1}]\,,k=0,1,\cdots,n$. Since $N_{n,a}\,(1,x)\equiv 1$, we have[1]

$$\begin{aligned}N_{n,a}\,(f,x)-f\,(x)&=\sum_{j=0}^{n}(f(x_j)-f\,(x))\,r_j\,(x)\\&=\sum_{j=0}^{k-1}(f(x_j)-f\,(x))\,r_j\,(x)+(f\,(x_k)-f\,(x))\,r_j\,(x)\\&+\sum_{j=k+1}^{n}(f(x_j)-f\,(x))\,r_j\,(x)\\&=:I_1+I_2+I_3.\end{aligned}\tag{2.2}$$

By (1.1) and noting that $\frac{e^{-A(n)|x-x_k|}}{\sum_{i=0}^n e^{-A(n)|x-x_i|}}\le 1$, we have

$$|I_2|\le\phi^{-1}\left(\frac{C}{n}\right).\tag{2.3}$$

[1]When $k=0$ (or $k=n$), I_1 (or I_3) vanishes.

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For I_1, we have

$$
\begin{aligned}
|I_1| &\le \sum_{j=0}^{k-1} |f(x_j) - f(x)|\, e^{-A(n)(|x-x_j|-|x-x_k|)} \\
&\le \sum_{j=0}^{k-1} |f(x_j) - f(x)|\, e^{-A(n)(x_k-x_j)} \\
&\le \sum_{j=0}^{k-1} |f(x_j) - f(x)|\, e^{-A(n)(k-j)\left(x_{j_0+1}-x_{j_0}\right)} \\
&\le \sum_{j=0}^{k-1} |f(x_j) - f(x)|\, e^{-(k-j)}.
\end{aligned}
$$

By (1.1), it is obvious that

$$
|f(x_j) - f(x)| \le |f(x_j) - f(x_{j+1})| + \cdots + |f(x_k) - f(x)| \le (k-j+1)\,\phi^{-1}\left(\frac{C}{n}\right).
$$

Therefore,

$$
\begin{aligned}
|I_1| &\le \phi^{-1}\left(\frac{C}{n}\right) \sum_{j=0}^{k-1} (k-j+1)\, e^{-(k-j)} \\
&\le \phi^{-1}\left(\frac{C}{n}\right) \left(\sum_{i=1}^{k} i e^{-i} + \sum_{i=1}^{k} e^{-i} \right) \\
&\le \frac{2e-1}{(e+1)^2} \phi^{-1}\left(\frac{C}{n}\right),
\end{aligned}
\tag{2.4}
$$

where in the last inequality, we used the identity

$$
\sum_{i=1}^{\infty} i q^i = \frac{q}{(1-q)^2},\ |q| < 1,
$$

which in turn can be derived by using the derivatives of the geometric series in the circle of convergence.

Similarly,

$$
\begin{aligned}
|I_3| &\le \sum_{j=k+1}^{n} |f(x_j) - f(x)|\, e^{-A(n)(|x-x_j|-|x-x_{k+1}|)} \\
&\le \phi^{-1}\left(\frac{C}{n}\right) \sum_{j=k+1}^{n} (j-k)\, e^{-A(n)(x_j-x_{k+1})} \\
&\le \phi^{-1}\left(\frac{C}{n}\right) \sum_{j=k+1}^{n} (j-k)\, e^{-A(n)(j-k-1)\left(x_{j_0+1}-x_{j_0}\right)}
\end{aligned}
$$

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$$\le \phi^{-1}\left(\frac{C}{n}\right)\sum_{j=k+1}^{n}(j-k)\,e^{-(j-k-1)}$$
$$\le \phi^{-1}\left(\frac{C}{n}\right)\sum_{i=1}^{n-k} ie^{-i+1}$$
$$\le \frac{e^2}{(e+1)^2}\phi^{-1}\left(\frac{C}{n}\right). \tag{2.5}$$

Combining (2.2)-(2.5), we have

$$|N_{n,a}(f,x)-f(x)| \le \frac{2e^2+4e}{(e+1)^2}\phi^{-1}\left(\frac{C}{n}\right),$$

which proves (2.1).

2.2. **Proof of Theorem 2.** If we consider the function $g(x)=f(a+(b-a)x)$, it is sufficient to take only the case when $[a,b]=[0,1]$.

In this part, we take $x_j=\frac{j}{n+1}$, $j=0,1,2,\cdots,n$, and $A(n)=n+1$ in (1.2). Then, we only need to prove the following:

$$\|f-N_{n,a}(f)\|_p \le C_p\,(n+1)^{-1}\,\|f'\|_p\,,\ 1\le p<\infty. \tag{2.6}$$

By Hlder's inequality, we deduce that

$$\int_0^1 |f(x)-N_{n,a}(f,x)|^p\,dx = \int_0^1\left|\sum_{j=0}^{n}\int_x^{x_j} f'(t)\,dt\,r_j(x)\right|^p dx$$
$$\le \int_0^1\sum_{j=0}^{n}\left|\int_x^{\frac{j}{n+1}} f'(t)\,dt\right|^p r_j(x)\,dx$$
$$\le \int_0^1\sum_{j=0}^{n}\left|\int_x^{\frac{j}{n+1}} |f'(t)|^p\,dt\right| |x-x_j|^{p-1}\,r_j(x)\,dx$$
$$\le \sum_{k=0}^{n}\sum_{j=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left|\int_{\frac{k^*}{n+1}}^{\frac{j}{n+1}}|f'(t)|^p\,dt\right|\left|\frac{k^*}{n+1}-\frac{j}{n+1}\right|^{p-1} r_j(x)\,dx,$$

where $k^*=k$ for $k>j$, $k^*=k+1$ for $k\le j$. Similar to the proof of Theorem 1, we have for $x\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]$ that

$$r_j(x)=\frac{e^{-(n+1)\left|x-\frac{j}{n+1}\right|}}{\sum_{j=0}^{n} e^{-(n+1)\left|x-\frac{j}{n+1}\right|}} \le e^{-(n+1)\left(\left|x-\frac{j}{n+1}\right|-\left|x-\frac{k}{n+1}\right|\right)} = e^{-(k-j)},\ \text{for } j\le k,$$

and

$$r_j(x)\le e^{-(n+1)\left(\left|x-\frac{j}{n+1}\right|-\left|x-\frac{k+1}{n+1}\right|\right)} = e^{-(j-k-1)},\ \text{for } j>k.$$

Hence,

$$\int_0^1 |f(x)-N_{n,a}(f,x)|^p\,dx \le (n+1)^{-p}\sum_{k=0}^{n}\sum_{j=0}^{n}\left|\int_{\frac{k^*}{n+1}}^{\frac{j}{n+1}}|f'(t)|^p\,dt\right|(|j-k|+1)^p\,e^{-(|j-k|+1)}$$
$$= (n+1)^{-p}\sum_{m=0}^{n}\sum_{0\le j,k\le n,|j-k|=m}\left|\int_{\frac{k^*}{n+1}}^{\frac{j}{n+1}}|f'(t)|^p\,dt\right|(m+1)^p\,e^{-(m+1)}. \tag{2.7}$$

For any given k $(0 \le k \le n)$, we have for $m \ge 1$ that (when $k > n-m$ (or $k < m$), the second last term (or the last term) in (2.8) should be understood as zero)

$$\begin{aligned}\sum_{|j-k|=m}\left|\int_{\frac{k^*}{n+1}}^{\frac{j}{n+1}}|f'(t)|^p\,dt\right| &= \int_{\frac{k}{n+1}}^{\frac{m+k}{n+1}}|f'(t)|^p\,dt+\int_{\frac{k-m}{n+1}}^{\frac{k}{n+1}}|f'(t)|^p\,dt\\ &= \sum_{i=0}^{m-1}\int_{\frac{i+k}{n+1}}^{\frac{i+k+1}{n+1}}|f'(t)|^p\,dt+\sum_{i=0}^{m-1}\int_{\frac{k-i-1}{n+1}}^{\frac{k-i}{n+1}}|f'(t)|^p\,dt, \qquad (2.8)\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{0\le j,k\le n,|j-k|=m}\left|\int_{\frac{k^*}{n+1}}^{\frac{j}{n+1}}|f'(t)|^p\,dt\right| &= \sum_{k=0}^{n+1-m}\sum_{i=0}^{m-1}\int_{\frac{i+k}{n+1}}^{\frac{i+k+1}{n+1}}|f'(t)|^p\,dt+\sum_{k=m}^{n}\sum_{i=0}^{m-1}\int_{\frac{k-i-1}{n+1}}^{\frac{k-i}{n+1}}|f'(t)|^p\,dt\\ &\le 2m\int_0^1|f'(t)|^p\,dt. \qquad (2.9)\end{aligned}$$

Substituting (2.9) into (2.7), we get

$$\begin{aligned}\int_0^1|f(x)-N_{n,a}(f,x)|^p\,dx &\le (n+1)^{-p}\left(\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f'(t)|^p\,dt+\sum_{m=1}^{n}2m(m+1)^p e^{-(m-1)}\int_0^1|f'(t)|^p\,dt\right)\\ &\le (n+1)^{-p}\left(1+2\sum_{m=1}^{n}(m+1)^{p+1}e^{-(m-1)}\right)\int_0^1|f'(t)|^p\,dt,\end{aligned}$$

which proves (2.6).

2.3. **Proof of Theorem 3.** We also take only the case when $[a,b]=[0,1]$. For any given $h(0 < h < 1)$, define the Steklov function $f_h(x)$ as follows:

$$f_h(x) := \frac{1}{h}\int_0^h f(x+u)du,\ 0 \le x \le 1-h.$$

Then, it is obvious that $f_h(x)$ is absolute continuous,

$$f_h'(x) = \frac{1}{h}\left(f(x+h)-f(x)\right)\text{ for almost every } x \in [0,1-h],$$

and

$$\left(\int_0^{1-h}|f(x)-f_h(x)|^p dx\right)^{1/p} \le \omega(f,h)_p,\ 1 \le p < \infty. \qquad (2.10)$$

Set

$$f_h^*(x) := \begin{cases} f_h(x), & 0 \le x \le 1-h,\\ f_h(1-h), & 1-h < x \le 1.\end{cases}$$

Then,

$$(f_h^*(x))' = \begin{cases} f_h'(x), & 0 \le x \le 1-h,\\ 0, & 1-h < x \le 1,\end{cases} \qquad (2.11)$$

almost for every $x \in [0,1]$.

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By the symmetry, we deduce that

$$\begin{aligned}\int_{1-h}^{1} dx \int_{1-h}^{1} |f(x)-f(y)|^p\, dy &= 2\int_{1-h}^{1} dx \int_{x}^{1} |f(x)-f(y)|^p\, dy \\ &= 2\int_{1-h}^{1} dx \int_{0}^{1-x} |f(x)-f(x+y)|^p\, dy \\ &= 2\int_{0}^{h} dy \int_{1-h}^{1-y} |f(x)-f(x+y)|^p\, dx \\ &\le 2\int_{0}^{h} \omega^p(f,y)\, dy \\ &\le 2h\omega^p(f,h). \end{aligned} \tag{2.12}$$

By (2.10) and (2.12), we have

$$\begin{aligned}\left(\int_0^1 |f(x)-f_h^*(x)|^p\, dx\right)^{1/p} &\le \left(\int_0^{1-h} |f(x)-f_h(x)|^p\, dx\right)^{1/p} + \left(\int_{1-h}^{1} |f(x)-f_h(1-h)|^p\, dx\right)^{1/p} \\ &\le \omega(f,h)_p + \left(\frac{1}{h^p}\int_{1-h}^{1}\left|\int_0^h (f(x)-f(1-h+y))\, dy\right|^p dx\right)^{1/p} \\ &\le \omega(f,h)_p + \left(\frac{1}{h}\int_{1-h}^{1}\int_0^h |f(x)-f(1-h+y)|^p\, dydx\right)^{1/p} \\ &= \omega(f,h)_p + \left(\frac{1}{h}\int_{1-h}^{1}\int_{1-h}^{1} |f(x)-f(y)|^p\, dydx\right)^{1/p} \\ &\le \left(1+2^{1/p}\right)\omega(f,h), \quad 1\le p<\infty. \end{aligned} \tag{2.13}$$

In what follows, we always take $h=\frac{1}{n+1}$, $x_j=\frac{j}{n+1}$, $j=0,1,2,\cdots,n$, and $A(n)=n+1$ in (1.2). Then, we only need to prove the following:

$$\|f-N_{n,a}(f_h^*)\|_p \le C_p\omega\left(f,\frac{1}{n+1}\right)_p, \ 1\le p<\infty. \tag{2.14}$$

By (2.13), we have

$$\begin{aligned}\|f-N_{n,a}(f_h^*)\|_p &\le \|f-f_h^*\|_p + \|f_h^*-N_{n,a}(f_h^*)\|_p \\ &\le \left(1+2^{1/p}\right)\omega\left(f,\frac{1}{n+1}\right)_p + \|f_h^*-N_{n,a}(f_h^*)\|_p. \end{aligned} \tag{2.15}$$

Now, by Theorem 2 and (2.11), we see that

$$\begin{aligned}\|f_h^*-N_{n,a}(f_h^*)\|_p &\le C_p(n+1)^{-1}\left\|(f_h^*)'\right\|_p \\ &\le C_p(n+1)^{-1}\left(\int_0^{1-\frac{1}{n+1}} |f_h'(x)|^p\, dx\right)^{1/p} \\ &= C_p\left(\int_0^{1-\frac{1}{n+1}}\left|f\left(x+\frac{1}{n+1}\right)-f(x)\right|^p dx\right)^{1/p} \\ &\le C_p\omega\left(f,\frac{1}{n+1}\right)_p. \end{aligned} \tag{2.16}$$

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We prove (2.14) by combining (2.15) and (2.16).

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Int-soft implicative filters in BE-algebras

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Abstract. The notion of int-soft implicative filters of a BE-algebra is introduced, and related properties are investigated. The problem of classifying int-soft implicative by their γ-inclusive filter is solved. Also, as a generalization of int-soft implicative filters, the foldness of int-soft implicative filters are considered. Characterizations of int-soft (n-fold) implicative filters are discussed.

1. Introduction

In 1966, Imai and Iséki [3] and Iséki [4] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [11]. In response to this situation Zadeh [12] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [13]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [9]. Maji et al. [8] and Molodtsov [9] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [9] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [8] described the application

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of soft set theory to a decision making problem. Maji et al. [7] also studied several operations on the theory of soft sets. Ahn et al. [1] introduced the notion of an implicative vague filter in BE-algebras, and investigate some properties of it.

In this paper, we introduce the notion of int-soft implicative filter of a BE-algebra, and investigate their properties. We solve the problem of classifying int-soft subalgebras by their γ-inclusive implicative filters. We provide conditions for an int-soft filter to be an int-soft implicative filter. We make a new int-soft implicative filter from old one. Also, as a generalization of int-soft implicative filters, we consider the foldness of int-soft implicative filters. We discuss characterizations of int-soft (n-fold) implicative filters.

2. Preliminaries

We recall some definitions and results discussed in [5].

An algebra $(X;*,1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x\leq y$ if and only if $x*y=1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$.

Definition 2.1.([5]) Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$

for all $x,y\in X$.

Definition 2.2. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called an *implicative filter* of X if

(F1) $1\in F$;
(F2) $x*(y*z)\in F$ and $x*y\in F$ imply $x*z\in F$

for all $x,y,z\in X$.

Proposition 2.3. *Let $(X;*,1)$ be a BE-algebra and let F be a filter of X. If $x\leq y$ and $x\in F$ for any $y\in X$, then $y\in F$.*

Proposition 2.4. *Let $(X;*,1)$ be a self distributive BE-algebra. Then following hold: for any $x,y,z\in X$,*

(i) *if $x\leq y$, then $z*x\leq z*y$ and $y*z\leq x*z$.*
(ii) $y*z\leq(z*x)*(y*z)$.

(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.4(iii).

A soft set theory is introduced by Molodtsov [9]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq X$.

Definition 2.5. A *soft set* (f, A) of X over U is defined to be the set of ordered pairs

$$(f, A) := \{(x, f(x)) : x \in X,\ f(x) \in \mathscr{P}(U)\},$$

where $f : X \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For a soft set (f, A) of X and a subset γ of U, the γ-*inclusive set* of (f, A), denoted by $i_A(f; \gamma)$, is defined to be the set

$$i_A(f; \gamma) := \{x \in A \mid \gamma \subseteq f(x)\}.$$

For any soft sets (f, X) and (g, X) of X, we call (f, X) a *soft subset* of (g, X), denoted by $(f, X) \tilde{\subseteq} (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cup} (g, X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by

$$(f \tilde{\cup} g)(x) = f(x) \cup g(x) \text{ for all } x \in M.$$

The *soft intersection* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cap} (g, X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by

$$(f \tilde{\cap} g)(x) = f(x) \cap g(x) \text{ for all } x \in S.$$

3. Int-soft implicative filters

In what follows, we take a BE-algebra X, as a set of parameters unless specified.

Definition 3.1.([1]) A soft set (f, X) of X over U is called an *intersection-soft filter* (briefly, *int-soft filter*) over U if it satisfies:

(IS1) $(\forall x \in X)\,(f(1) \supseteq f(x))$,
(IS2) $(\forall x, y \in X)\,(f(x * y) \cap f(x) \subseteq f(y))$.

Proposition 3.2.([1]) Every int-soft filter (f, X) of X over U satisfies the following properties:

(i) $(\forall x, y \in X)\,(x \leq y \Rightarrow f(x) \subseteq f(y))$.
(ii) $(\forall x, y, z \in X)\,(f(x * z) \supseteq f(x * (y * z)) \cap f(y))$.

Definition 3.3. A soft set (f, X) of X over U is called an *intersection-soft implicative filter* (briefly, *int-soft implicative filter*) over U if it satisfies (IS1) and

(IS3) $(\forall x, y, z \in X)\,(f(x * (y * z)) \cap f(x * y) \subseteq f(x * z))$.

Example 3.4. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra ([5]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, a, b\} \\ \gamma_1 & \text{if } x \in \{c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is an int-soft implicative filter of X.

Proposition 3.5. *Every int-soft implicative filter over U is an int-soft filter over U.*

Proof. Let (f, X) be an int-soft implicative filter over U. Using (BE4) and (IS3), we have

$$\begin{aligned} f(y * (x * z)) \cap f(x * y) =& f(x * (y * z)) \cap f(x * y) \\ \subseteq& f(x * z) \end{aligned} \tag{3.1}$$

for any $x, y, z \in X$. Putting $x := 1$ in (3.1), we get $f(y * z) \cap f(y) \subseteq f(z)$. Hence (IS2) holds. Therefore (f, X) is an int-soft filter over U. $\square$

The converse of Proposition 3.5 is not true in general as seen in the following example.

Example 3.6. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set where $X = \{1, a, b, c, d, 0\}$ is a BE-algebra as in Example 3.4. Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is an int-soft filter of X. But it is not an int-soft implicative filter over U, since $f(d * (a * 0)) \cap f(d * a) = \gamma_2 \nsubseteq \gamma_1 = f(d * 0)$.

We provide conditions for an int-soft filter to be an int-soft implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let (f, X) be a soft filter over U satisfying*

$$(\forall x, y, z \in X)(f(y * z) \supseteq f(x * (y * (y * z))) \cap f(y * x)). \tag{3.2}$$

Then (f, X) is an int-soft implicative filter over U.

Proof. Since $x*(y*z) = y*(x*z) \le (x*y)*(x*(x*z)) = x*(y*(x*z)) = y*(x*(x*z))$ for all $x, y \in X$, we have $f(x*(y*z)) \subseteq f(y*(x*(x*z)))$ by Proposition 3.2(i). Using (3.2), we have $f(x*z) \supseteq f(y*(x*(x*z)) \cap f(x*y) \supseteq f(x * (y * z)) \cap f(x * y)$. Thus (f, X) is an int-soft implicative filter over U. $\square$

Theorem 3.8. *Let X be a transitive BE-algebra. For any int-soft filter (f, X) over U, the following are equivalent:*

(i) *(f, X) is an int-soft implicative filter,*
(ii) $(\forall x, y \in X)\,(f(x * y) \supseteq f(x * (x * y)))$,
(iii) $(\forall x, y, z \in X)\,(f((x * y) * (x * z)) \supseteq f(x * (y * z)))$.

Proof. (i)⇒(ii) Assume that (f, X) is an int-soft implicative filter over U. Putting $z := y, y := x$ in (IS3), we have

$$\begin{aligned} f(x*y) \supseteq & f(x*(x*y)) \cap f(x*x) \\ = & f(x*(x*y)) \cap f(1) \\ = & f(x*(x*y)). \end{aligned}$$

Hence (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Since $x*(y*z) \leq x*((x*y)*(x*z)) = x*(x*((x*y)*z))$, using Proposition 3.2(i) we have $f(x*((x*y)*(x*z))) = f(x*(x*((x*y)*z))) \supseteq f(x*(y*z))$. Using (ii), we get

$$\begin{aligned} f((x*y)*(x*z)) = & f(x*((x*y)*z)) \\ \supseteq & f(x*(x*((x*y)*z))) \\ \supseteq & f(x*(y*z)). \end{aligned}$$

Thus (iii) holds.

(iii)⇒(ii) Assume that (iii) holds. Using (IS2) and (iii), we have

$$\begin{aligned} f(x*z) \supseteq & f((x*y)*(x*z)) \cap f(x*y) \\ \supseteq & f(x*(y*z)) \cap f(x*y). \end{aligned}$$

Therefore (f, X) is an int-soft implicative filter over U. □

Theorem 3.9. *Let X be a self distributive BE-algebra. Then the soft set (f, X) over U is an int-soft implicative filter over U if and only if it is an int-soft filter over U.*

Proof. By Proposition 3.5, every int-soft implicative filter over U is an int-soft filter over U.

Conversely, Suppose that (f, X) is an int-soft filter over U. For any $x, y, z \in X$, using (IS2) we have

$$\begin{aligned} f(x*z) \supseteq & f((x*y)*(x*z)) \cap f(x*y) \\ = & f(x*(y*z)) \cap f(x*y). \end{aligned}$$

Hence (f, X) is an int-soft implicative filter over U.

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x*(\cdots*(x*(x*y))\cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.10. A soft set (f, X) over U is called an *int-soft n-fold implicative filter* over U if it satisfies (IS1) and

(IS4) $(\forall x, y, z \in X)\left(f(x^n * z) \supseteq f(x^n * (y*z)) \cap f(x^n * y)\right)$.

Note that an int-soft 1-fold implicative filter over U is an int-soft implicative filter over U.

Example 3.11. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([11]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, b, c\} \\ \gamma_1 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is an int-soft n-fold implicative filter over U.

Theorem 3.12. *Every int-soft n-fold implicative filter over U is an int-soft filter over U.*

Proof. Taking $x := 1$ in (IS4) and (BE3), we have $f(z) \supseteq f(y * z) \cap f(y)$. Hence (f, X) is an int-soft filter over U. □

The converse of Theorem 3.12 is not true in general as seen the following example.

Example 3.13. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra as in Example 3.11. Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is an int-soft filter of X. But it is not an int-soft 1-fold implicative filter over U, since $f(d * c) = f(b) = \gamma_1 \not\supseteq \gamma_2 = f(1) = f(d * (b * c)) \cap f(d * b)$.

Theorem 3.14. *Let X be a transitive BE-algebra. For any int-soft filter (f, X) over U, the following are equivalent:*

(i) (f, X) *is an int-soft n-fold implicative filter,*

(ii) $(\forall x, y \in X) \left(f(x^n * y) \supseteq f(x^{n+1} * y)\right)$,

(iii) $(\forall x, y, z \in X) \left(f((x^n * y) * (x^n * z)) \supseteq f(x^n * (y * z))\right)$.

Proof. (i)⇒(ii) Assume that (f, X) is an int-soft n-fold implicative filter over U. Putting $z := y, y := x$ in (IS4), we have

$$\begin{aligned} f(x^n * y) \supseteq& f(x^n * (x * y)) \cap f(x^n * x) \\ =& f(x^{n+1} * y) \cap f(1) \\ =& f(x^{n+1} * y). \end{aligned}$$

Hence (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $f(x^n * ((x^n * y) * (x^n * z))) \supseteq f(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y)) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} f(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) =& f(x^n * (x^{n-1} * ((x^n * y) * z)) \\ \supseteq& f(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ =& f(x^n * ((x^n * y) * (x^n * z))) \\ \supseteq& f(x^n * (y * z)). \end{aligned} \tag{3.3}$$

It follows form (ii) and (3.3) that

$$\begin{aligned} f(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& f(x^n * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& f(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& f(x^n * (y * z)). \end{aligned}$$

Repeating this process, we conclude that

$$\begin{aligned} f((x^n * y) * (x^n * z)) =& f(x^n * ((x^n * y) * z)) \\ \supseteq& f(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. Using (iii), we have

$$\begin{aligned} f(x^n * z) \supseteq& f((x^n * y) * (x^n * z)) \cap f(x^n * y) \\ \supseteq& f((x^n * (y * z)) \cap f(x^n * y). \end{aligned}$$

Hence (f, X) is an int-soft n-fold implicative filter □

Definition 3.15. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.16. *In an n-fold implicative BE-algebra, the notion of int-soft filters and int-soft n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.17. *Let X be a BE-algebra. A soft set (f, X) over U is an int-soft n-fold implicative filter over U if and only if it satisfies (IS1) and*

(IS5) $(\forall x, y, z \in X) \left(f(x^n * z) \supseteq f(x^n * (y^{n+1} * z)) \cap f(x) \right)$.

Proof. Suppose that a soft set (f, X) over U is an int-soft n-fold implicative filter. By Theorem Theorem 3.14, 3.12 and (IS2), we have

$$\begin{aligned} f(y^n * z) \supseteq& f(y^{n+1} * z) \\ \supseteq& f(x * (y^{n+1} * z)) \cap f(x) \end{aligned}$$

for any $x, y, z \in X$. Hence (IS5) holds.

Conversely, assume that (f, X) satisfies (IS1) and (IS5). Using (BE3), we obtain

$$\begin{aligned} f(y) =& f(1^n * y) \\ \supseteq& f(x * (1^{n+1} * y)) \cap f(x) \\ =& f(x * y) \cap f(x). \end{aligned}$$

Hence (IS2) holds and so (f, X) is an int-soft filter over U. By (IS5), (IS1) and (BE3), we get

$$\begin{aligned} f(x^n * y) \supseteq& f(1 * (x^{n+1} * y)) \cap f(1) \\ =& f(x^{n+1} * y). \end{aligned}$$

By Theorem 3.14, (f, X) is an int-soft n-fold implicative filter over U. □

Theorem 3.18. *A soft set (f, X) of X over U is an int-soft implicative filter of X over U if and only if the γ-inclusive set $i_X(f; \gamma)$ is an implicative filter of X over U for all $\gamma \in \mathscr{P}(U)$ with $i_X(f; \gamma) \neq \emptyset$.*

The filter $i_X(f; \gamma)$ in Theorem 3.18 is called the *inclusive filter* of X over U.

Proof. Assume that (f, X) is an int-soft implicative filter over U. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x*(y*z) \in i_X(f;\gamma)$ and $x*y \in i_X(f;\gamma)$. Then $\gamma \subseteq f(x*(y*z))$ and $\gamma \subseteq f(x*y)$. It follows from(IS1) and (IS3) that $\gamma \subseteq f(1)$ and $\gamma \subseteq f(x*(y*z)) \cap f(x*y) \subseteq f(x*z)$ for $x, y, z \in X$. Hence $1 \in i_X(f;\gamma)$ and $x*z \in i_X(f;\gamma)$. Thus $i_X(f;\gamma)$ is an implicative filter of X over U.

Conversely, suppose that $i_X(f;\gamma)$ is an implicative filter of X over U for all $\gamma \in \mathscr{P}(U)$ with $i_X(f;\gamma) \neq \emptyset$. For any $x \in X$, let $f(x) = \gamma$. Since $i_X(f;\gamma)$ is an implicative filter of X, we have $1 \in i_X(f;\gamma)$ and so $f(x) = \gamma \subseteq f(1)$. For any $x, y \in X$, let $f(x*(y*z)) = \gamma_{x*(y*z)}$ and $f(x*y) = \gamma_{x*y}$. Take $\gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y}$. Then $x*(y*z) \in i_X(f;\gamma)$ and $x*y \in i_X(f;\gamma)$ which imply that $x*z \in i_X(f;\gamma)$. Hence

$$f(x*z) \supseteq \gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y} = f(x*(y*z)) \cap f(x*y).$$

Thus (f, X) is an int-soft implicative filter of X over U. □

We make a new int-soft implicative filter from one.

Theorem 3.19. *Let $(f, X) \in S(U)$ and define a soft set (f^*, X) of X over U by*

$$f^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} f(x) & \text{if } x \in i_X(f;\gamma), \\ \delta & \text{otherwise} \end{cases}$$

where γ is any subset of U and δ is a subset of U satisfying $\delta \subsetneq \bigcap_{x \notin i_X(f;\gamma)} f(x)$. If (f, X) is an int-soft implicative filter of X, then so is (f^*, X).

Proof. Assume that (f, X) is an int-soft implicative filter of X. Then $i_X(f;\gamma)(\neq \emptyset)$ is an implicative filter of X over U for all $\gamma \subseteq U$ by Theorem 3.18. Hence $1 \in i_X(f;\gamma)$, and so $f^*(1) = f(1) \supseteq f(x) \supseteq f^*(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x*(y*z) \in i_X(f;\gamma)$ and $x*y \in i_X(f;\gamma)$, then $x*z \in i_X(f;\gamma)$. Hence

$$\begin{aligned} f^*(x*z) = f(x*z) &\supseteq f(x*(y*z)) \cap f(x*y) \\ &= f^*(x*(y*z)) \cap f^*(x*y). \end{aligned}$$

If $x*(y*z) \notin i_X(f;\gamma)$ or $x*y \notin i_X(f;\gamma)$, then $f^*(x*(y*z)) = \delta$ or $f^*(x*y) = \delta$. Thus

$$f^*(x*z) \supseteq \delta = f^*(x*(y*z)) \cap f^*(x*y).$$

Therefore (f^*, X) is an int-soft implicative filter of X. □

Theorem 3.20. *Every filter of a BE-algebra can be represented as a γ-inclusive set of an int-soft implicative filter.*

Proof. Let F be a filter of a BE-algebra X. For a subset γ of U, define a soft set (f, X) over U by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ \emptyset & \text{if } x \notin F. \end{cases}$$

Obviously, $F = i_X(f;\gamma)$. We now prove that (f, X) is an int-soft implicative filter of X. Since $1 \in F = i_X(f;\gamma)$, we have $f(1) = \gamma \supseteq f(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x*(y*z), x*y \in F$, then $x*z \in F$ because F is an implicative filter of X. Hence $f(x*(y*z)) = f(x*y) = f(x*z) = \gamma$, and so $f(x*(y*z)) \cap f(x*y) \subseteq f(x*z)$. If $x*(y*z) \in F$ and $x*y \notin F$, then $f(x*(y*z)) = \gamma$ and $f(x*y) = \emptyset$ which imply that

$$f(x*(y*z)) \cap f(x*y) = \gamma \cap \emptyset = \emptyset \subseteq f(x*z).$$

Similarly, if $x*(y*z) \notin F$ and $x*y \in F$, then $f(x*(y*z)) \cap f(x*y) \subseteq f(x*z)$. Obviously, if $x*(y*z) \notin F$ and $x*y \notin F$, then $f(x*(y*z)) \cap f(x*y) \subseteq f(x*z)$. Therefore (f, X) is an int-soft implicative filter of X. □

For two elements a and b of X, consider a soft set (f_a^b, X) over U where

$$f_a^b : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } a*(b*x)=1, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_2 \subsetneq \gamma_1$. In the following example, we know that there exist $a, b \in X$ such that (f_a^b, X) is not an int-soft implicative filter of X.

Example 3.21. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.4. Then (f_1^a, X) is not an int-soft implicative filter of X over U since

$$f_1^a(1*(a*b)) \cap f_1^a(1*a) = \gamma_1 \nsubseteq f_1^a(1*b) = \gamma_2.$$

Now we provide a condition for the soft set (f_a^b, X) to be an int-soft implicative filter of X over U for all $a, b \in X$.

Theorem 3.22. *If X is a self distributive BE-algebra, then the soft set (f_a^b, X) is an int-soft implicative filter of X over U for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $f_a^b(1) \supseteq f_a^b(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $a*(b*(x*(y*z))) \neq 1$ or $a*(b*(x*y)) \neq 1$. Then $f_a^b(x*(y*z)) = \gamma_2$ or $f_a^b(x*y) = \gamma_2$. Hence

$$f_a^b(x*(y*z)) \cap f_a^b(x*y) = \gamma_2 \subseteq f_a^b(x*z).$$

Assume that $a*(b*(x*(y*z))) = 1$ and $a*(b*(x*y)) = 1$. Then

$$\begin{aligned} 1 &= a*(b*(x*(y*z))) \\ &= a*(b*((x*y)*(x*z))) \\ &= a*((b*(x*y))*(b*(x*z))) \\ &= (a*(b*(x*y)))*(a*(b*(x*z))) \\ &= 1*(a*(b*(x*z))) \\ &= a*(b*(x*z)), \end{aligned}$$

and so $f_a^b(x*(y*z)) \cap f_a^b(x*y) = \gamma_1 = f_a^b(x*z)$. Therefore (f_a^b, X) is an int-soft implicative filter of X over U for all $a, b \in X$. □

Theorem 3.23. *If (f, X) and (g, X) are int-soft implicative filters of X, then the soft intersection $(f, X) \tilde{\cap} (g, X)$ of (f, X) and (g, X) is an int-soft implicative filter of X.*

Proof. For any $x \in X$, we have

$$(f \tilde{\cap} g)(1) = f(1) \cap g(1) \supseteq f(x) \cap g(x) = (f \tilde{\cap} g)(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned} (f \tilde{\cap} g)(x*z) &= f(x*z) \cap g(x*z) \\ &\supseteq (f(x*(y*z)) \cap f(x*y)) \cap (g(x*(y*z)) \cap g(x*y)) \\ &= (f(x*(y*z)) \cap g(x*(y*z))) \cap (f(x*y) \cap g(x*y)) \\ &= (f \tilde{\cap} g)(x*(y*z)) \cap (f \tilde{\cap} g)(x*y). \end{aligned}$$

Hence $(f, X) \tilde{\cap} (g, X)$ is an int-soft implicative filter of X. □

The following example shows that the soft union of int-soft implicative filters of X may not be an int-soft implicative filter of X.

Example 3.24. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set, where $X = \{1, a, b, c, d\}$ is a BE-algebra with the following Cayley table ([5]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let (f, X) and (g, X) be soft sets of X over U defined, respectively, as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{1, b\} \\ \gamma_1 & \text{if } x \in \{a, c, d\} \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, c\} \\ \gamma_2 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that (f, X) and (g, X) are int-soft implicative filters of X over U. But $(f, X) \tilde{\cup} (g, X) = (f \tilde{\cup} g, X)$ is not an int-soft implicative filter of X over U, since

$$\begin{aligned}(f \tilde{\cup} g)(1 * (c * d)) \cap (f \tilde{\cup} g)(1 * c) &= (f \tilde{\cup} g)(b) \cap (f \tilde{\cup} g)(c) \\ &= (f(b) \cup g(b)) \cap (f(c) \cup g(c)) \\ &= \gamma_3 \cap \gamma_4 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ &= f(1 * d) \cup g(1 * d).\end{aligned}$$

Let (f, X) be a soft set of X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$f[a^k; b] := \left\{x \in X \mid f\left(a^k * (b * x)\right) = f(1)\right\}$$

where $f(a^k * x) = f(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in f[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.25. *Let (f, X) be a soft set of X over U such that the condition (IS1) and $f(x * y) = f(x) \cup f(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in f[a^k; b]$, then $y * x \in f[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in f[a^k; b]$. Then $f(a^k * (b * x)) = f(1)$, and so

$$\begin{aligned}f(a^k * (b * (y * x))) &= f(a^k * (y * (b * x))) \\ &= f(y * (a^k * (b * x))) \\ &= f(y) \cup f(a^k * (b * x)) \\ &= f(y) \cup f(1) = f(1)\end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$. Hence $y * x \in f[a^k; b]$ for all $y \in X$. □

Proposition 3.26. *For any soft set (f, X) of X, let $a \in X$ satisfy the following assertion:*

$$(\forall x \in X)\,(a * x = 1)\,.$$

Then $f[a^k; b] = X = f[b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.

Proof. For any $x \in X$, we have

$$\begin{aligned} f(a^k * (b * x)) &= f(a^{k-1} * (a * (b * x))) \\ &= f(a^{k-1} * (b * (a * x))) \\ &= f(a^{k-1} * (b * 1)) \\ &= f(1), \end{aligned}$$

and so $x \in f[a^k; b]$. Similarly, $x \in f[b^k; a]$. □

Proposition 3.27. *Let X be a self distributive BE-algebra and let (f, X) be an order-preserving soft set of X over U with the property (IS1). If $b \leq c$ in X, then $f[a^k; c] \subseteq f[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in f[a^k; c]$, then

$$\begin{aligned} f(1) = f(a^k * (c * x)) &= f(c * (a^k * x)) \\ \subseteq f(b * (a^k * x)) &= f(a^k * (b * x)) \end{aligned}$$

by Proposition 2.4(i), Proposition 3.2(i) and (BE4), and so $f(a^k * (b * x)) = f(1)$. Thus $x \in f[a^k; b]$, which completes the proof. □

The following example shows that there exists a soft set (f, X) of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $f[a^k; b]$ is not a filter of X.

Example 3.28. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set where $X = \{1, a, b, c\}$ is a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then it is a soft set of X over U. But $f[c; b] = \{x \in X | f(c*(b*x)) = f(1)\} = \{1, a, b\}$ is not an implicative filter, since $a * (1 * c) = a \in f[c; b]$, $a * 1 \in f[c; b]$ and $1 * c = c \notin f[c; b]$.

We provide conditions for a set $f[a^k; b]$ to be an implicative filter.

Theorem 3.29. *Let (f, X) be a soft set over U. If X is a self distributive BE-algebra and f is injective, then $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and f is injective. Obviously, $1 \in f[a^k; b]$. Let $a, b, x, y, z \in X$ and $k \in \mathbb{N}$ be such that $x * (y * z) \in f[a^k; b]$ and $x * y \in f[a^k; b]$. Then $f(a^k * (b * (x * y))) = f(1)$ which implies that $a^k * (b * (x * y)) = 1$ since f is injective. Since X is a self distributive BE-algebra, we have

$$\begin{aligned} f(1) &= f(a^k * (b * (x * (y * z)))) \\ &= f(a^{k-1} * (a * (b * (x * (y * z))))) \\ &= f(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\ &= \cdots \\ &= f((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\ &= f(1 * (a^k * (b * (x * z)))) \\ &= f(a^k * (b * (x * z))), \end{aligned}$$

which implies that $x * z \in f[a^k; b]$. Therefore $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.30. *Let X be a self distributive BE-algebra. Let (f, X) be a soft set of X over U satisfying the condition (IS1) and*

$$(\forall x, y \in X)\, (f(x * y) = f(x) \cap f(y))\,. \tag{3.4}$$

Then $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in f[a^k; b]$. Let $x, y, z \in X$ be such that $x * (y * z) \in f[a^k; b]$ and $x * y \in f[a^k; b]$. Then $f\left(a^k * (b * (x * y))\right) = f(1)$, which implies from (3.4) and (IS1) that

$$\begin{aligned} f(1) &= f(a^k * (b * (x * (y * z)))) \\ &= f(a^{k-1} * (a * (b * (x * (y * z))))) \\ &= f(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\ &= \cdots \\ &= f((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\ &= f(a^k * (b * (x * y))) \cap f(a^k * (b * (x * z))) \\ &= f(1) \cap f(a^k * (b * (x * z))) \\ &= f(a^k * (b * (x * z))). \end{aligned}$$

Hence $x * z \in f[a^k; b]$ and therefore $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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TWISTED PETERS POLYNOMIALS ARISING FROM MULTIVARIATE p-ADIC INTEGRAL ON $\mathbb{Z}_p$

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Abstract. Recently, several authors have studied the twisted Bernoulli and Euler polynomials. In this paper, we study the properties of the twisted Peters polynomials arising from multivariate p-adic integral on $\mathbb{Z}_p$.

1. Introduction

For $n \geq 0$, let $C_{p^n} = \{\zeta \in \mathbb{C}_p | \zeta^{p^n} = 1\}$ be the cyclic group of the p^n-th root of unity and let $T_p = \cup_{n \geq 0} C_{p^n} = \varinjlim_{n} C_{p^n}$ be the space of locally constant functions. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined by $|p|_p = \frac{1}{p}$. For any positive integer k and $\zeta \in T_p$, the twisted Euler polynomials of order k are defined by the generating function to be

$$\left(\frac{2}{\zeta e^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\zeta}^{(k)}(x) \frac{t^n}{n!}, \ (see \ [1,7,10,13]). \tag{1.1}$$

Recently, several authors have studied the twisted Euler numbers and polynomials (see [1-13]). When $x = 0$, $E_{n,\zeta}^{(k)} = E_{n,\zeta}^{(k)}(0)$ are called the twisted Euler numbers of order k. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n - 1} f(x)(-1)^x, \ (see \ [7,8,9,10]). \tag{1.2}$$

Let f_1 be the translation of f given by $f_1(x) = f(x+1)$. Then, by (1.2), we get

$$I(f_1) = -I(f) + 2f(0) \ (see \ [7,8]). \tag{1.3}$$

As is well known, the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \ (n \geq 0), \ (see \ [7,8]), \tag{1.4}$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}, \ (see \ [7,8]). \tag{1.5}$$

In this paper, we study the properties of the twisted Peters polynomials arising from multivariate p-adic integral on $\mathbb{Z}_p$ and give some relations between twisted Peters polynomials and special polynomials.

2. Twisted Peters polynomials

In this section, we assume that $\zeta \in T_p$ and $\lambda \in \mathbb{Z}_p$. For any positive integer k, let us define the twisted Peters polynomials of the first kind as follows:

$$\left(\frac{1}{1+(1+t\zeta)^{\lambda}}\right)^{k}(1+\zeta t)^{x}=\sum_{n=0}^{\infty}P_{n,\zeta}^{(k)}(x|\lambda)\frac{t^n}{n!},\ (k\in\mathbb{N}). \tag{2.1}$$

Note that $P_n^{(k)}(x|\lambda)=P_{n,1}^{(k)}(x|\lambda)$ are the Peters polynomials. Let us take $f(x)=(1+\zeta t)^{\lambda x}$. Then, by (1.3), we get

$$\int_{\mathbb{Z}_p}(1+\zeta t)^{x\lambda}d\mu(x)=\frac{2}{(1+\zeta t)^{\lambda}+1}. \tag{2.2}$$

From (2.2), we can derive the following equation (2.3):

$$\begin{aligned}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-times}(1+\zeta t)^{\lambda x_1+\cdots+\lambda x_k+x}d\mu(x_1)\dots d\mu(x_k)&=\left(\frac{2}{(1+\zeta t)^{\lambda}+1}\right)^{k}(1+\zeta t)^{x}\\&=2^k\sum_{n=0}^{\infty}P_{n,\zeta}^{(k)}(x|\lambda)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\zeta t)^{\lambda x_1+\cdots+\lambda x_k+x}d\mu(x_1)\dots d\mu(x_k)\\&=\sum_{n=0}^{\infty}\zeta^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_k+x)_n d\mu(x_1)\dots d\mu(x_k)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$2^kP_{n,\zeta}^{(k)}(x|\lambda)=\zeta^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_k+x)_n d\mu(x_1)\dots d\mu(x_k).$$

From Theorem 2.1, we note that

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$$\begin{aligned}2^k P_{n,\zeta}^{(k)}(x|\lambda) &= \zeta^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_k + x)^l d\mu(x_1) \ldots d\mu(x_k) \\ &= \zeta^n \sum_{l=0}^{n} S_1(n,l)\lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + \frac{x}{\lambda})^l d\mu(x_1) \ldots d\mu(x_k) \\ &= \zeta^n \sum_{l=0}^{n} S_1(n,l)\lambda^l E_l^{(k)}(\frac{x}{\lambda}),\end{aligned} \tag{2.5}$$

where $E_l^{(k)}(x)$ is the l-th Euler polynomial of order k.
Replacing t by $e^t - \frac{1}{\zeta}$ in (2.1), we get

$$\begin{aligned}\left(\frac{2}{\zeta^\lambda e^{t\lambda}+1}\right)^k (\zeta e^t)^x &= 2^k \sum_{n=0}^{\infty} P_{n,\zeta}^{(k)}(x|\lambda) \frac{(e^t - \frac{1}{\zeta})^n}{n!} \\ &= 2^k \sum_{n=0}^{\infty} \zeta^{-n} P_{n,\zeta}^{(k)}(x|\lambda) \frac{1}{n!}(\zeta e^t - 1)^n \\ &= 2^k \sum_{n=0}^{\infty} \zeta^{-n} P_{n,\zeta}^{(k)}(x|\lambda) \frac{1}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{(\log\zeta + t)^m}{m!} \\ &= 2^k \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \zeta^{-n} P_{n,\zeta}^{(k)}(x|\lambda) S_2(m,n) \right) \frac{t^m}{m!},\end{aligned} \tag{2.6}$$

and

$$\left(\frac{2}{\zeta^\lambda e^{\lambda t}+1}\right)^k \zeta^x e^{xt} = \zeta^x \sum_{m=0}^{\infty} \lambda^m E_{m,\zeta}^{(k)} \left(\frac{x}{\lambda}\right) \frac{t^m}{m!}, \tag{2.7}$$

where $E_{n,\zeta}^{(k)}(x)$ are the twisted Euler polynomials of order k. Therefore, by (2.5), (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $m \geq 0$, we have*

$$\lambda^m E_{m,\zeta}^{(k)} \left(\frac{x}{\lambda}\right) = 2^k \sum_{n=0}^{m} \zeta^{-x-n} P_{n,\zeta}^{(k)}(x|\lambda) S_2(m,n), \ k \in \mathbb{N},$$

and

$$2^k P_{n,\zeta}^{(k)}(x|\lambda) = \zeta^n \sum_{l=0}^{n} S_1(n,l)\lambda^l E_l^{(k)} \left(\frac{x}{\lambda}\right).$$

For $n \geq 0$, the rising factorial sequence is defined by

$$(x)_n = x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n.$$

Now, we consider the twisted Peters polynomials of the second kind as follows:

$$\left(\frac{(1+\zeta t)^\lambda}{1+(1+\zeta t)^\lambda}\right)^k (1+\zeta t)^x = \sum_{n=0}^{\infty} \widehat{P}_{n,\zeta}^{(k)}(x|\lambda) \frac{t^n}{n!}. \tag{2.8}$$

From (1.3), we have

$$\int_{\mathbb{Z}_p}(1+\zeta t)^{-\lambda x}d\mu(x)=\frac{2(1+\zeta t)^{\lambda}}{1+(1+\zeta t)^{\lambda}}. \tag{2.9}$$

Thus, by (2.9), we get

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-times}(1+\zeta t)^{-(\lambda x_1+\cdots+\lambda x_k)+x}d\mu(x_1)\dots d\mu(x_k)\\&=\left(\frac{2(1+\zeta t)^{\lambda}}{1+(1+\zeta t)^{\lambda}}\right)^k(1+\zeta t)^x=2^k\sum_{n=0}^{\infty}\widehat{P}_{n,\zeta}^{(k)}(x|\lambda)\frac{t^n}{n!},\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\zeta t)^{-(\lambda x_1+\cdots+\lambda x_k)+x}d\mu(x_1)\dots d\mu(x_k)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-(\lambda x_1+\cdots+\lambda x_k)+x)_n d\mu(x_1)\dots d\mu(x_k)\zeta^n\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$\zeta^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda(x_1+\cdots+x_k)+x)_n d\mu(x_1)\dots d\mu(x_k)=2^k\widehat{P}_{n,\zeta}^{(k)}(x|\lambda).$$

From (1.4) and Theorem 1.3, we have

$$\begin{aligned}2^k\widehat{P}_{n,\zeta}^{(k)}(x|\lambda)&=\zeta^n\sum_{l=0}^{n}S_1(n,l)(-\lambda)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k-\frac{x}{\lambda})^l d\mu(x_1)\dots d\mu(x_k)\\&=\zeta^n\sum_{l=0}^{n}S_1(n,l)\lambda^l(-1)^lE_l^{(k)}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{2.12}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\zeta}^{(k)}(k-x)\frac{t^n}{n!}&=\left(\frac{2}{\zeta e^t+1}\right)^k e^{(k-x)t}\\&=\zeta^{-k}\left(\frac{2}{1+\zeta^{-1}e^{-t}}\right)^k e^{-xt}\\&=\zeta^{-k}\sum_{n=0}^{\infty}(-1)^nE_{n,\zeta^{-1}}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

By comparing the coefficients of the both sides of (2.13), we get

$$E_{n,\zeta}^{(k)}(k-x)=\zeta^{-k}(-1)^nE_{n,\zeta^{-1}}^{(k)}(x). \tag{2.14}$$

From (2.12) and (2.14), we have

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$$\begin{aligned} 2^k \widehat{P}_{n,\zeta}^{(k)}(x|\lambda) &= \zeta^{n+k} \sum_{l=0}^{n} S_1(n,l)\lambda^l \zeta^{-k}(-1)^l E_l^{(k)}\left(-\frac{x}{\lambda}\right) \\ &= \zeta^{n+k} \sum_{l=0}^{n} S_1(n,l)\lambda^l E_l^{(k)}\left(k+\frac{x}{\lambda}\right). \end{aligned} \tag{2.15}$$

Replacing t by $e^t - \frac{1}{\zeta}$ in (2.10), we get

$$\begin{aligned} \left(\frac{2(\zeta e^t)^\lambda}{(\zeta e^t)^\lambda + 1}\right)^k (\zeta e^t)^x &= 2^k \sum_{n=0}^{\infty} \widehat{P}_{n,\zeta}^{(k)}(x|\lambda)\frac{1}{n!}\left(e^t - \frac{1}{\zeta}\right)^n \\ &= 2^k \sum_{n=0}^{\infty} \zeta^{-n} \widehat{P}_{n,\zeta}^{(k)}(x|\lambda)\frac{1}{n!}\left(\zeta e^t - 1\right)^n \\ &= 2^k \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} S_2(m,n)\zeta^{-n}\widehat{P}_{n,\zeta}^{(k)}(x|\lambda)\right)\frac{t^m}{m!}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \left(\frac{2(\zeta e^t)^\lambda}{\zeta^\lambda e^{t\lambda} + 1}\right)^k (\zeta e^t)^x &= \left(\frac{2}{\zeta^\lambda e^{t\lambda}+1}\right)^k e^{xt+\lambda k t}\zeta^{k+x} \\ &= \zeta^{k+x}\sum_{m=0}^{\infty} \lambda^m E_{m,\zeta}^{(k)}\left(k+\frac{x}{\lambda}\right)\frac{t^m}{m!}. \end{aligned} \tag{2.17}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.4. *For $m \geq 0$, we have*

$$\lambda^m E_{m,\zeta}^{(k)}\left(k+\frac{x}{\lambda}\right) = 2^k \sum_{n=0}^{m} \zeta^{-x-k-n} S_2(m,n)\widehat{P}_{n,\zeta}^{(k)}(x|\lambda),$$

and

$$2^k \widehat{P}_{n,\zeta}^{(k)}(x|\lambda) = \zeta^{n+k}\sum_{l=0}^{n} S_1(n,l)\lambda^l E_l^{(k)}\left(k+\frac{x}{\lambda}\right).$$

We observe that

$$
\begin{aligned}
(-1)^n \frac{2^k P_{n,\zeta}^{(k)}(x|\lambda)}{n!} &= \zeta^n (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x+\lambda x_1+\cdots+\lambda x_k}{n} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-(\lambda x_1+\cdots+\lambda x_k)-x+n-1}{n} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-(\lambda x_1+\cdots+\lambda x_k)-x}{m} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} m! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-(\lambda x_1+\cdots+\lambda x_k)-x}{m} d\mu(x_1)\dots d\mu(x_k) \\
&= \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{2^k \widehat{P}_{m,\zeta}^{(k)}(x|\lambda)}{m!}.
\end{aligned}
\tag{2.18}
$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$
(-1)^n \frac{P_{n,\zeta}^{(k)}(x|\lambda)}{n!} = \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{\widehat{P}_{m,\zeta}^{(k)}(x|\lambda)}{m!}.
$$

Remark 1. By the same method as (2.18), we get

$$
\begin{aligned}
&(-1)^n \frac{2^k \widehat{P}_{n,\zeta}^{(k)}(x|\lambda)}{n!} = \zeta^n (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-(\lambda x_1+\cdots+\lambda x_k)+x}{n} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda x_1+\cdots+\lambda x_k-x+n-1}{n} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda x_1+\cdots+\lambda x_k-x}{m} d\mu(x_1)\dots d\mu(x_k) \\
&= \zeta^n \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} m! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda x_1+\cdots+\lambda x_k-x}{m} d\mu(x_1)\dots d\mu(x_k) \\
&= \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{2^k P_{m,\zeta}^{(k)}(-x|\lambda)}{m!}.
\end{aligned}
\tag{2.19}
$$

Therefore, by (2.19), we get

$$
(-1)^n \frac{\widehat{P}_{n,\zeta}^{(k)}(x|\lambda)}{n!} = \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{P_{m,\zeta}^{(k)}(-x|\lambda)}{m!}.
$$

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2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

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Average value problems for nonlinear second-order impulsive q-difference equations

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Abstract

This article studies the existence and uniqueness of solutions for an average value problem of nonlinear second-order impulsive q_k-difference equations. Two results are obtained by applying Banach's contraction mapping principle and Leray-Schauder's nonlinear alternative. Some examples are presented to illustrate the results.

Keywords: q_k-derivative; q_k-integral; impulsive q_k-difference equation; existence; uniqueness

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1 Introduction and Preliminaries

In this paper, we consider the following average value problem for nonlinear second-order impulsive q_k-difference equation of the form

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$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t)), \quad t \in J := [0, T], \ t \neq t_k, \\ \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \dots, m, \\ D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*\left(x(t_k)\right), \quad k = 1, 2, \dots, m, \\ \displaystyle\sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} x(s) d_{q_j} s = \alpha, \quad x(\eta) = \beta, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1, 2, \dots, m$, $x(t_k^+) = \lim_{t \to 0+} x(t_k + h)$, $0 < q_k < 1$ for $k = 0, 1, 2, \dots, m$, $\eta \in (t_i, t_{i+1})$ a fixed constant for some $i \in \{0, 1, \dots, m\}$, and α, β are given constants.

The notions of q_k-derivative and q_k-integral on finite intervals were introduced in [1]. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

Definition 1.1. *Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression*

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1 - q_k) t_k)}{(1 - q_k)(t - t_k)}, t \neq t_k, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t), \tag{2}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (2), then $D_{q_k} f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \tag{3}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 1.2. *Let $f : J_k \to \mathbb{R}$ is a continuous function, we call the second-order q_k-derivative $D_{q_k}^2 f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D_{q_k}^2 f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D_{q_k}^n : J_k \to \mathbb{R}$.*

The q_k-integral is defined as follows:

Definition 1.3. *Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by*

$$\int_{t_k}^{t} f(s)d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \tag{4}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s)d_{q_k}s &= \int_{t_k}^{t} f(s)d_{q_k}s - \int_{t_k}^{a} f(s)d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \\ &\quad -(1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1-q_k^n)t_k).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (4) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

For the basic properties of q_k-derivative and q_k-integral we refer to [1].

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3]-[15] and the references cited therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [16]-[18].

In this paper we prove existence and uniqueness results for the impulsive boundary value problem (1) by using Banach's contraction mapping principle and Leray-Schauder's nonlinear alternative. The rest of this paper is organized as follows: In Section 2 we present an auxiliary lemma which is used to convert the impulsive boundary value problem (1) into an equivalent integral equation. The main results are given in Section 3, while examples illustrating the results are presented in Section 4.

2 An auxiliary lemma

Let $J=[0,T]$, $J_0=[t_0,t_1]$, $J_k=(t_k,t_{k+1}]$ for $k=1,2,\ldots,m$. Let $PC(J,\mathbb{R})=\{x:J\to\mathbb{R}: x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-)=x(t_k)$, $k=1,2,\ldots,m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\|_{PC}=\sup\{|x(t)|;\, t\in J\}$.

Lemma 2.1. *Let $\Lambda\neq\eta T$. The unique solution of problem (1) is given by*

$$
\begin{aligned}
x(t) \;=\;& \frac{(t-\eta)}{\eta T-\Lambda}\Bigg\{-\alpha+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^{-}\\
&+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j)\\
&+\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}u d_{q_j}\tau d_{q_j}s\Bigg\}\\
&+\frac{(tT-\Lambda)}{\eta T-\Lambda}\Bigg\{\beta-\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)\\
&-\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \qquad (5)\\
&-\int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau,x(\tau))d_{q_i}\tau d_{q_i}s\Bigg\}\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(t-t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s} f(\tau,x(\tau))d_{q_k}\tau d_{q_k}s,
\end{aligned}
$$

with $\sum_{0<0}(\cdot)=0$, where

$$\Lambda=\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)(t_{j+1}+q_jt_j)}{1+q_j}, \tag{6}$$

and

$$H_{jk}^{-}=\frac{(t_{j+1}-t_j)(t_{j+1}+q_jt_j-t_k(1+q_j))}{1+q_j}, k=1,\ldots,m, j=0,\ldots,m. \tag{7}$$

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Proof. For $t \in J_0$, taking q_0-integral for the first equation of (1), we get

$$D_{q_0}x(t) = D_{q_0}x(0) + \int_0^t f(s, x(s))d_{q_0}s, \tag{8}$$

which yields

$$D_{q_0}x(t_1) = D_{q_0}x(0) + \int_0^{t_1} f(s, x(s))d_{q_0}s. \tag{9}$$

For $t \in J_0$, by q_0-integrating (8) and setting $x(0) = A$, $D_{q_0}x(0) = B$, we obtain

$$\begin{aligned} x(t) &= x(0) + D_{q_0}x(0)t + \int_0^t \int_0^s f(\tau, x(\tau))d_{q_0}\tau d_{q_0}s \\ &= A + Bt + \int_0^t \int_0^s f(\tau, x(\tau))d_{q_0}\tau d_{q_0}s. \end{aligned}$$

In particular, for $t = t_1$

$$x(t_1) = A + Bt_1 + \int_0^{t_1} \int_0^s f(\tau, x(\tau))d_{q_0}\tau d_{q_0}s. \tag{10}$$

For $t \in J_1 = (t_1, t_2]$, q_1-integrating (1), we have

$$D_{q_1}x(t) = D_{q_1}x(t_1^+) + \int_{t_1}^t f(s, x(s))d_{q_1}s.$$

Using the third condition of (1) with (9), it follows that

$$D_{q_1}x(t) = B + \int_0^{t_1} f(s, x(s))d_{q_0}s + I_1^*(x(t_1)) + \int_{t_1}^t f(s, x(s))d_{q_1}s. \tag{11}$$

Taking q_1-integral to (11) for $t \in J_1$, we obtain

$$\begin{aligned} x(t) &= x(t_1^+) + \left[B + \int_0^{t_1} f(s, x(s))d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \int_{t_1}^s f(\tau, x(\tau))d_{q_1}\tau d_{q_1}s. \end{aligned} \tag{12}$$

Applying the second equation of (1) with (10) and (12), we get

$$\begin{aligned} x(t) &= A + Bt_1 + \int_0^{t_1} \int_0^s f(\tau, x(\tau))d_{q_0}\tau d_{q_0}s + I_1(x(t_1)) \\ &\quad + \left[B + \int_0^{t_1} f(s, x(s))d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \int_{t_1}^s f(\tau, x(\tau))d_{q_1}\tau d_{q_1}s \end{aligned}$$

$$\begin{aligned} &= A + Bt + \int_0^{t_1}\int_0^s f(\tau, x(\tau))d_{q_0}\tau d_{q_0}s + I_1(x(t_1)) \\ &\quad + \left[\int_0^{t_1} f(s, x(s))d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^{t}\int_{t_1}^{s} f(\tau, x(\tau))d_{q_1}\tau d_{q_1}s. \end{aligned}$$

Repeating the above process, for $t \in J$, we get

$$\begin{aligned} x(t) &= A + Bt + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau, x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) \\ &\quad + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s, x(s))d_{q_{k-1}}s + I_k^*\left(x(t_k)\right)\right)(t - t_k) \qquad (13) \\ &\quad + \int_{t_k}^{t}\int_{t_k}^{s} f(\tau, x(\tau))d_{q_k}\tau d_{q_k}s. \end{aligned}$$

From (13), we have

$$\begin{aligned} &\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}} x(s)d_{q_j}s \\ &= AT + B\Lambda \\ &\quad + \sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau, x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right)(t_{j+1} - t_j) \\ &\quad + \sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s, x(s))d_{q_{k-1}}s + I_k^*\left(x(t_k)\right)\right)H_{jk}^{-} \qquad (14) \\ &\quad + \sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u, x(u))d_{q_j}u d_{q_j}\tau d_{q_j}s, \end{aligned}$$

with $\sum_{0<0}(\cdot) = 0$.

Further, for $t = \eta \in (t_i, t_{i+1})$, we have

$$\begin{aligned} x(\eta) &= A + B\eta \\ &\quad + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau, x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) \\ &\quad + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s, x(s))d_{q_{k-1}}s + I_k^*\left(x(t_k)\right)\right)(\eta - t_k) \qquad (15) \\ &\quad + \int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau, x(\tau))d_{q_i}\tau d_{q_i}s. \end{aligned}$$

6

Using the boundary conditions of (1) with (14) and (15), we get

$$\begin{aligned} & AT+B\Lambda \\ = \quad & \alpha-\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j) \\ & -\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^{-} \qquad (16) \\ & -\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}ud_{q_j}\tau d_{q_j}s, \end{aligned}$$

and

$$\begin{aligned} A+B\eta \quad = \quad & \beta-\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right) \\ & -\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \qquad (17) \\ & -\int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau,x(\tau))d_{q_i}\tau d_{q_i}s. \end{aligned}$$

Solving the linear system of equations (16), (17) for unknown constants A and B, we have

$$\begin{aligned} A \quad = \quad & \frac{1}{\eta T-\Lambda}\Bigg\{\eta\alpha-\eta\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^{-} \\ & -\eta\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j) \\ & -\eta\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}ud_{q_j}\tau d_{q_j}s-\Lambda\beta \\ & +\Lambda\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right) \\ & +\Lambda\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \\ & +\Lambda\int_{t_k}^{\eta}\int_{t_k}^{s} f(\tau,x(\tau))d_{q_k}\tau d_{q_k}s\Bigg\}, \qquad (18) \end{aligned}$$

7

and

$$
\begin{aligned}
B \;=\; & \frac{1}{\eta T-\Lambda}\Bigg\{-\alpha+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^{-} \\
& +\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j) \\
& +\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}ud_{q_j}\tau d_{q_j}s \\
& +T\beta-T\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right) \\
& -T\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \\
& -T\int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau,x(\tau))d_{q_i}\tau d_{q_i}s\Bigg\}. \qquad (19)
\end{aligned}
$$

Substituting the values of A and B in (13), we obtain the solution (5). □

3 Main results

In view of Lemma 2.1, we define an operator $\mathcal{F}: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
& (\mathcal{F}x)(t) \\
= \; & \frac{(t-\eta)}{\eta T-\Lambda}\Bigg\{-\alpha+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^{-} \\
& +\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j) \\
& +\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}ud_{q_j}\tau d_{q_j}s\Bigg\} \\
& +\frac{(tT-\Lambda)}{\eta T-\Lambda}\Bigg\{\beta-\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right) \qquad (20) \\
& -\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \\
& -\int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau,x(\tau))d_{q_i}\tau d_{q_i}s\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, x(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s + I_k\left(x(t_k)\right) \right) \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s, x(s)) d_{q_{k-1}} s + I_k^*\left(x(t_k)\right) \right) (t - t_k) \\
&+ \int_{t_k}^{t} \int_{t_k}^{s} f(\tau, x(\tau)) d_{q_k} \tau d_{q_k} s,
\end{aligned}
$$

with $\Lambda \neq \eta T$. It should be noticed that problem (1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

For convenience, we set:

$$
H_{jk}^{+} := \frac{(t_{j+1} - t_j)(t_{j+1} + q_j t_j + t_k(1 + q_j))}{1 + q_j}, \tag{21}
$$

$$
\begin{aligned}
\Omega \quad := \quad & \frac{(T+\eta)}{|\eta T - \Lambda|} \left\{ \sum_{j=1}^{m} \sum_{k=1}^{j} \left(L_1(t_k - t_{k-1}) + L_3\right) H_{jk}^{+} \right. \\
& + \sum_{j=1}^{m} \sum_{k=1}^{j} \left(L_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 \right) (t_{j+1} - t_j) \\
& \left. + \sum_{j=0}^{m} L_1 \frac{(t_{j+1} - t_j)^3}{(1 + q_j + q_j^2)} \right\} + \frac{(T^2 + |\Lambda|)}{|\eta T - \Lambda|} \left\{ \sum_{k=1}^{i} \left(L_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 \right) \right. \\
& \left. + \sum_{k=1}^{i} \left(L_1(t_k - t_{k-1}) + L_3\right)(\eta - t_k) + L_1 \frac{(\eta - t_i)^2}{1 + q_i} \right\} \qquad\qquad (22) \\
& + \sum_{k=1}^{m} \left(L_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 \right) + \sum_{k=1}^{m} \left(L_1(t_k - t_{k-1}) + L_3\right)(T - t_k) \\
& + L_1 \frac{(T - t_m)^2}{1 + q_m},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma \quad := \quad & \frac{(T+\eta)}{|\eta T - \Lambda|} \left\{ |\alpha| + \sum_{j=1}^{m} \sum_{k=1}^{j} \left(M_1(t_k - t_{k-1}) + M_3\right) H_{jk}^{+} \right. \\
& \left. + \sum_{j=1}^{m} \sum_{k=1}^{j} \left(M_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + M_2 \right) (t_{j+1} - t_j) + \sum_{j=0}^{m} M_1 \frac{(t_{j+1} - t_j)^3}{(1 + q_j + q_j^2)} \right\} \\
& + \frac{(T^2 + |\Lambda|)}{|\eta T - \Lambda|} \left\{ |\beta| + \sum_{k=1}^{i} \left(M_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + M_2 \right) \right.
\end{aligned}
$$

$$+\sum_{k=1}^{i}\left(M_1(t_k-t_{k-1})+M_3\right)(\eta-t_k)+M_1\frac{(\eta-t_i)^2}{1+q_i}\Bigg\} \tag{23}$$
$$+\sum_{k=1}^{m}\left(M_1\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+M_2\right)+\sum_{k=1}^{m}\left(M_1(t_k-t_{k-1})+M_3\right)(T-t_k)$$
$$+M_1\frac{(T-t_m)^2}{1+q_m}.$$

Theorem 3.1. *Assume that:*

(H_1) $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function and there exists a constant* $L_1>0$ *such that* $|f(t,x)-f(t,y)|\le L_1|x-y|$, *for each* $t\in J$ *and* $x,y\in\mathbb{R}$.

(H_2) *The functions* $I_k, I_k^*:\mathbb{R}\to\mathbb{R}$ *are continuous and there exist constants* L_2, $L_3>0$ *such that* $|I_k(x)-I_k(y)|\le L_2|x-y|$ *and* $|I_k^*(x)-I_k^*(y)|\le L_3|x-y|$ *for each* $x,y\in\mathbb{R}$, $k=1,2,\ldots,m$.

If

$$\Omega\le\delta<1, \tag{24}$$

then average value problem (1) *has a unique solution on* J.

Proof. We transform the problem (1) into a fixed point problem, $x=\mathcal{F}x$, where the operator $\mathcal{F}$ is defined by (20). By using the Banach's contraction mapping principle, we shall show that $\mathcal{F}$ has a fixed point which is a unique solution of problem (1).

Setting $\sup_{t\in J}|f(t,0)|=M_1<\infty$, $\sup\{|I_k(0)|\,;\,k=1,2,\ldots,m\}=M_2<\infty$ and $\sup\{|I_k^*(0)|\,;\,k=1,2,\ldots,m\}=M_3<\infty$, and choosing $r\ge\dfrac{\gamma}{1-\varepsilon}$, where $\delta\le\varepsilon<1$ we show that $\mathcal{F}B_r\subset B_r$, where $B_r=\{x\in PC(J,\mathbb{R}):\|x\|\le r\}$. For $x\in B_r$, we have

$$\begin{aligned}&\|\mathcal{F}x\|\\ \le\ &\sup_{t\in J}\Bigg\{\frac{|t-\eta|}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k))|\right)|H_{jk}^-|\\ &+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(x(t_k))|\right)(t_{j+1}-t_j)\\ &+\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau}|f(u,x(u))|d_{q_j}ud_{q_j}\tau d_{q_j}s\Bigg\}\end{aligned}$$

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$$
\begin{aligned}
& + \frac{|tT-\Lambda|}{|\eta T-\Lambda|}\Bigg\{|\beta| + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s + |I_k(x(t_k))|\right) \\
& + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s + |I_k^*(x(t_k))|\right)(\eta - t_k) \\
& + \int_{t_i}^{\eta}\int_{t_i}^{s}|f(\tau,x(\tau))|d_{q_i}\tau d_{q_i}s\Bigg\} \\
& + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s + |I_k(x(t_k))|\right) \\
& + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s + |I_k^*(x(t_k))|\right)(t - t_k) \\
& + \int_{t_k}^{t}\int_{t_k}^{s}|f(\tau,x(\tau))|d_{q_k}\tau d_{q_k}s\Bigg\} \\
\leq\ & \frac{(T+\eta)}{|\eta T-\Lambda|}\Bigg\{|\alpha| + \sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)d_{q_{k-1}}s\right. \\
& \left. + |I_k^*(x(t_k)) - I_k^*(0)| + |I_k^*(0)|\right)|H_{jk}^-| \\
& + \sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(|f(\tau,x(\tau)) - f(\tau,0)|\right. \\
& \left. + |f(\tau,0)|)d_{q_{k-1}}\tau d_{q_{k-1}}s + |I_k(x(t_k)) - I_k(x(0))| + |I_k(x(0))|\right)(t_{j+1} - t_j) \\
& + \sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau}(|f(u,x(u)) - f(u,0)| + |f(u,0)|)d_{q_j}ud_{q_j}\tau d_{q_j}s\Bigg\} \\
& + \frac{(T^2+|\Lambda|)}{|\eta T-\Lambda|}\Bigg\{|\beta| + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|)d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
& \left. + |I_k(x(t_k)) - I_k(0)| + |I_k(0)|\right) \\
& + \sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)d_{q_{k-1}}s\right. \\
& \left. + |I_k^*(x(t_k)) - I_k^*(0)| + |I_k^*(0)|\right)(\eta - t_k) \\
& + \int_{t_i}^{\eta}\int_{t_i}^{s}(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|)d_{q_i}\tau d_{q_i}s\Bigg\}
\end{aligned}
$$

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$$
\begin{aligned}
& + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (|f(\tau, x(\tau)) - f(\tau, 0)| + |f(\tau, 0)|) d_{q_{k-1}} \tau d_{q_{k-1}} s \right. \\
& \left. + |I_k(x(t_k)) - I_k(0)| + |I_k(0)| \right) \\
& + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) d_{q_{k-1}} s \right. \\
& \left. + |I_k^*(x(t_k)) - I_k^*(0)| + |I_k^*(0)| \right) (T - t_k) \\
& + \int_{t_m}^{T} \int_{t_m}^{s} (|f(\tau, x(\tau)) - f(\tau, 0)| + |f(\tau, 0)|) d_{q_m} \tau d_{q_m} s \\
\leq \; & \frac{(T+\eta)}{|\eta T - \Lambda|} \left\{ |\alpha| + \sum_{j=1}^{m} \sum_{k=1}^{j} \big((L_1 r + M_1)(t_k - t_{k-1}) + L_3 r + M_3 \big) H_{jk}^{+} \right. \\
& + \sum_{j=1}^{m} \sum_{k=1}^{j} \left((L_1 r + M_1) \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 r + M_2 \right) (t_{j+1} - t_j) \\
& \left. + \sum_{j=0}^{m} (L_1 r + M_1) \frac{(t_{j+1} - t_j)^3}{(1 + q_j + q_j^2)} \right\} \\
& + \frac{(T^2 + |\Lambda|)}{|\eta T - \Lambda|} \left\{ |\beta| + \sum_{k=1}^{i} \left((L_1 r + M_1) \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 r + M_2 \right) \right. \\
& \left. + \sum_{k=1}^{i} \Big((L_1 r + M_1)(t_k - t_{k-1}) + L_3 r + M_3 \Big) (\eta - t_k) + (L_1 r + M_1) \frac{(\eta - t_i)^2}{1 + q_i} \right\} \\
& + \sum_{k=1}^{m} \left((L_1 r + M_1) \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + L_2 r + M_2 \right) \\
& + \sum_{k=1}^{m} ((L_1 r + M_1)(t_k - t_{k-1}) + L_3 r + M_3) (T - t_k) + (L_1 r + M_1) \frac{(T - t_m)^2}{1 + q_m} \\
= \; & r\Omega + \gamma \leq (\delta + 1 - \varepsilon) r \leq r.
\end{aligned}
$$

It follows that $\mathcal{F}B_r \subset B_r$.

For $x, y \in PC(J, \mathbb{R})$ and for each $t \in J$, we have

$$
\begin{aligned}
& |\mathcal{F}x(t) - \mathcal{F}y(t)| \\
\leq \; & \frac{|t - \eta|}{|\eta T - \Lambda|} \left\{ \sum_{j=1}^{m} \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} |f(s, x(s)) - f(s, y(s))| d_{q_{k-1}} s \right. \right.
\end{aligned}
$$

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$$
\begin{aligned}
& + |I_k^*(x(t_k)) - I_k^*(y(t_k))| \Big) H_{jk}^+ \\
& + \sum_{j=1}^{m}\sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} |f(\tau,x(\tau)) - f(\tau,y(\tau))| d_{q_{k-1}}\tau d_{q_{k-1}} s \right. \\
& \left. + |I_k(x(t_k)) - I_k(y(t_k))| \right) (t_{j+1} - t_j) \\
& \left. + \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \int_{t_j}^{s}\int_{t_j}^{\tau} |f(u,x(u)) - f(u,y(u))| d_{q_j} u d_{q_j}\tau d_{q_j} s \right\} \\
& + \frac{|tT-\Lambda|}{|\eta T - \Lambda|} \left\{ \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} |f(\tau,x(\tau)) - f(\tau,y(\tau))| d_{q_{k-1}}\tau d_{q_{k-1}} s \right. \right. \\
& \left. + |I_k(x(t_k)) - I_k(y(t_k))| \right) + \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} |f(s,x(s)) - f(s,y(s))| d_{q_{k-1}} s \right. \\
& \left. \left. + |I_k^*(x(t_k)) - I_k^*(y(t_k))| \right) (\eta - t_k) + \int_{t_i}^{\eta}\int_{t_i}^{s} |f(\tau,x(\tau)) - f(\tau,y(\tau))| d_{q_i}\tau d_{q_i} s \right\} \\
& + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} |f(\tau,x(\tau)) - f(\tau,y(\tau))| d_{q_{k-1}}\tau d_{q_{k-1}} s \right. \\
& \left. + |I_k(x(t_k)) - I_k(y(t_k))| \right) \\
& + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} |f(s,x(s)) - f(s,y(s))| d_{q_{k-1}} s + |I_k^*(x(t_k)) - I_k^*(y(t_k))| \right) (T - t_k) \\
& + \int_{t_m}^{T}\int_{t_m}^{s} |f(\tau,x(\tau)) - f(\tau,y(\tau))| d_{q_m}\tau d_{q_m} s \\
\leq \quad & \|x-y\| \left\{ \frac{(T+\eta)}{|\eta T - \Lambda|} \left[\sum_{j=1}^{m}\sum_{k=1}^{j} \left(L_1(t_k - t_{k-1}) + L_3 \right) H_{jk}^+ \right. \right. \\
& \left. + \sum_{j=1}^{m}\sum_{k=1}^{j} \left(L_1 \frac{(t_k - t_{k-1})^2}{1+q_{k-1}} + L_2 \right) (t_{j+1} - t_j) + \sum_{j=0}^{m} L_1 \frac{(t_{j+1} - t_j)^3}{(1+q_j+q_j^2)} \right] \\
& + \frac{(T^2 + |\Lambda|)}{|\eta T - \Lambda|} \left[\sum_{k=1}^{i} \left(L_1 \frac{(t_k - t_{k-1})^2}{1+q_{k-1}} + L_2 \right) \right. \\
& \left. \left. + \sum_{k=1}^{i} \left(L_1(t_k - t_{k-1}) + L_3 \right) (\eta - t_k) + L_1 \frac{(\eta - t_i)^2}{1+q_i} \right] \right\} \\
& + \|x-y\| \sum_{k=1}^{m} \left(L_1 \frac{(t_k - t_{k-1})^2}{1+q_{k-1}} + L_2 \right) + \|x-y\| \sum_{k=1}^{m} \left(L_1(t_k - t_{k-1}) + L_3 \right) (T - t_k)
\end{aligned}
$$

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$$\begin{aligned} & + \|x-y\|L_1\frac{(T-t_m)^2}{1+q_m} \\ = \quad & \Omega\|x-y\|. \end{aligned}$$

This implies that $\|\mathcal{F}x - \mathcal{F}y\| \leq \Omega\|x-y\|$. As $\Omega < 1$, therefore $\mathcal{F}$ is a contraction. Hence, by the Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is the unique solution of the problem (1). □

Now, our second existence result is based on Leray-Schauder's nonlinear alternative.

Lemma 3.2. *(Nonlinear alternative for single-valued maps) [19]. Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Theorem 3.3. *Assume that:*

(H_3) *There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C(J, \mathbb{R}^+)$ such that*

$$|f(t,x)| \leq p(t)\psi(|x|) \quad for\ each\ (t,x) \in J \times \mathbb{R}.$$

(H_4) *There exist continuous nondecreasing functions $\varphi, \varphi^* : [0,\infty) \to (0,\infty)$ such that*

$$|I_k(x)| \leq \varphi(|x|), \quad |I_k^*(x)| \leq \varphi^*(|x|) \quad for\ all\ x \in \mathbb{R}.$$

(H_5) *There exists a constant $M^* > 0$ such that*

$$\frac{M^*}{p_0\psi(M^*)Q_0 + \varphi(M^*)Q_1 + \varphi^*(M^*)Q_2 + Q_3} > 1,$$

where $p_0 = \max\{p(t)\,;\, t \in J\}$ and

$$Q_0 \quad = \quad \frac{T+\eta}{|\eta T - \Lambda|}\left[\sum_{j=1}^{m}\sum_{k=1}^{j}\left((t_k - t_{k-1})H_{jk}^{+} + \frac{(t_k-t_{k-1})^2}{1+q_{k-1}}(t_{j+1}-t_j)\right)\right.$$

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$$+\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)^3}{1+q_j+q_j^2}\Bigg]+\sum_{k=1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+(t_k-t_{k-1})(T-t_k)\right)$$
$$+\frac{T^2+|\Lambda|}{|\eta T-\Lambda|}\left[\frac{(\eta-t_i)^2}{1+q_i}+\sum_{k=1}^{i}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+(t_k-t_{k-1})(\eta-t_k)\right)\right]$$
$$+\frac{(T-t_m)^2}{1+q_m},$$

$$Q_1 = m+\frac{i(T^2+|\Lambda|)}{|\eta T-\Lambda|}+\frac{T+\eta}{|\eta T-\Lambda|}\sum_{j=1}^{m}j(t_{j+1}-t_j),$$

$$Q_2 = \frac{T+\eta}{|\eta T-\Lambda|}\sum_{j=1}^{m}\sum_{k=1}^{j}H_{jk}^{+}+\frac{T^2+|\Lambda|}{|\eta T-\Lambda|}\sum_{k=1}^{i}(\eta-t_k)+\sum_{k=1}^{m}(T-t_k),$$

$$Q_3 = \frac{T+\eta}{|\eta T-\Lambda|}|\alpha|+\frac{T^2+|\Lambda|}{|\eta T-\Lambda|}|\beta|.$$

Then average value problem (1) *has at least one solution on J.*

Proof. Firstly, we shall show that $\mathcal{F}$, defined by (20), *maps bounded sets (balls) into bounded sets in $PC(J,\mathbb{R})$.* For a positive number ρ, let $B_\rho=\{x\in PC(J,\mathbb{R}):\|x\|\le\rho\}$ be a bounded ball in $PC(J,\mathbb{R})$. Then for $t\in J$ we have

$$\begin{aligned}
&|\mathcal{F}x(t)|\\
\le\;& \frac{(T+\eta)}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k))|\right)|H_{jk}^{-}|\\
&+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(x(t_k))|\right)(t_{j+1}-t_j)\\
&+\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau}|f(u,x(u))|d_{q_j}ud_{q_j}\tau d_{q_j}s\Bigg\}\\
&+\frac{(T^2+|\Lambda|)}{|\eta T-\Lambda|}\Bigg\{|\beta|+\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(x(t_k))|\right)\\
&+\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k))|\right)(\eta-t_k)\\
&+\int_{t_i}^{\eta}\int_{t_i}^{s}|f(\tau,x(\tau))|d_{q_i}\tau d_{q_i}s\Bigg\}
\end{aligned}$$

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$$
\begin{aligned}
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(x(t_k))|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k))|\right)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,x(\tau))|d_{q_m}\tau d_{q_m}s\\
\le\ & \frac{(T+\eta)}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s+\varphi^*(\|x\|)\right)H_{jk}^{+}\\
&+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}\tau d_{q_{k-1}}s+\varphi(\|x\|)\right)(t_{j+1}-t_j)\\
&+p_0\psi(\|x\|)\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau}d_{q_j}ud_{q_j}\tau d_{q_j}s\Bigg\}\\
&+\frac{(T^2+|\Lambda|)}{|\eta T-\Lambda|}\Bigg\{|\beta|+\sum_{k=1}^{i}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}\tau d_{q_{k-1}}s+\varphi(\|x\|)\right)\\
&+\sum_{k=1}^{i}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s+\varphi^*(\|x\|)\right)(\eta-t_k)\\
&+p_0\psi(\|x\|)\int_{t_i}^{\eta}\int_{t_i}^{s}d_{q_i}\tau d_{q_i}s\Bigg\}+\psi(\|x\|)\int_{t_m}^{T}\int_{t_m}^{s}p(\tau)d_{q_m}\tau d_{q_m}s\\
&+\sum_{k=1}^{m}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}\tau d_{q_{k-1}}s+\varphi(\|x\|)\right)\\
&+\sum_{k=1}^{m}\left(p_0\psi(\|x\|)\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s+\varphi^*(\|x\|)\right)(T-t_k)\\
\le\ & \frac{(T+\eta)}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\rho)(t_k-t_{k-1})+\varphi^*(\rho)\right)H_{jk}^{+}\\
&+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\rho)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\rho)\right)(t_{j+1}-t_j)\\
&+p_0\psi(\rho)\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)^3}{1+q_j+q_j^2}\Bigg\}\\
&+\frac{(T^2+|\Lambda|)}{|\eta T-\Lambda|}\Bigg\{|\beta|+\sum_{k=1}^{i}\left(p_0\psi(\rho)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\rho)\right)\\
&+\sum_{k=1}^{i}\left(p_0\psi(\rho)(t_k-t_{k-1})+\varphi^*(\rho)\right)(\eta-t_k)+p_0\psi(\rho)\frac{(\eta-t_i)^2}{1+q_i}\Bigg\}
\end{aligned}
$$

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$$
\begin{aligned}
& +\psi(\rho)\frac{(T-t_m)^2}{1+q_m}+\sum_{k=1}^{m}\left(p_0\psi(\rho)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\rho)\right) \\
& +\sum_{k=1}^{m}\left(p_0\psi(\rho)(t_k-t_{k-1})+\varphi^*(\rho)\right)(T-t_k) \\
:= \quad & K.
\end{aligned}
$$

Therefore, we conclude that $\mathcal{F}x \leq K$.

Next we show that $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$. Let $\sup_{(t,x)\in J\times B_\rho}|f(t,x)| = \overline{f} < \infty$, $\nu_1, \nu_2 \in (t_l, t_{l+1})$ for some $l \in \{0,1,\ldots,m\}$ with $\nu_1 < \nu_2$ and $x \in B_\rho$. Then we have

$$
\begin{aligned}
& |(\mathcal{F}x)(\nu_2)-(\mathcal{F}x)(\nu_1)| \\
= \quad & \Bigg|\frac{(\nu_2-\nu_1)}{|\eta T-\Lambda|}\Bigg\{-\alpha+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)H_{jk}^- \\
& +\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)(t_{j+1}-t_j) \\
& +\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\int_{t_j}^{s}\int_{t_j}^{\tau} f(u,x(u))d_{q_j}u d_{q_j}\tau d_{q_j}s\Bigg\} \\
& +\frac{(\nu_2 T-\nu_1 T)}{|\eta T-\Lambda|}\Bigg\{\beta-\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right) \\
& -\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\eta-t_k) \\
& -\int_{t_i}^{\eta}\int_{t_i}^{s} f(\tau,x(\tau))d_{q_i}\tau d_{q_i}s\Bigg\} \\
& +\sum_{k=1}^{l}\left(\int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(\nu_2-\nu_1) \\
& +\int_{t_l}^{\nu_2}\int_{t_l}^{s} f(\tau,x(\tau))d_{q_l}\tau d_{q_l}s-\int_{t_l}^{\nu_1}\int_{t_l}^{s} f(\tau,x(\tau))d_{q_l}\tau d_{q_l}s\Bigg| \\
\leq \quad & |\nu_2-\nu_1|\left[\frac{1}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left((t_k-t_{k-1})\overline{f}+\varphi^*(\rho)\right)H_{jk}^+\right. \\
& +\sum_{j=1}^{m}\sum_{k=1}^{j}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\overline{f}+\varphi(\rho)\right)(t_{j+1}-t_j)
\end{aligned}
$$

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$$
\begin{aligned}
&+\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)^3}{(1+q_j+q_j^2)}\overline{f}+T\left(|\beta|+\sum_{k=1}^{i}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\overline{f}+\varphi(\rho)\right)\right.\\
&\left.+\sum_{k=1}^{i}\left((t_k-t_{k-1})\overline{f}+\varphi^*(\rho)\right)(\eta-t_k)+\frac{(\eta-t_i)^2}{1+q_i}\overline{f}\right)\Bigg\}\\
&+\sum_{k=1}^{l}\left((t_k-t_{k-1})\overline{f}+\varphi^*(\rho)\right)+\frac{(\nu_2+\nu_1+2t_l)}{1+q_l}\overline{f}\Bigg].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\nu_2-\nu_1 \to 0$. As $\mathcal{F}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in J$, and following similar computations as in the first step, we have

$$
\begin{aligned}
&\|x\|\\
\leq\ &\frac{(T+\eta)}{|\eta T-\Lambda|}\Bigg\{|\alpha|+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\|x\|)(t_k-t_{k-1})+\varphi^*(\|x\|)\right)H_{jk}^{+}\\
&+\sum_{j=1}^{m}\sum_{k=1}^{j}\left(p_0\psi(\|x\|)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\|x\|)\right)(t_{j+1}-t_j)\\
&+p_0\psi(\|x\|)\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)^3}{(1+q_j+q_j^2)}\Bigg\}\\
&+\frac{(T^2+|\Lambda|)}{|\eta T-\Lambda|}\Bigg\{|\beta|+\sum_{k=1}^{i}\left(p_0\psi(\|x\|)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\|x\|)\right)\\
&+\sum_{k=1}^{i}\left(p_0\psi(\|x\|)(t_k-t_{k-1})+\varphi^*(\|x\|)\right)(\eta-t_k)+p_0\psi(\|x\|)\frac{(\eta-t_i)^2}{1+q_i}\Bigg\}\\
&+\sum_{k=1}^{m}\left(p_0\psi(\|x\|)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi(\|x\|)\right)\\
&+\sum_{k=1}^{m}\left(p_0\psi(\|x\|)(t_k-t_{k-1})+\varphi^*(\|x\|)\right)(T-t_k)+p_0\psi(\|x\|)\frac{(T-t_m)^2}{1+q_m}\\
=\ &p_0\psi(\|x\|)Q_0+\varphi(\|x\|)Q_1+\varphi^*(\|x\|)Q_2+Q_3.
\end{aligned}
$$

Consequently, we have

$$\frac{\|x\|}{p_0\psi(\|x\|)Q_0+\varphi(\|x\|)Q_1+\varphi^*(\|x\|)Q_2+Q_3}\leq 1.$$

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In view of (H_5), there exists M^* such that $\|x\| \neq M^*$. Let us set

$$U = \{x \in PC(J, \mathbb{R}) : \|x\| < M^*\}. \tag{25}$$

Note that the operator $\mathcal{F} : \overline{U} \to PC(J, \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda \mathcal{F} x$ for some $\lambda \in (0, 1)$. Consequently, by nonlinear alternative of Leray-Schauder type (Lemma 3.2) we deduce that $\mathcal{F}$ has a fixed point in $\overline{U}$, which is a solution of the average problem (1). This completes the proof. $\square$

4 Examples

Example 4.1. *Consider the following average value problem for second-order impulsive q_k-difference equation*

$$\begin{cases} D^2_{\frac{2+k}{3+2k}} x(t) = \dfrac{e^{-\cos^2 t}|x(t)|}{(10+t)^2(1+|x(t)|)}, \quad t \in J = [0,1], \ t \neq t_k, \\ \Delta x(t_k) = \dfrac{|x(t_k)|}{10(9+|x(t_k)|)}, \quad t_k = \dfrac{k}{10}, \quad k = 1, 2, \ldots, 9, \\ D_{\frac{2+k}{3+2k}} x(t_k^+) - D_{\frac{1+k}{1+2k}} x(t_k) = \dfrac{1}{8}\tan^{-1}\left(\dfrac{1}{11}x(t_k)\right), \quad t_k = \dfrac{k}{10}, \quad k = 1, 2, \ldots, 9, \\ \displaystyle\sum_{j=0}^{9} \int_{t_j}^{t_{j+1}} x(s) d_{\frac{2+j}{3+2j}} s = 5, \quad x\left(\dfrac{9}{40}\right) = 3, \end{cases} \tag{26}$$

Here $q_k = (2+k)/(3+2k)$, $k = 0, 1, 2, \ldots, 9$, $m = 9$, $T = 1$, $i = 2$, $\eta = 9/40$, $f(t, x) = (e^{-\cos^2 t}|x|)/((10+t)^2(1+|x|))$, $I_k(x) = |x|/(10(9+|x|))$ and $I_k^*(x) = (1/8)\tan^{-1}(x/11)$. Since

$$|f(t,x) - f(t,y)| \leq (1/100)|x-y|,$$

$$|I_k(x) - I_k(y)| \leq (1/90)|x-y| \ \text{ and } |I_k^*(x) - I_k^*(y)| \leq (1/88)|x-y|,$$

then (H_1) and (H_2) are satisfied with $L_1 = (1/100)$, $L_2 = (1/90)$, $L_3 = (1/88)$. We can show that

$$\Lambda \approx 0.514187, \quad \Omega \approx 0.7472621 < 1.$$

Hence, by Theorem 3.1, the average value problem (26) has a unique solution on $[0, 1]$.

Example 4.2. *Consider the following average value problem for second-order impulsive q_k-difference equation*

$$\begin{cases} D^2_{\frac{1+k}{3+2k}} x(t) = \dfrac{\sin(\pi x/2)}{13\pi^2 + \sin^2(\pi x)} + \dfrac{1+\sin(\pi t/2)}{25\pi}, \quad t \in J = [0,1],\ t \neq t_k, \\ \Delta x(t_k) = \dfrac{\sin(\pi x/2)}{15\pi^2}, \quad t_k = \dfrac{k}{10}, \quad k = 1,2,\ldots,9, \\ D_{\frac{1+k}{3+2k}} x(t_k^+) - D_{\frac{k}{1+2k}} x(t_k) = \dfrac{x}{35\pi + x^2}, \quad t_k = \dfrac{k}{10}, \quad k = 1,2,\ldots,9, \\ \displaystyle\sum_{j=0}^{9} \int_{t_j}^{t_{j+1}} x(s) d_{\frac{1+j}{3+2j}} s = 1, \quad x\left(\frac{1}{4}\right) = \frac{1}{2}, \end{cases} \tag{27}$$

Here $q_k = (1+k)/(3+2k)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $i = 2$, $\eta = 1/4$, $\alpha = 1$, $\beta = 1/2$, $f(t,x) = (\sin(\pi x/2)) / \left(13\pi^2 + \sin^2(\pi x)\right) + (1+\sin(\pi t/2))/(25\pi)$, $I_k(x) = (\sin(\pi x/2)) / \left(15\pi^2\right)$ and $I_k^*(x) = (x)/\left(35\pi + x^2\right)$. Clearly,

$$|f(t,x)| = \left| \frac{\sin(\pi x/2)}{13\pi^2 + \sin^2(\pi x)} + \frac{1+\sin(\pi t/2)}{25\pi} \right| \leq (1+\sin(\pi t/2)) \left(\frac{|x|+1}{25\pi} \right),$$

$$|I_k(x)| = \left| \frac{\sin(\pi x/2)}{15\pi^2} \right| \leq \frac{|x|}{30\pi},$$

and

$$|I_k^*(x)| = \left| \frac{x}{35\pi + x^2} \right| \leq \frac{|x|}{35\pi}.$$

Choosing $p(t) = 1 + \sin(\pi t/2)$, $\psi(|x|) = (|x|+1)/(25\pi)$, $\varphi(|x|) = (|x|)/(30\pi)$ and $\varphi^*(|x|) = (|x|)/(35\pi)$, we can show that

$$\Lambda \approx 0.519461129, \quad Q_0 \approx 3.976411236,$$

$$Q_1 \approx 41.15277206, \quad Q_2 \approx 27.92869898, \quad Q_3 \approx 7.458332013,$$

and

$$\frac{M^*}{(2)\left(\frac{M^*+1}{25\pi}\right)(3.976411236) + \left(\frac{M^*}{30\pi}\right)(41.15277206) + \left(\frac{M^*}{35\pi}\right)(27.92869898) + 7.458332013} > 1,$$

which implies that $M^* > 36.32714112$. Hence, by Theorem 3.3, the average value problem (27) has at least one solution on $[0,1]$.

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ESSENTIAL NORMS OF INTEGRAL-TYPE OPERATORS FROM WEIGHTED BERGMAN SPACES INTO ZYGMUND-TYPE SPACES

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Abstract. For a unit disk $\mathbb{D}$ in the complex plane, by $H(\mathbb{D})$ denote the class of all holomorphic functions on $\mathbb{D}$. We estimate the essential norms of integral-type operators $(C^m_{\varphi,g}f)(z) = \int_0^z f^{(m)}(\varphi(\xi))g(\xi)d\xi$ from weighted bergman spaces A^p_α into zygmund-type spaces $\mathcal{Z}_\mu$, where $f, g \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. Moreover, we obtain a new characterization for the boundedness and its essential norm in terms of the $n-$th power of φ of $C^m_{\varphi,g} : A^p_\alpha \to \mathcal{Z}_\mu$ on $\mathbb{D}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $\mathbb{N}$ the set of all nonnegative integers. For $p > 0$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of $f \in H(\mathbb{D})$ with the following notation

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\right)^{\frac{1}{p}} < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, and $dA(z)$ is the 2-dimentional Lebesgue area measure on $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. Note that when $1 \le p < \infty$ the space A^p_α is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. If $0 < p < 1$, the space A^p_α is a complete metric space with the distance: $d(f, g) = \|f - g\|^p_{p,\alpha}$.

A positive continuous function v on $[0, 1)$ is called normal (see [8]), if there exist three positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that for $r \in [\delta, 1)$

$$(i)\quad \frac{v(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1),\quad \lim_{r\to 1} \frac{v(r)}{(1-r)^a} = 0;$$

$$(ii)\quad \frac{v(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1),\quad \lim_{r\to 1} \frac{v(r)}{(1-r)^b} = \infty.$$

If we say a function μ is normal we will also assume that it's radial, that is, $\mu(z) = \mu(|z|), z \in \mathbb{D}$. In this paper, we suppose μ is normal.

Recall that the Zygmund-type space, denoted by $\mathcal{Z}_\mu$, consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Under the following norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|. \tag{1.1}$$

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Let $\mathcal{Z}_{\mu,0}$ denote the subspace of $\mathcal{Z}_\mu$ consisting of those $f \in H(\mathbb{D})$ for which

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0.$$

This space is called little Zygmund-type space. Both $\mathcal{Z}_\mu$ and $\mathcal{Z}_{\mu,0}$ are Banach spaces with the norm $\|\cdot\|_{\mathcal{Z}_\mu}$ defined in (1.1).

Moreover, we need the weighted Banach spaces of analytic functions

$$H_\nu^\infty = \{f \in H(\mathbb{D}) : \|f\|_\nu := \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty\},$$

endowed with the norm $\|\cdot\|_\nu$. The so-called associated weight is an important tool to handle problems in the setting of weighted Banach spaces of analytic functions. For a weight ν the associated weight $\widetilde{\nu}$ is defined by $\widetilde{\nu}(z) := (\sup\{|f(z)|; f \in H_\nu^\infty, \|f\|_\nu \le 1\})^{-1},\ z \in \mathbb{D}$. For standard weights $\nu_\alpha(z) = (1-|z|^2)^\alpha$, where $0 < \alpha < \infty$ and $\widetilde{\nu}_\alpha = \nu_\alpha$.

If φ is an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, the linear weighted composition operator associated to φ and u is defined by $uC_\varphi f = uf(\varphi)$ for any $f \in H(\mathbb{D})$. This operator is well studied by many researchers, and readers interested in this topic can refer to [2, 3, 4, 7, 13].

Recently interest has arisen to characterize boundedness and compactness of composition operators acting some spaces in terms of the $n-$th power of φ on $\mathbb{D}$. More clearly, in [10], Wulan, Zheng and Zhu obtained a new result about the compactness of the composition operator on the Bloch space in the unit disk. In [12], Zhao gave a new characterization of boundedness and compactness between Bloch type spaces in terms of φ^n.

Let $g \in H(\mathbb{D})$, m be a positive integer and φ be a holomorphic self-map of $\mathbb{D}$. The following integral-type operator

$$(C_{\varphi,g}^m f)(z) = \int_0^z f^{(m)}(\varphi(\xi))g(\xi)d\xi,\ \ f \in H(\mathbb{D}),\ z \in \mathbb{D},$$

which was firstly introduced by Zhu in [14]. $C_{\varphi,g}^1$ is the generalized composition operator. $C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [5].

Motivated by [9, 14], we first study the boundedness and compactness of $C_{\varphi,g}^m : A_\alpha^p \to \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$ as well as the the essential norm of $C_{\varphi,g}^m : A_\alpha^p \to \mathcal{Z}_\mu$. Apart from these, we get similar estimates about the boundedness and the essential norm of such operators in terms of μ, g and the $n-$th power φ^n of φ. Our approach is inspired by Esmaeili and Lindström in [2], some estimates in [3, 4] make it possible to obtain our results with very easy proofs.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Notation and Lemmas

In order to prove the main results of this paper, we need some auxiliary results.

For $p > 0$, $\alpha > -1$, $j \in \mathbb{N}$ and $a \in \mathbb{D}$. Define the test functions

$$f_{1,a}(z) = \alpha_2 h_{a,1}(z) - \alpha_1 h_{a,2}(z),\ f_{2,a}(z) = \beta_2 h_{a,1}(z) - \beta_1 h_{a,2}(z),$$

where $h_{a,j}(z) = \{\frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j+1}}\}^{\frac{\alpha+2}{p}}$, $\alpha_{i-1} = i \cdot \frac{2+\alpha}{p}(i \cdot \frac{2+\alpha}{p} + 1) \cdot \cdots \cdot (i \cdot \frac{2+\alpha}{p} + m - 1)$, and $\beta_{i-1} = \alpha_{i-1}(i \cdot \frac{2+\alpha}{p} + m)$, for $i = 2,\ 3$. By [11, Proposition 1.4.10], we see that $f_{i,a} \in A_\alpha^p$.

Lemma 2.1. *Suppose $p > 0$, $\alpha > -1$, $m \in \mathbb{N}$, and φ is a holomorphic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^m : A_\alpha^p \to \mathcal{Z}_\mu$ is compact if and only if $C_{\varphi,g}^m : A_\alpha^p \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A_α^p which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|C_{\varphi,g}^m f_k\|_\infty \to 0$ as $k \to \infty$.*

Proof. Similarly to [1, Prop 3.11]. Hence the details are omitted here. □

The following lemma can be proved similarly to [6, Lemma 1].

Lemma 2.2. *Let μ be normal. A closed subset K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies* $\lim\limits_{|z|\to 1}\sup\limits_{f\in K}\mu(z)|f(z)|=0.$

Lemma 2.3 ([3], Theorem 2.4)**.** *Let ν and ω be radial, non-increasing weights tending to 0 at the boundary of $\mathbb{D}$. Then*

(i) the weighted composition operators uC_φ maps H^∞_ν into H^∞_ω if and only if

$$\sup_{n\geq 0}\frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu}\asymp\sup_{z\in\mathbb{D}}\frac{\omega(z)}{\widetilde{\nu}(\varphi(z))}|u(z)|<\infty,$$

with norm comparable to the above supremum.

(ii) $\|uC_\varphi\|_{e,H^\infty_\nu\to H^\infty_\omega}=\lim\limits_{n\to\infty}\sup\frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu}=\lim\limits_{|\varphi(z)|\to 1}\sup\frac{\omega(z)}{\widetilde{\nu}(\varphi(z))}|u(z)|.$

Lemma 2.4 ([4], Lemma 2.1)**.** *For $\forall\ 0<\alpha<\infty$, we have* $\lim\limits_{n\to\infty}(n+1)^\alpha\|z^n\|_{\nu_\alpha}=(\frac{2\alpha}{e})^\alpha.$

3. The boundedness of $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$.

Theorem 3.1. *Suppose $p>0$, $\alpha>-1$, $m\in\mathbb{N}$, and φ is a holomorphic self-map of $\mathbb{D}$, $t=\frac{\alpha+2}{p}+m+1$. Then $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu$ is bounded if and only if*

$$\sup_{n\geq 0}(n+1)^t\|\varphi' g\|_\mu\asymp\sup_{z\,\in\mathbb{D}}\frac{\mu(z)\,|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}<\infty, \tag{3.2}$$

$$\sup_{n\geq 0}(n+1)^{t-1}\|g'\|_\mu\asymp\sup_{z\,\in\mathbb{D}}\frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}}<\infty. \tag{3.3}$$

Proof. Suppose first $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu$ is bounded, that is to say $\|C^m_{\varphi,g}f\|_{\mathcal{Z}_\mu}\leq C\|f\|_{p,\alpha}$. Taking the test functions given by $f_1(z)=z^m$ and $f_2(z)=z^{m+1}$. For $|\varphi(z)|\leq 1$ and $\|f_i\|_{p,\alpha}\leq C,\ i=1,\ 2$, it's clear that

$$\sup_{z\,\in\mathbb{D}}\mu(z)|g'(z)|<\infty, \tag{3.4}$$

$$\sup_{z\,\in\mathbb{D}}\mu(z)|g(z)\varphi'(z)|<\infty. \tag{3.5}$$

Let $a=\varphi(z)$ in $f_{1,a}(z)$ and $f_{2,a}(z)$, from the boundedness of $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu$, we get

$$\infty>\|C^m_{\varphi,g}f_{1,a}\|_{\mathcal{Z}_\mu}\geq\frac{\alpha_1\alpha_2\frac{\alpha+2}{p}\mu(z)|g(z)\varphi'(z)||\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^t},$$

and

$$\infty>\|C^m_{\varphi,g}f_{2,a}\|_{\mathcal{Z}_\mu}\geq\frac{\alpha_1\alpha_2\frac{\alpha+2}{p}\mu(z)|g'(z)||\varphi(z)|^m}{(1-|\varphi(z)|^2)^{t-1}}.$$

If $|\varphi(z)|>r$, for any fixed $r\in(0,1)$, from the above, we obtain

$$\sup_{|\varphi(z)|>r}\frac{\mu(z)\,|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}<\sup_{|\varphi(z)|>r}\frac{\mu(z)\,|\varphi'(z)g(z)||\varphi(z)|^{m+1}}{r^{m+1}(1-|\varphi(z)|^2)^t}<\infty, \tag{3.6}$$

and

$$\sup_{|\varphi(z)|>r}\frac{\mu(z)\,|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}}<\sup_{|\varphi(z)|>r}\frac{\mu(z)\,|\varphi'(z)g(z)||\varphi(z)|^m}{r^m(1-|\varphi(z)|^2)^{t-1}}<\infty. \tag{3.7}$$

If $|\varphi(z)|\leq r$, for any fixed $r\in(0,1)$, it follows from (3.4) and (3.5) that

$$\sup_{|\varphi(z)|\leq r}\frac{\mu(z)\,|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}\leq\frac{1}{(1-r^2)^t}\sup_{|\varphi(z)|\leq r}\mu(z)|g(z)\varphi'(z)|<\infty, \tag{3.8}$$

and

$$\sup_{|\varphi(z)|\le r} \frac{\mu(z)\,|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}} \le \frac{1}{(1-r^2)^{t-1}} \sup_{|\varphi(z)|\le r} \mu(z)|g'(z)| < \infty. \tag{3.9}$$

On the other hand, (i) of Lemma 2.3 and Lemma 2.4 tell us

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\,|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{t}} \asymp \sup_{n\ge 0} \frac{\|\varphi' g\varphi^n\|_\mu}{\|z^n\|_{\nu_t}} \asymp \sup_{n\ge 0}(n+1)^t\|\varphi' g\|_\mu < \infty, \tag{3.10}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}} \asymp \sup_{n\ge 0} \frac{\|g'\varphi^n\|_\mu}{\|z^n\|_{\nu_{t-1}}} \asymp \sup_{n\ge 0}(n+1)^{t-1}\|g'\|_\mu < \infty. \tag{3.11}$$

Combining (3.6), (3.8) with (3.10), the desired result (3.2) follows. From (3.7), (3.9) and (3.11), we obtain (3.3).

For the converse, we assume (3.2) and (3.3) hold. A well-known inequality (see [9]) for $\forall f\in A^p_\alpha,\ z\in\mathbb{D}$,

$$|f^{(m)}(z)| \le \frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}+m}}. \tag{3.12}$$

Hence

$$\begin{aligned} \mu(z)|(C^m_{\varphi,g}f)''(z)| &= \mu(z)|g(z)\varphi'(z)||f^{(m+1)}(\varphi(z))| + \mu(z)|g'(z)||f^{(m)}(\varphi(z))| \\ &\le \frac{C\mu(z)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^t}\|f\|_{p,\alpha} + \frac{C\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}}\|f\|_{p,\alpha} < \infty. \end{aligned}$$

Moreover, $|(C^m_{\varphi,g}f)(0)|=0$, and $|(C^m_{\varphi,g}f)'(0)| = |f^{(m)}(\varphi(0))g(0)| \le \frac{|g(0)|\|f\|_{p,\alpha}}{(1-|\varphi(0)|^2)^m} < \infty$. So $\|C^m_{\varphi,g}f\|_{\mathcal{Z}_\mu} = |(C^m_{\varphi,g}f)(0)| + |(C^m_{\varphi,g}f)'(0)| + \sup_{z\in\mathbb{D}}\mu(z)(C^m_{\varphi,g}f)''(z)| < \infty$. Thus $C^m_{\varphi,g}: A^p_\alpha \to \mathcal{Z}_\mu$ is bounded. This completes the proof. □

Theorem 3.2. *Suppose $p>0$, $\alpha>-1$, $m\in\mathbb{N}$, and φ is a holomorphic self-map of $\mathbb{D}$. Then operator $C^m_{\varphi,g}: A^p_\alpha \to \mathcal{Z}_{\mu,0}$ is bounded if and only if $C^m_{\varphi,g}: A^p_\alpha\to\mathcal{Z}_\mu$ is bounded,*

$$\lim_{|z|\to 1}\mu(z)\,|\varphi'(z)g(z)| = 0, \tag{3.13}$$

$$\lim_{|z|\to 1}\mu(z)|g'(z)| = 0. \tag{3.14}$$

Proof. Assume that $C^m_{\varphi,g}: A^p_\alpha\to\mathcal{Z}_{\mu,0}$ is bounded. Since $\mathcal{Z}_{\mu,0}$ is the subspace of $\mathcal{Z}_\mu$, it is easy to see $C^m_{\varphi,g}: A^p_\alpha\to\mathcal{Z}_\mu$ is bounded. From this and since $f_1(z)=z^m$ and $f_2(z)=z^{m+1}$ are in A^p_α, we can easily show that (3.13) and (3.14) hold.

Conversely, suppose that $C^m_{\varphi,g}: A^p_\alpha\to\mathcal{Z}_\mu$ is bounded and (3.13) and (3.14) hold. Using the condition that polynomials are dense in A^p_α, we see that

$$\mu(z)(C^m_{\varphi,g}p)''(z)| \le \mu(z)|\varphi'(z)g(z)|\|p^{(m+1)}\|_\infty + \mu(z)|g'(z)|\|p^{(m)}\|_\infty,$$

where p denotes any polynomial, $\|\cdot\|_\infty$ is the supremum norm of function on $\mathbb{D}$. Applying (3.13) and (3.14) to the above inequality, we have $C^m_{\varphi,g}p\in\mathcal{Z}_{\mu,0}$. Then $C^m_{\varphi,g}f\in\mathcal{Z}_{\mu,0}$, for any $f\in A^p_\alpha$. By the closed graph theorem $C^m_{\varphi,g}: A^p_\alpha\to\mathcal{Z}_{\mu,0}$ is bounded. □

4. The essential norm of $C^m_{\varphi,g}: A^p_\alpha \to \mathcal{Z}_\mu$.

The essential norm of a bounded operator T is the distance from T to the compact $\mathcal{K}$, that is $\|T\|_e = \inf\{\|T-\mathcal{K}\| : \mathcal{K} \;\; is \;\; compact\}$. Notice that $\|T\|_e = 0$ if and only if T is compact.

Theorem 4.1. *Suppose $p>0$, $\alpha>-1$, $m\in\mathbb{N}$, and $\|\varphi\|_\infty = 1$, $t=\frac{\alpha+2}{p}+m+1$. If the operator $C^m_{\varphi,g}: A^p_\alpha \to \mathcal{Z}_\mu$ is bounded, then*

$$\begin{aligned}\|C^m_{\varphi,g}\|_e &\asymp \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\;|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}} \\ &\asymp \limsup_{n\to\infty}(n+1)^t\|\varphi' g\|_\mu + \limsup_{n\to\infty}(n+1)^{t-1}\|g'\|_\mu. \end{aligned} \tag{4.15}$$

Proof. We start with the lower bound. Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Considering the functions $f_{1,j}(z)$ and $f_{2,j}(z)$ separately,

$$f_{1,j}(z) = \alpha_2 h_{\varphi(z_j),1}(z) - \alpha_1 h_{\varphi(z_j),2}(z), \quad f_{2,j}(z) = \beta_2 h_{\varphi(z_j),1}(z) - \beta_1 h_{\varphi(z_j),2}(z).$$

For $i = 1,\ 2$, a calculation shows that $\{f_{i,j}\}\to 0$ uniformly on compact subsets of $\mathbb{D}$ in A^p_α. Since the duality of A^p_α verifies $f_{i,j}\to 0$ weakly in A^p_α, for every compact operator $\mathcal{K}: A^p_\alpha\to\mathcal{Z}_\mu$, $\;\|\mathcal{K}f_{i,j}\|_{\mathcal{Z}_\mu}, \to 0$ as $j\to\infty$. Hence we obtain that

$$\|C^m_{\varphi,g}\|_e \;\geq\; C\limsup_{j\to\infty}(\|C^m_{\varphi,g}f_{1,j}\|_{\mathcal{Z}_\mu} - \|\mathcal{K}f_{1,j}\|_{\mathcal{Z}_\mu}) \asymp \limsup_{j\to\infty}\frac{\mu(z)|\varphi'(z_j)g(z_j)|}{(1-|\varphi(z_j)|^2)^t}. \tag{4.16}$$

$$\|C^m_{\varphi,g}\|_e \;\geq\; C\limsup_{j\to\infty}(\|C^m_{\varphi,g}f_{2,j}\|_{\mathcal{Z}_\mu} - \|\mathcal{K}f_{2,j}\|_{\mathcal{Z}_\mu}) \asymp \limsup_{j\to\infty}\frac{\mu(z)|g'(z_j)|}{(1-|\varphi(z_j)|^2)^{t-1}}. \tag{4.17}$$

Inequalities (4.16) and (4.17) establish the lower estimate in (4.15).

For the upper bound. Let $L_kf(z) = f(\frac{k}{k+1}z)$ for each positive integer k. Since $C^m_{\varphi,g}$ is bounded and L_k is compact on A^p_α, $C^m_{\varphi,g}L_k: A^p_\alpha\to\mathcal{Z}_\mu$ is also compact. Thus

$$\begin{aligned}\|C^m_{\varphi,g}\|_e &\leq \|C^m_{\varphi,g} - C^m_{\varphi,g}L_k\| \\ &= |(C^m_{\varphi,g}(I-L_k)f)(0)| + |(C^m_{\varphi,g}(I-L_k)f)'(0)| \\ &\quad + \sup_{\|f\|_{p,\alpha}\leq 1}\mu(z)|(C^m_{\varphi,g}(I-L_k)f)''(z)| \\ &= \sup_{\|f\|_{p,\alpha}\leq 1}\mu(z)|(C^m_{\varphi,g}(I-L_k)f)''(z)|, \end{aligned}\tag{4.18}$$

where I denotes the identity operator on A^p_α.
On the other hand, for any $r\in(0,1)$, the last term in (4.18) is equal to the following

$$\sup_{\|f\|_{p,\alpha}\leq 1}\sup_{|\varphi(z)|\leq r}\mu(z)|(C^m_{\varphi,g}(I-L_k)f)''(z)| + \sup_{\|f\|_{p,\alpha}\leq 1}\sup_{|\varphi(z)|> r}\mu(z)|(C^m_{\varphi,g}(I-L_k)f)''(z)|.$$

Morover

$$\begin{aligned}(C^m_{\varphi,g}(I-L_k)f)''(z) &= \left\{f^{(m+1)}(\varphi(z)) - (\frac{k}{k+1})^{m+1}f^{(m+1)}(\frac{k\varphi(z)}{k+1})\right\}\varphi'(z)g(z) \\ &\quad + \left\{f^{(m)}(\varphi(z)) - (\frac{k}{k+1})^{m}f^{(m)}(\frac{k\varphi(z)}{k+1})\right\}g'(z). \end{aligned}\tag{4.19}$$

When $|\varphi(z)|>r$, by (3.12) and (4.19)

$$\sup_{|\varphi(z)|>r}\mu(z)|(C^m_{\varphi,g}(I-L_k)f)''(z)| \leq \sup_{|\varphi(z)|>r}(\frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z_j)|^2)^t} + \frac{\mu(z)|g'(z)|}{(1-|\varphi(z_j)|^2)^{t-1}}). \tag{4.20}$$

Next assume $|\varphi(z)| < r$. In order to calculate conveniently, during the next proof of this theorem, let $\omega = \varphi(z)$. By Lagrange's mean value theorem, there exist $\xi_1(\omega), \xi_2(\omega) \in [\frac{k}{k+1}\omega, \omega]$, such that

$$|f^{(m+1)}(\omega) - f^{(m+1)}(\frac{k\omega}{k+1})| \leq \frac{|\omega|}{k+1}|f^{(m+2)}(\xi_1(\omega))| \leq \frac{(m+2)!}{(k+1)R^{m+2}} \max_{|\zeta|=R+r} |f(\zeta)|, \quad (4.21)$$

$$|f^{(m)}(\omega) - f^{(m)}(\frac{k\omega}{k+1})| \leq \frac{|\omega|}{k+1}|f^{(m+1)}(\xi_2(\omega))| \leq \frac{(m+1)!}{(k+1)R^{m+1}} \max_{|\zeta|=R+r} |f(\zeta)|. \quad (4.22)$$

The last inequalities are applying Cauchy's estimate to $f^{(m+1)}$ and $f^{(m+2)}$ on the circle with center at $\xi_1(\omega)$, $\xi_2(\omega)$ and radius $R \in (0, 1-r)$, using (4.21) and (3.12), we obtain

$$\begin{aligned} & |f^{(m+1)}(\omega) - (\frac{k}{k+1})^{m+1} f^{(m+1)}(\frac{k}{k+1}\omega)| \\ \leq \quad & \frac{r(m+1)!\|f\|_{p,\alpha}}{(k+1)R^{m+1}(1-(R+r)^2)^{\frac{2+\alpha}{p}}} + \frac{(k+1)^{m+1} - k^{m+1}}{(k+1)^{m+1}} \cdot \frac{\|f\|_{p,\alpha}}{\{1-(\frac{kr}{k+1})^2\}^t}. \end{aligned} \quad (4.23)$$

The same argument in the above, we get

$$\begin{aligned} & |f^{(m)}(\omega) - (\frac{k}{k+1})^{m} f^{(m)}(\frac{k}{k+1}\omega)| \\ \leq \quad & \frac{rm!}{(k+1)R^{m}} \cdot \frac{\|f\|_{p,\alpha}}{(1-(R+r)^2)^{\frac{2+\alpha}{p}}} + \frac{(k+1)^{m+1} - k^{m+1}}{(k+1)^{m+1}} \cdot \frac{\|f\|_{p,\alpha}}{\{1-(\frac{kr}{k+1})^2\}^{t-1}}. \end{aligned} \quad (4.24)$$

We have

$$\sup_{\|f\|_{p,\alpha}\leq 1} \sup_{|\varphi(z)|\leq r} \mu(|z|)|(C^m_{\varphi,g}(I-L_k)f)''(z)| \to 0 \ \text{ as } \ k \to \infty. \quad (4.25)$$

By (4.18), (4.20) and (4.25), since $r \in (0,1)$ is arbitrary, we obtain the desired upper bound in (4.15).

According to (i) of Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} & \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\ |\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}} \\ \asymp \quad & \limsup_{n\to\infty} \frac{\|\varphi' g \varphi^n\|_\mu}{\|z^n\|_{\nu_t}} + \limsup_{n\to\infty} \frac{\|g'\varphi^n\|_\mu}{\|z^n\|_{\nu_{t-1}}} \\ \asymp \quad & \limsup_{n\to\infty} (n+1)^t \|\varphi' g\|_\mu + \limsup_{n\to\infty} (n+1)^{t-1}\|g'\|_\mu. \end{aligned}$$

This completes the proof of the theorem. □

5. The compactness of $C^m_{\varphi,g} : A^p_\alpha \to \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$.

Theorem 5.1. *Suppose $p > 0$, $\alpha > -1$, $m \in \mathbb{N}$, and $\|\varphi\|_\infty = 1$, $t = \frac{\alpha+2}{p} + m + 1$. The operator $C^m_{\varphi,g} : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded. Then $C^m_{\varphi,g} : A^p_\alpha \to \mathcal{Z}_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\ |\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t} = 0, \quad (5.26)$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}} = 0. \quad (5.27)$$

Theorem 5.2. *Suppose $p>1$, $\alpha>-1$, $m\in\mathbb{N}$, and $\|\varphi\|_\infty<1$. Then the following conditions are equivalent:*

(a) $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu$ *is bounded.*

(b) $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_\mu$ *is compact.*

(c) $\sup_{z\in\mathbb{D}}\mu(z)|g'(z)|<\infty,\ \sup_{z\in\mathbb{D}}\mu(z)|g(z)\varphi'(z)|<\infty.$

Proof. $(b)\Rightarrow(a)$ is clearly true. By Theorem 3.1, $(a)\Rightarrow(c)$ is obtained. So we only need to prove $(c)\Rightarrow(b)$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a bounded sequence in A^p_α and converge to zero weakly as $j\to\infty$. Since the duality of A^p_α, it verifies that $f_j\to0$ uniformly on compact subsets of $\mathbb{D}$ and so does $\{f_j^{(m)}\}$. Therefore for a fixed $z\in\mathbb{D}$,

$$\begin{aligned}\mu(z)|(C^m_{\varphi,g}f_j)''(z)| &= \mu(|z|)|g(z)\varphi'(z)||f_j^{(m+1)}(\varphi(z))|+\mu(z)|g'(z)||f_j^{(m)}(\varphi(z))|\\ &\le \mu(z)|g(z)\varphi'(z)|\max_{|\omega|\le\|\varphi\|_\infty}|f_j^{(m+1)}(\omega)|+\mu(z)|g'(z)|\max_{|\omega|\le\|\varphi\|_\infty}|f_j^{(m)}(\omega)|,\end{aligned}$$

which implies that $\lim_{j\to\infty}\sup_{z\in\mathbb{D}}\mu(z)|(C^m_{\varphi,g}f_j)''(z)|=0$. Consequently, $\|C^m_{\varphi,g}f_j\|_\infty\to0$ as $j\to\infty$. By lemma 2.1, (b) follows from (c), as desired. ☐

Theorem 5.3. *Suppose $p>1$, $\alpha>-1$, $m\in\mathbb{N}$, and μ is a normal weight on $\mathbb{D}$, $t=\frac{\alpha+2}{p}+m+1$. If $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_{\mu,0}$ is bounded. Then $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to1}\frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}=0,\tag{5.28}$$

$$\lim_{|z|\to1}\frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}}=0.\tag{5.29}$$

Proof. First assume $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_{\mu,0}$ is compact, then the operator is bounded. Hence, by Theorem 3.2, conditions (3.13) and (3.14) are true. Since $\mathcal{Z}_{\mu,0}$ is the subset of $\mathcal{Z}_\mu$, along with Theorem 5.1, we get (5.26) and (5.27).
By (3.13) and (5.26), it follows that, for any $\varepsilon>0$, there exist $\delta,\ r\in(0,1)$

$$\mu(z)|\varphi'(z)g(z)|<\varepsilon,\quad\text{if}\quad |z|\in(\delta,1),$$

$$\frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}<\varepsilon,\quad\text{if}\quad |\varphi(z)|\in(r,1).$$

Therefore, when $\delta<|z|<1$ and $r<|\varphi(z)|<1$, we have that

$$\frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}<\varepsilon.$$

When $\delta<|z|<1$ and $|\varphi(z)|\le r$, we obtain

$$\frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^t}<\frac{\varepsilon}{(1-r^2)^t}.$$

The above two inequalities yields (5.28) easily.
By replacing conditions (3.14) and (5.27) with (3.13) and (5.26) in the above argument, we obtain (5.29).

Conversely, suppose that (5.28) and (5.29) hold. Let $\mathbb{B}_{A^p_\alpha}$ be the unit ball in A^p_α. Applying the boundedness of $C^m_{\varphi,g}:A^p_\alpha\to\mathcal{Z}_{\mu,0}$, $C^m_{\varphi,g}\mathbb{B}_{A^p_\alpha}$ is closed in $\mathcal{Z}_{\mu,0}$. Combining this with (3.12),

(5.28) and (5.29) yield that

$$\begin{aligned}\mu(z)|(C^m_{\varphi,g}f)''(z)| &= \mu(z)|g(z)\varphi'(z)||f^{(m+1)}(\varphi(z))| + \mu(z)|g'(z)||f^{(m)}(\varphi(z))| \\ &\le \frac{C\mu(z)|g(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^t}\|f\|_{p,\alpha} + \frac{C\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{t-1}}\|f\|_{p,\alpha} \\ &\to 0, \quad as \quad |z|\to 1,\end{aligned}$$

for every $f \in \mathbb{B}_{A^p_\alpha}$.

By Lemma 2.2, $C^m_{\varphi,g}\mathbb{B}_{A^p_\alpha}$ is a compact subset in $\mathcal{Z}_{\mu,0}$. Hence $C^m_{\varphi,g} : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ is compact. □

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Five-Dimensional Discrete Green's Function and Its Estimates

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In this article we first introduce definitions of the regularized Green's function, the discrete Green's function, the discrete δ function, and the L^2 projection operator in five dimensions. Then the $W^{2,1}$-norm estimate for the regularized Green's function is derived. Finally, we obtain the $W^{2,1}$-seminorm estimate for the discrete Green's function.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence of the finite element approximation (see [1–5]). Especially, the estimates for the Green's function are usually used to bound the maximum-norm error of the finite element approximation. For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [5]). For three-dimensional elliptic problems, we also obtained the optimal estimates for the discrete derivative Green's function and the discrete Green's function (see [6, 7]). Recently, we derived the $W^{1,1}$-seminorm estimate with order $O(|\ln h|^{\frac{7}{5}})$ for the discrete derivative Green's function for the 5D Poisson equation (see [8]). This article will discuss the $W^{2,1}$-seminorm estimate for the discrete Green's function.

In this article, we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

The model problem that we study in this paper is

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

where $\Omega \subset \mathcal{R}^5$ is a bounded polytopic domain. Let $\{\mathcal{T}^h\}$ be a regular family of uniform partitions of $\bar{\Omega}$, and $S_0^h(\Omega) \subset H_0^1(\Omega)$ the linear finite element space. Discretizing the above elliptic equation using S_0^h as approximating space means

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finding $u_h \in S_0^h$ such that $a(u_h\,,\,v) = (f\,,\,v)$ for all $v \in S_0^h$, where

$$a(u_h\,,\,v) \equiv \int_\Omega \nabla u_h \cdot \nabla v\, dX,$$

and

$$(f\,,\,v) \equiv \int_\Omega f v\, dX.$$

In addition, we also assume that the given function f is smooth enough.

For every $Z \in \Omega$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that (see [7] for the three-dimensional setting)

$$(v,\,\delta_Z^h) = v(Z) \quad \forall\, v \in S_0^h(\Omega), \tag{1.2}$$

$$(u - P_h u,\, v) = 0 \quad \forall\, v \in S_0^h(\Omega). \tag{1.3}$$

Let $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the elliptic problem $-\Delta G_Z^* = \delta_Z^h$. We may call G_Z^* the regularized Green's function. Further, let the discrete Green's function $G_Z^h \in S_0^h(\Omega)$ be the finite element approximation to G_Z^*. Thus,

$$a(G_Z^* - G_Z^h, v) = 0 \quad \forall\, v \in S_0^h(\Omega). \tag{1.4}$$

The purpose of this article is to bound the term $\left|G_Z^h\right|_{2,\,1,\,\Omega}^h$.

The rest of this article is organized as follows. In Section 2 we derive the $W^{2,1}$-norm optimal estimate for G_Z^*. Section 3 is devoted to the estimate for G_Z^h.

2 Weight Function and Estimates for G_Z^*

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2\right)^{-\frac{5}{2}} \quad \forall\, X \in \bar{\Omega},$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [5, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} \left|D^\beta v\right|^2, |\nabla^n v|_{\phi^\alpha,\,\Omega} = \left(\int_\Omega \phi^\alpha\, |\nabla^n v|^2\, dX\right)^{\frac{1}{2}},$$

$$\|v\|_{m,\,\phi^\alpha,\,\Omega}^2 = \sum_{n=0}^{m} |\nabla^n v|_{\phi^\alpha,\,\Omega}^2\,,$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, $|\beta| = \beta_1+\beta_2+\beta_3+\beta_4+\beta_5$, and $\beta_i \geq 0,\ i = 1, \cdots, 5$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha,\,\Omega} = \left(\int_\Omega \phi^\alpha |v|^2\, dX\right)^{\frac{1}{2}}.$$

We assume that there exists an q_0 $(1 < q_0 \leq \infty)$ such that

$$-\Delta : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \longrightarrow L^q(\Omega)\ (1 < q < q_0)$$

is a homeomorphism (see [5]). Thus, for $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, we have the so-called a priori estimate:

$$\|v\|_{2,q,\Omega} \leq C(q)\| - \Delta v\|_{0,q,\Omega}, \tag{2.2}$$

where $C(q)$ denotes a constant depending on q. The weight function ϕ possesses the following properties.

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n)\phi^{\alpha+\frac{n}{5}},\ \alpha \in \mathcal{R},\ n = 1, 2. \tag{2.3}$$

$$\int_\Omega \phi^\alpha \, dX \leq C(\alpha - 1)^{-1}\theta^{-5(\alpha-1)}\ \forall\, \alpha > 1, \tag{2.4}$$

$$\int_\Omega \phi \, dX \leq C(k)|\ln\theta|,\ \theta \leq k < 1. \tag{2.5}$$

For L^2-projection operator P_h, we have

$$\|P_h w\|_{1,q,\Omega} \leq C(q)\|w\|_{1,q,\Omega} \quad \text{for } 1 \leq q \leq \infty. \tag{2.6}$$

As for δ_Z^h, we have the following estimate.

Lemma 2.1. *For δ_Z^h the discrete δ function defined by (1.2), we have the weighted-norm estimate*

$$\left\|\delta_Z^h\right\|_{\phi^{-\alpha}} \leq Ch^{\frac{5(\alpha-1)}{2}} \quad \forall\, \alpha > 0. \tag{2.7}$$

Remark 1. The proof of the result (2.7) is similar to that of (2.27) in [7].

Lemma 2.2. *For $q_0 > 2$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a constant $C = C(\alpha, \Omega) > 0$ such that*

$$\left\|\nabla^2 v\right\|_{\phi^{-\alpha}}^2 \leq C\left(\|\Delta v\|_{\phi^{-\alpha}}^2 + |v|_{1,\phi^{-\alpha+\frac{2}{5}}}^2 + \|v\|_{\phi^{-\alpha+\frac{4}{5}}}^2\right)\ \forall\, \alpha \in \mathcal{R}. \tag{2.8}$$

Proof.

$$\begin{aligned}
&\left\|\nabla^2 v\right\|_{\phi^{-\alpha}}^2 = \int_\Omega \phi^{-\alpha}\left|\nabla^2 v\right|^2 dx = \int_\Omega \left(\phi^{-\frac{\alpha}{2}}\left|\nabla^2 v\right|\right)^2 dx \\
\leq\ & C\left(\int_\Omega \left|\nabla^2\left(\phi^{-\frac{\alpha}{2}} v\right)\right|^2 dx + \int_\Omega \left|v\nabla^2\phi^{-\frac{\alpha}{2}}\right|^2 dx + \int_\Omega \left|\nabla\phi^{-\frac{\alpha}{2}}\right|^2 |\nabla v|^2 dx\right) \\
\leq\ & C\left(\left\|\nabla^2\left(\phi^{-\frac{\alpha}{2}} v\right)\right\|_0^2 + \|v\|_{\phi^{-\alpha+\frac{4}{5}}}^2 + |v|_{1,\phi^{-\alpha+\frac{2}{5}}}^2\right) \\
\leq\ & C\left(\left\|\Delta\left(\phi^{-\frac{\alpha}{2}} v\right)\right\|_0^2 + \|v\|_{\phi^{-\alpha+\frac{4}{5}}}^2 + |v|_{1,\phi^{-\alpha+\frac{2}{5}}}^2\right) \\
\leq\ & C\left(\int_\Omega \phi^{-\alpha}|\Delta v|^2 dx + \int_\Omega \left|\nabla\phi^{-\frac{\alpha}{2}}\right|^2 |\nabla v|^2 dx\right. \\
&\left. + \int_\Omega \left|\Delta\phi^{-\frac{\alpha}{2}}\right|^2 |v|^2 dx + \|v\|_{\phi^{-\alpha+\frac{4}{5}}}^2 + |v|_{1,\phi^{-\alpha+\frac{2}{5}}}^2\right) \\
\leq\ & C\left(\|\Delta v\|_{\phi^{-\alpha}}^2 + |v|_{1,\phi^{-\alpha+\frac{2}{5}}}^2 + \|v\|_{\phi^{-\alpha+\frac{4}{5}}}^2\right).
\end{aligned}$$

The proof of the result (2.8) is completed.

Lemma 2.3. *Suppose $q_0 > 2$ and $\alpha \in \mathcal{R}$. For G_Z^* and δ_Z^h, the regularized Green's function and the discrete δ function, respectively, we have*

$$\left\|\nabla^2 G_Z^*\right\|_{\phi^{-\alpha}}^2 \le C\left\|\delta_Z^h\right\|_{\phi^{-\alpha}}^2 + C\left\|G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}}^2 . \tag{2.9}$$

Proof. Taking $v = G_Z^*$ in (2.8), and by the definition of G_Z^*, we have

$$\begin{aligned}
\left\|\nabla^2 G_z^*\right\|_{\phi^{-\alpha}}^2 &\le C\left(\|\Delta G_z^*\|_{\phi^{-\alpha}}^2 + |G_z^*|_{1,\,\phi^{-\alpha+\frac{2}{5}}}^2 + \|G_z^*\|_{\phi^{-\alpha+\frac{4}{5}}}^2\right)\\
&\le C\left\|\delta_z^h\right\|_{\phi^{-\alpha}}^2 + C\left|a\left(G_z^*, \phi^{-\alpha+\frac{2}{5}}G_z^*\right)\right| + C\left\|G_z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}}^2\\
&\le C\left\|\delta_z^h\right\|_{\phi^{-\alpha}}^2 + C\left|\left(\delta_z^h, \phi^{-\alpha+\frac{2}{5}}G_z^*\right)\right| + C\left\|G_z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}}^2\\
&\le C\left\|\delta_z^h\right\|_{\phi^{-\alpha}}^2 + C\left\|\delta_z^h\right\|_{\phi^{-\alpha}}\left\|G_z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}} + C\left\|G_z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}}^2\\
&\le C\left\|\delta_z^h\right\|_{\phi^{-\alpha}}^2 + C\left\|G_z^*\right\|_{\phi^{-\alpha+\frac{4}{5}}}^2 ,
\end{aligned}$$

which completes the proof of the result (2.9).

Lemma 2.4. *Suppose $q_0 > \frac{5}{2}$. For G_Z^* the regularized Green's function, we have*

$$\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}}^2 \le C\left|\ln h\right|^{\frac{11}{5}} + C\left|\ln h\right|^{\frac{4}{5}}\left|G_Z^*\right|_{1,\frac{5}{4}} . \tag{2.10}$$

Proof. Consider the problem

$$\begin{cases} -\Delta w = \phi^{-\frac{1}{5}}G_Z^*, & \text{in } \Omega\\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

By the stability estimate (2.6), a priori estimate (2.2), and the Sobolev Embedding Theorem [9],

$$\begin{aligned}
&\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}}^2 = (\phi^{-\frac{1}{5}}G_Z^*\,,\,G_Z^*) = a(w\,,\,G_Z^*) = (\delta_Z^h\,,\,w) = P_h w(Z)\\
\le\;& |P_h w|_{0,\,\infty} \le C\left|\ln h\right|^{\frac{4}{5}}|P_h w|_{1,\,5} \le C\left|\ln h\right|^{\frac{4}{5}}|w|_{1,\,5}\\
\le\;& C\left|\ln h\right|^{\frac{4}{5}}\|w\|_{2,\,\frac{5}{2}} \le C\left|\ln h\right|^{\frac{4}{5}}\left\|\phi^{-\frac{1}{5}}G_Z^*\right\|_{0,\,\frac{5}{2}}\\
\le\;& C\left|\ln h\right|^{\frac{4}{5}}\left\|\phi^{-\frac{1}{5}}G_Z^*\right\|_{2,\,\frac{5}{4}} \le C\left|\ln h\right|^{\frac{4}{5}}\left\|-\Delta(\phi^{-\frac{1}{5}}G_Z^*)\right\|_{0,\,\frac{5}{4}}\\
\le\;& C\left|\ln h\right|^{\frac{4}{5}}\left(\left\|\phi^{-\frac{1}{5}}\delta_Z^h\right\|_{0,\,\frac{5}{4}} + \|\nabla G_Z^*\|_{0,\frac{5}{4}} + \left\|\phi^{\frac{1}{5}}G_Z^*\right\|_{0,\,\frac{5}{4}}\right).
\end{aligned} \tag{2.11}$$

In addition,

$$\left\|\phi^{\frac{1}{5}}G_Z^*\right\|_{0,\,\frac{5}{4}} \le \left(\int_\Omega \phi\,dX\right)^{\frac{3}{10}}\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}} \le C\left|\ln\theta\right|^{\frac{3}{10}}\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}} . \tag{2.12}$$

Combining (2.11), (2.12), and the *Young* inequality yields

$$\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}}^2 \le C\left|\ln h\right|^{\frac{4}{5}} + C\left|\ln h\right|^{\frac{4}{5}}\|\nabla G_Z^*\|_{0,\frac{5}{4}} + C\varepsilon\left\|G_Z^*\right\|_{\phi^{-\frac{1}{5}}}^2 + C(\varepsilon)\left|\ln h\right|^{\frac{11}{5}} . \tag{2.13}$$

Choosing a suitable ε in (2.13), we immediately obtain the result (2.10).

Remark 2. To derive (2.13), we used the fact that $\left\|\phi^{-\frac{1}{5}}\delta_Z^h\right\|_{0,\frac{5}{4}} \leq C$.

Lemma 2.5. *Suppose $q_0 > \frac{5}{2}$. For G_Z^* the regularized Green's function, we have*

$$\|G_Z^*\|_{2,1} \leq C\,|\ln h|^{\frac{9}{5}}. \tag{2.14}$$

Proof. Obviously,

$$\|G_Z^*\|_{2,1}^2 \leq \int_\Omega \phi\, dX \cdot \left\|\nabla^2 G_Z^*\right\|_{\phi^{-1}}^2. \tag{2.15}$$

Taking $\alpha = 1$ in (2.7) and (2.9), and combining (2.5), (2.7), (2.9), and (2.15) yield

$$\|G_Z^*\|_{2,1}^2 \leq C\,|\ln\theta| + C\,|\ln\theta|\,\|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2. \tag{2.16}$$

From (2.10) and (2.16),

$$\|G_Z^*\|_{2,1}^2 \leq C\,|\ln h|^{\frac{16}{5}} + C\,|\ln h|^{\frac{9}{5}}\,|G_Z^*|_{1,\frac{5}{4}}. \tag{2.17}$$

By the Sobolev Embedding Theorem [9] and the *Young* inequality,

$$\begin{aligned} \|G_Z^*\|_{2,1}^2 &\leq C\,|\ln h|^{\frac{16}{5}} + C\,|\ln h|^{\frac{9}{5}}\,\|G_Z^*\|_{2,1} \\ &\leq C\,|\ln h|^{\frac{16}{5}} + C(\varepsilon)\,|\ln h|^{\frac{18}{5}} + C\varepsilon\,\|G_Z^*\|_{2,1}^2. \end{aligned} \tag{2.18}$$

Taking a suitable ε in (2.18) yields the result (2.14).

3 Finite Element Approximation to G_Z^* and Its Estimates

In this section, we will bound the discrete Green's function, G_Z^h, the finite element approximation to G_Z^*.

Lemma 3.1. *For the discrete Green's function $G_Z^h \in S_0^h(\Omega)$, the finite element approximation to $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-1}}^2 \leq Ch^2\left|\nabla^2 G_Z^*\right|_{\phi^{-1}}^2 + C\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2. \tag{3.1}$$

Remark 3. The proof of the result (3.1) is similar to that of (2.43) in [7].

Lemma 3.2. *For G_Z^* and G_Z^h, the regularized Green's function and the discrete Green's function, respectively, we have the following estimate:*

$$\left|G_Z^* - G_Z^h\right|_{1,1} \leq Ch|\ln h|^{\frac{9}{5}}. \tag{3.2}$$

Proof. Obviously,

$$\left|G_Z^* - G_Z^h\right|_{1,1}^2 \leq \int_\Omega \phi\, dX \cdot \left|G_Z^* - G_Z^h\right|_{1,\phi^{-1}}^2. \tag{3.3}$$

From (3.1),

$$\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 \le Ch^2 \left|\nabla^2 G_Z^*\right|_{\phi^{-1}}^2 + C\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2. \tag{3.4}$$

By the *Young* inequality and the interpolation error estimate with the weighted-norm, we have

$$\begin{aligned}\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2 &= (\phi^{-\frac{3}{5}}(G_Z^* - G_Z^h),\, G_Z^* - G_Z^h) = a(v,\, G_Z^* - G_Z^h)\\ &= a(v - \Pi v,\, G_Z^* - G_Z^h) \le \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}} \cdot |v - \Pi v|_{1,\,\phi}\\ &\le \varepsilon \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 + C(\varepsilon)\,|v - \Pi v|_{1,\,\phi}^2\\ &\le \varepsilon \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 + C(\varepsilon)h^2 \left|\nabla^2 v\right|_{\phi}^2,\end{aligned} \tag{3.5}$$

where $-\Delta v = \phi^{-\frac{3}{5}}(G_Z^* - G_Z^h)$ and Π is a linear interpolation operator. Similar to (2.44) in [8], we have

$$\begin{aligned}&\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2 \le \varepsilon \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2\\ &+C(\varepsilon)\gamma^{-2}\left(h^2 + \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 + \left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2\right).\end{aligned} \tag{3.6}$$

For a fixed $\varepsilon \in (0,1)$, choosing $\gamma \in [5, +\infty)$ in (3.6) such that $0 < C(\varepsilon)\gamma^{-2} < \min(\varepsilon, \frac{1}{2})$, we then have

$$\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{3}{5}}}^2 \le 4\varepsilon \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 + h^2. \tag{3.7}$$

From (3.4) and (3.7),

$$\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 \le Ch^2 \left|\nabla^2 G_Z^*\right|_{\phi^{-1}}^2 + 4\varepsilon C \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 + Ch^2. \tag{3.8}$$

Taking a suitable $\varepsilon \in (0,1)$ in (3.8) such that $4\varepsilon C < 1$, we have

$$\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 \le Ch^2 \left|\nabla^2 G_Z^*\right|_{\phi^{-1}}^2 + Ch^2. \tag{3.9}$$

In addition,

$$\begin{aligned}&\left|\nabla^2 G_Z^*\right|_{\phi^{-1}}^2 = \int_\Omega \phi^{-1}\left|\nabla^2 G_Z^*\right|^2 dX = \int_\Omega \left(\phi^{-\frac{1}{2}}\left|\nabla^2 G_Z^*\right|\right)^2 dX\\ \le\ & C\left(\int_\Omega \left|\nabla^2(\phi^{-\frac{1}{2}} G_Z^*)\right|^2 dX + \int_\Omega \left|G_Z^* \nabla^2 \phi^{-\frac{1}{2}}\right|^2 dX + \int_\Omega \left|\nabla \phi^{-\frac{1}{2}} \cdot \nabla G_Z^*\right|^2 dX\right)\\ \le\ & C\left(\left\|\nabla^2(\phi^{-\frac{1}{2}} G_Z^*)\right\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{3}{5}}}^2\right)\\ \le\ & C\left(\left\|-\Delta(\phi^{-\frac{1}{2}} G_Z^*)\right\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{3}{5}}}^2\right)\\ \le\ & C\left(\|\Delta G_Z^*\|_{\phi^{-1}}^2 + \|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{3}{5}}}^2\right)\\ \le\ & C\left(\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + \|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + a(G_Z^*,\, \phi^{-\frac{3}{5}} G_Z^*)\right)\\ =\ & C\left(\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + \|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + (\delta_Z^h,\, \phi^{-\frac{3}{5}} G_Z^*)\right)\\ \le\ & C\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + C\|G_Z^*\|_{\phi^{-\frac{1}{5}}}^2,\end{aligned}$$

namely,

$$\left|\nabla^2 G_Z^*\right|^2_{\phi^{-1}} \leq C \left\|\delta_Z^h\right\|^2_{\phi^{-1}} + C \left\|G_Z^*\right\|^2_{\phi^{-\frac{1}{5}}}. \tag{3.10}$$

From (2.10), (2.14), and the Sobolev Embedding Theorem [9],

$$\begin{aligned} \|G_Z^*\|^2_{\phi^{-\frac{1}{5}}} &\leq C\,|\ln h|^{\frac{11}{5}} + C\,|\ln h|^{\frac{4}{5}}\,|G_Z^*|_{1,\frac{5}{4}} \\ &\leq C\,|\ln h|^{\frac{11}{5}} + C\,|\ln h|^{\frac{4}{5}}\,\|G_Z^*\|_{2,1} \\ &\leq C\,|\ln h|^{\frac{13}{5}}. \end{aligned} \tag{3.11}$$

Taking $\alpha = 1$ in (2.7), we have

$$\left\|\delta_Z^h\right\|^2_{\phi^{-1}} \leq C. \tag{3.12}$$

Combining (2.5), (3.3), and (3.9)-(3.12) yields the desired result (3.2).

Theorem 3.1. *For G_Z^h the discrete Green's function, we have the following estimate:*

$$\left|G_Z^h\right|^h_{2,\,1} \leq C|\ln h|^{\frac{9}{5}}. \tag{3.13}$$

Proof. Obviously, by the triangle inequality, the inverse inequality and the interpolation error estimate, we have

$$\begin{aligned} \left|G_Z^* - G_Z^h\right|^h_{2,\,1} &\leq |G_Z^* - \Pi G_Z^*|^h_{2,\,1} + \left|\Pi G_Z^* - G_Z^h\right|^h_{2,\,1} \\ &\leq C\,\|G_Z^*\|_{2,\,1} + Ch^{-1}\left|\Pi G_Z^* - G_Z^h\right|_{1,\,1} \\ &\leq C\,\|G_Z^*\|_{2,\,1} + Ch^{-1}\,|\Pi G_Z^* - G_Z^*|_{1,\,1} + Ch^{-1}\left|G_Z^* - G_Z^h\right|_{1,\,1} \\ &\leq C\,\|G_Z^*\|_{2,\,1} + Ch^{-1}\left|G_Z^* - G_Z^h\right|_{1,\,1}, \end{aligned}$$

where Π is a linear interpolation operator. By the triangle inequality,

$$\left|G_Z^h\right|^h_{2,\,1} \leq \left|G_Z^* - G_Z^h\right|^h_{2,\,1} + \|G_Z^*\|_{2,\,1} \leq C\,\|G_Z^*\|_{2,\,1} + Ch^{-1}\left|G_Z^* - G_Z^h\right|_{1,\,1}. \tag{3.14}$$

Combining (2.14), (3.2), and (3.14) yields the result (3.13).

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On the geometric convergence of the Arrow-Hurwicz algorithm for steady incompressible Navier-Stokes equations [1]

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Abstract

In the seminal monograph [11], Temam proposed an Arrow-Hurwicz algorithm for solving steady incompressible Navier-Stokes equations and proved its convergence. However, the convergence rate analysis remains open until now. Using some techniques in [8] combined with a very tricky analysis, we prove in this paper that the previous method converges geometrically. This result is important in theory, showing a fundamental property of the Arrow-Hurwicz algorithm. Moreover, it motivates us to devise efficient algorithms for numerically solving steady incompressible Navier-Stokes equations.

Keywords. Navier-Stokes equations, Arrow-Hurwicz algorithm, convergence rate analysis

1 Introduction

Consider the steady incompressible Navier-Stokes equations:

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in}\,\Omega, \\ \nabla\cdot\boldsymbol{u} = 0 \quad \text{in}\,\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ (dimension $d = 2$ or 3) is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu > 0$ is the viscosity coefficient of the flow under consideration, and the $\boldsymbol{f}$ prescribes body force; $\boldsymbol{u}$ and p are the corresponding velocity field and pressure field, respectively. The above equations are the well-known mathematical model describing the steady flow of an incompressible Newtonian fluid, such as air or water (cf. [6, 9, 11]). To simplify the presentation, we impose the non-slip condition for the equations (1.1),

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on}\,\partial\Omega. \tag{1.2}$$

For the uniqueness of p we assume that

$$p \in L_0^2(\Omega) := \{q \in L^2(\Omega);\ \int_\Omega q dx = 0\}. \tag{1.3}$$

Next, we introduce some notation for later uses. Let $(\cdot,\cdot)$ denote the scalar product over $L^2(\Omega)$. Given a non-negative integer s, let $H^s(\Omega)$ be the usual Sobolev space consisting of all functions $v \in L^2(\Omega)$ whose generalized derivatives with the total degree no more than s are still $L^2(\Omega)$-integrable. We equip $H^s(\Omega)$ with the standard norm $\|\cdot\|_s$ and seminorm $|\cdot|_s$ (cf. [1]). The closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_s$ is denoted by $H_0^s(\Omega)$. The dual of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. Let $\boldsymbol{H}^s(\Omega)$ be the product space $(H^s(\Omega))^d$, whose induced norm, seminorm, and scalar product are expressed with the same symbols over

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$H^s(\Omega)$, when there is no confusion caused. The similar conventions also apply to $H_0^s(\Omega)$ and $H^{-s}(\Omega)$.

Furthermore, write $\boldsymbol{V} := \boldsymbol{H}_0^1(\Omega)$, $P := L_0^2(\Omega)$, and let $\boldsymbol{V}^3$ denote the product space $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$. Define over $\boldsymbol{V}^3$ two trilinear forms as follows:

$$a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} dx,$$

$$N(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) - \frac{1}{2} a_1(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}).$$

For $\boldsymbol{u} \in \boldsymbol{V}$, $\boldsymbol{v} \in \boldsymbol{V}$, and $\boldsymbol{w} \in \boldsymbol{V}$, since $\operatorname{div} \boldsymbol{u} = 0$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ vanish on $\partial\Omega$, we can easily deduce that $a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) = -a_1(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})$ by integration by parts. Hence, the above two trilinear forms are identical over $\boldsymbol{V}^3$, and $N(\cdot; \cdot, \cdot)$ can be viewed as the anti-symmetrization of $a_1(\cdot; \cdot, \cdot)$. Then the variational formulation of problem (1.1)-(1.3) can be described as follows (cf. [7, 9, 11]):

Problem Q. Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ such that

$$\begin{cases} N(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, & (1.4) \\ (\operatorname{div} \boldsymbol{u}, q) = 0 \quad \forall\, p \in P, & (1.5) \end{cases}$$

where $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $\langle \cdot, \cdot \rangle$ stands for the bilinear form between the dual pair $\boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$.

As shown in [7, 9, 11], there exists a positive number $\mathcal{N}$ such that

$$a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) \le \mathcal{N} |\boldsymbol{u}|_1 |\boldsymbol{v}|_1 |\boldsymbol{w}|_1 \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \tag{1.6}$$

and there exits a generic constant $\beta > 0$ such that

$$\inf_{q \in P} \sup_{\boldsymbol{v} \in V} \frac{(\operatorname{div} \boldsymbol{v}, q)}{|\boldsymbol{v}|_1 \|q\|_0} \ge \beta. \tag{1.7}$$

Throughout this paper, define

$$\Lambda = \nu^{-2} \mathcal{N} \|\boldsymbol{f}\|_{-1}, \tag{1.8}$$

where

$$\|\boldsymbol{f}\|_{-1} := \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle}{|\boldsymbol{v}|_1}.$$

Then, it is well-known that problem Q has a solution for any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, and the solution is unique whenever $\Lambda < 1$. We refer the reader to the references [7, 9, 11] for details about these results.

The problem (1.4)-(1.5) is a nonlinear saddle-point system, so it is very challenging to get the solution efficiently (cf. [6]). In the seminal monograph [11], following the ideas in [2], Temam proposed an iterative method, called the Arrow-Hurwicz algorithm, for solving problem (1.1) (or equivalently, problem (1.4)-(1.5)). The key advantage of the method is that the original system is decoupled, and at each iteration step it is only required to solve an equation involving the unknown velocity field. Moreover, the convergence of the method was proved in the previous monograph; but no convergence rate analysis was investigated. As commented in [10], the analysis in [11] did not allows us to estimate the convergence rate of the Arrow-Hurwicz algorithm. We mention in passing that some algorithms of the Arrow-Hurwicz type were also given and thoroughly discussed in [3, 5, 10] for finite dimensional linear saddle-point systems.

2

In this paper, we are concerned with the convergence rate analysis of the Arrow-Hurwicz algorithm in [11]. We first show the iteration sequence generated by the algorithm is bounded. Then, using some techniques in [8] combined with a very tricky analysis, we prove that the algorithm is convergent geometrically. The contribution of our result is twofold. First, it is important in theory, showing a fundamental property of the Arrow-Hurwicz algorithm. Second, it motivates us to devise efficient algorithms for numerically solving steady incompressible Navier-Stokes equations. We remark that using the similar arguments we can also prove an Uzawa algorithm proposed in [11] converges geometrically. The discrete analogue and the modified algorithm of the Uzawa algorithm were devised in [4] to solve steady incompressible Navier-Stokes equation discretized by mixed element methods.

2 The Arrow-Hurwicz algorithm and some basic results

The Arrow-Hurwicz algorithm was first proposed in [2] for solving constrained optimization problems based on the method of (discrete) gradient flow. For the quadratic optimization problem with the affine constraint, by using the method of Lagrange multipliers, the solution (including the primal variable and dual variable) satisfies the following linear saddle-point system:

$$\begin{cases} Au + Bp = f, \\ B^T u = g. \end{cases}$$

Then the Arrow-Hurwicz algorithm in this case can be described as follows. Given any initial guess u^0 and p^0, for $k = 0, 1, 2, \cdots$, when u^k and p^k are available, we get u^{k+1} and p^{k+1} by solving the following equations:

$$\begin{cases} J\dfrac{u^{k+1} - u^k}{\omega} + Au^k + Bp^k = f, \\ -K\dfrac{p^{k+1} - p^k}{\tau} + B^T u^{k+1} = g, \end{cases}$$

where J and K denote the preconditioned operators, ω and τ are iterative parameters. From the computational point of view, the Arrow-Hurwicz algorithm can be viewed as a variant of the following Uzawa method

$$\begin{cases} Au^{k+1} + Bp^k = f, \\ p^{k+1} = p^k + \alpha(B^T u^{k+1} - g), \end{cases}$$

with the subproblem $Aw = d$ solved inexactly. We refer the reader to [10] for details along this line. The other point to be emphasized is that it is very challenging and problem-oriented to construct feasible preconditioners J and K to increase computational efficiency. Some discussions were given in [3, 5, 10].

Following the ideas in [2], Temam proposed in [11] an Arrow-Hurwicz algorithm for solving steady incompressible Navier-Stokes equations, described as Algorithm 1.

Under the assumptions that

$$\nu^* := \nu - 2\nu^{-1}\mathcal{N}\|\boldsymbol{f}\|_{-1} - 4\nu^{-2}\mathcal{N}^2\|\boldsymbol{f}\|_{-1}^2 > 0 \tag{2.1}$$

and

$$0 < \rho < \frac{\alpha\nu^*}{2(1 + \nu^2\alpha)},$$

3

Temam proved in [11] that $\boldsymbol{u}^n$ converges weakly to $\boldsymbol{u}$ in $\boldsymbol{V}$, and p^n converges weakly to p in P as $n \to \infty$, where $(\boldsymbol{u}, p)$ is the unique solution of (1.1).

According to the notation mentioned in the last section, condition (2.1) can be rewritten as $\nu - 2\nu\Lambda - 4\nu^2\Lambda^2 > 0$, which implies

$$0 < \Lambda < (1 + \sqrt{1 + 4\nu})^{-1} < 1/2.$$

That means only if $0 < \Lambda < (1 + \sqrt{1 + 4\nu})^{-1}$, the above result can ensure convergence of Algorithm 1. In this paper, we will develop the convergence rate analysis of Algorithm 1. As a direct consequence, we will find Algorithm 1 is convergent for $0 \leq \Lambda < 1$ as long as the parameters ρ and α are well chosen.

Algorithm 1 The Arrow-Hurwicz algorithm.

Construct a sequence of couples $(\boldsymbol{u}^n, p^n)$ as follows.
Start with arbitrary $u^0 \in \boldsymbol{V}$ and $p^0 \in P$, when $\boldsymbol{u}^n$ and p^n are available, derive $\boldsymbol{u}^{n+1} \in \boldsymbol{V}$ and $p^{n+1} \in P$ by solving

$$\begin{aligned}(\nabla(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n), \nabla\boldsymbol{v}) + \rho\nu(\nabla\boldsymbol{u}^n, \nabla\boldsymbol{v}) + \rho N(\boldsymbol{u}^n; \boldsymbol{u}^{n+1}, \boldsymbol{v}) - \rho(p^n, \operatorname{div}\boldsymbol{v})& \\ = \rho\langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},&\end{aligned} \tag{2.2}$$

$$\alpha(p^{n+1} - p^n, q) + \rho(\operatorname{div}\boldsymbol{u}^{n+1}, q) = 0 \quad \forall\, q \in P, \tag{2.3}$$

where ρ and α are two positive numbers.

For later uses, we present some basic and useful results in the following lemma.

Lemma 2.1 *The following statements hold:*

1. *For all* $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, $\|\operatorname{div}\boldsymbol{v}\|_0 \leq \|\nabla\boldsymbol{v}\|_0$. *And* $\beta \in (0, 1]$, *where* β *is the generic constant given in* (1.7).

2. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ *be a solution of problem* Q, *if* $\Lambda < 1$ *and inf-sup condition holds, then*
$$|\boldsymbol{u}|_1 \leq \nu^{-1}\|\boldsymbol{f}\|_{-1}, \quad \beta\|p\|_0 < 3\|\boldsymbol{f}\|_{-1} \tag{2.4}$$

Proof. The first statement is due to [8]. The first estimate in (2.4) is well known (cf. [7,11]), and let us prove the second estimate of (2.4). It follows from (1.4) and (1.6) that

$$(p, \operatorname{div}\boldsymbol{v}) = N(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle \leq \mathcal{N}|\boldsymbol{u}|_1^2|v|_1 + \nu|\boldsymbol{u}|_1|\boldsymbol{v}|_1 + \|\boldsymbol{f}\|_{-1}|\boldsymbol{v}|_1,$$

which, in conjunction with the inf-sup condition (1.7) and the estimate for $\boldsymbol{u}$, gives

$$\beta\|p\|_0 \leq \sup_{\boldsymbol{v}\in V} \frac{(p, \operatorname{div}\boldsymbol{v})}{|\boldsymbol{v}|_1} \leq \mathcal{N}|\boldsymbol{u}|_1^2 + \nu|\boldsymbol{u}|_1 + \|\boldsymbol{f}\|_{-1} \leq \Lambda\|\boldsymbol{f}\|_{-1} + 2\|\boldsymbol{f}\|_{-1} < 3\|\boldsymbol{f}\|_{-1},$$

as required. ■

3 Convergence rate analysis

Define
$$\boldsymbol{E}^n = \boldsymbol{u} - \boldsymbol{u}^n, \quad e^n = p - p^n, \tag{3.1}$$

which represent the errors between the solution of problem Q and the iteration sequence generated by Algorithm 1.

Lemma 3.1 *Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ be the unique solution of problem Q. Let $(\boldsymbol{u}^n, p^n)$ be the function sequence generated by Algorithm 1. If the parameter Λ given in (1.8) is less than 1 and the parameters ρ and α in Algorithm 1 satisfy*

$$|1-\rho\nu| + \rho\nu\Lambda + \frac{\rho^2}{2\alpha} < 1,$$

then the sequences $\{|\boldsymbol{u}^n|_1\}$ and $\{\|p^n\|_0\}$ are bounded.

Proof. Because of (2.4), it suffices to prove the boundedness of the sequences $\{|\boldsymbol{E}^n|_1\}$ and $\{\|e^n\|_0\}$.

For this, recalling the definitions of $\boldsymbol{E}^n$ and e^n (cf. (3.1)), multiplying (1.4) by ρ and taking subtraction from (2.2), we get

$$(\nabla(\boldsymbol{E}^{n+1} - \boldsymbol{E}^n), \nabla \boldsymbol{v}) + \rho\nu(\nabla \boldsymbol{E}^n, \nabla \boldsymbol{v}) - \rho(e^n, \operatorname{div} \boldsymbol{v}) = -\rho N(\boldsymbol{E}^n; \boldsymbol{u}, \boldsymbol{v}) - \rho N(\boldsymbol{u}^n; \boldsymbol{E}^{n+1}, \boldsymbol{v}). \tag{3.2}$$

Choosing $\boldsymbol{v} = \boldsymbol{E}^{n+1}$ in the above equation and noting that $N(\boldsymbol{u}^n; \boldsymbol{E}^{n+1}, \boldsymbol{E}^{n+1}) = 0$, we have

$$|\boldsymbol{E}^{n+1}|_1^2 - \rho(e^n, \operatorname{div} \boldsymbol{E}^{n+1}) = (1-\rho\nu)(\nabla \boldsymbol{E}^n, \nabla \boldsymbol{E}^{n+1}) - \rho N(\boldsymbol{E}^n; \boldsymbol{u}, \boldsymbol{E}^{n+1}). \tag{3.3}$$

On the other hand, it follows from (1.5) and (2.3) that

$$-\rho(e^n, \operatorname{div} \boldsymbol{E}^{n+1}) = \rho(e^n, \operatorname{div} \boldsymbol{u}^{n+1}) = \alpha(e^{n+1} - e^n, e^n) = \frac{\alpha}{2}(\|e^{n+1}\|_0^2 - \|e^n\|_0^2 - \|e^{n+1} - e^n\|_0^2). \tag{3.4}$$

Now, we infer that

$$\|e^{n+1} - e^n\|_0 \le \rho\alpha^{-1}|\boldsymbol{E}^{n+1}|_1. \tag{3.5}$$

As a matter of fact, it follows from (1.5) and (2.3) that

$$(e^{n+1} - e^n, q) = -\rho\alpha^{-1}(\operatorname{div}\boldsymbol{E}^{n+1}, q).$$

Choosing $q = e^{n+1} - e^n$ in the above equation, and then using the Cauchy-Schwarz inequality and Lemma 2.1, we can derive (3.5) readily.

Inserting (3.4) into (3.3), in view of (3.5) and the inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$, we further have

$$\begin{aligned}
2|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 =& \alpha\|e^n\|_0^2 + \alpha\|e^{n+1} - e^n\|_0^2 + 2(1-\rho\nu)(\nabla \boldsymbol{E}^n, \nabla \boldsymbol{E}^{n+1}) - 2\rho N(\boldsymbol{E}^n; \boldsymbol{u}, \boldsymbol{E}^{n+1}) \\
\le& \alpha\|e^n\|_0^2 + \rho^2\alpha^{-1}|\boldsymbol{E}^{n+1}|_1^2 + 2|1-\rho\nu||\boldsymbol{E}^n|_1|\boldsymbol{E}^{n+1}|_1 + 2\rho\nu\Lambda|\boldsymbol{E}^n|_1|\boldsymbol{E}^{n+1}|_1 \\
\le& \alpha\|e^n\|_0^2 + \rho^2\alpha^{-1}|\boldsymbol{E}^{n+1}|_1^2 + (|1-\rho\nu| + \rho\nu\Lambda)(\varepsilon|\boldsymbol{E}^{n+1}|_1^2 + \varepsilon^{-1}|\boldsymbol{E}^n|_1^2).
\end{aligned}$$

where $\varepsilon > 0$ is a parameter to be determined later on.

Define number $r = |1-\rho\nu| + \rho\nu\Lambda$. Then the above estimate can be expressed as

$$(2 - \rho^2\alpha^{-1} - r\varepsilon)|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \le r\varepsilon^{-1}|\boldsymbol{E}^n|_1^2 + \alpha\|e^n\|_0^2. \tag{3.6}$$

We try to find ε such that

$$2 - \rho^2\alpha^{-1} - r\varepsilon = r\varepsilon^{-1}, \tag{3.7}$$

or equivalently, ε is a positive root of the quadratic equation

$$r\varepsilon^2 - (2 - \rho^2\alpha^{-1})\varepsilon + r = 0.$$

Note that the discriminant of the equation is

$$\Delta = (2 - \rho^2\alpha^{-1})^2 - 4r^2 = 4\left(1 - \frac{\rho^2}{2\alpha}\right)^2 - 4r^2,$$

which is positive as long as the condition $|1-\rho\nu| + \rho\nu\Lambda + \frac{\rho^2}{2\alpha} = r + \frac{\rho^2}{2\alpha} < 1$ holds. Hence, under the assumption of the lemma, $\varepsilon_* := \frac{2-\rho^2\alpha^{-1}-\sqrt{\Delta}}{2r} > 0$, which is what we find. Now, letting

$$\mathcal{D} = 2 - \rho^2\alpha^{-1} - r\varepsilon_* = r\varepsilon_*^{-1} = 1 - \frac{\rho^2}{2\alpha} + \frac{1}{2}\sqrt{\Delta},$$

we are able to rewrite (3.6) as

$$\mathcal{D}|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \le \mathcal{D}|\boldsymbol{E}^n|_1^2 + \alpha\|e^n\|_0^2,$$

which implies

$$\mathcal{D}|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \le \mathcal{D}|\boldsymbol{E}^0|_1^2 + \alpha\|e^0\|_0^2.$$

The proof is complete. ■

Now we are ready to develop the convergence rate analysis for Algorithm 1.

Theorem 3.1 *Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ be the unique solution of problem Q. Let $(\boldsymbol{u}^n, p^n)$ be the function sequence generated by Algorithm 1. If the parameter Λ given in* (1.8) *is less than* 1 *and the parameters ρ and α in Algorithm 1 satisfy*

$$|1-\rho\nu| + \rho\nu\Lambda + \frac{\rho^2}{2\alpha} < 1,$$

then we have

$$\mathcal{F}|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \le \gamma(\mathcal{F}|\boldsymbol{E}^n|_1^2 + \alpha\|e^n\|_0^2),$$

where $\mathcal{F} \in (0,1)$ and $\gamma \in (0,1)$ are two generic constants independent of n, and $\boldsymbol{E}^n$ and e^n are given as in (3.1).

Proof. From Lemma 3.1, there is a positive number $\mathcal{F}_1$ such that $|\boldsymbol{u}^n|_1 \le \mathcal{F}_1$ for all natural numbers n. On the other hand, we can rewrite (3.2) in the form

$$(e^n, \operatorname{div}\boldsymbol{v}) = \nu(\nabla\boldsymbol{E}^n, \nabla\boldsymbol{v}) + \rho^{-1}(\nabla(\boldsymbol{E}^{n+1} - \boldsymbol{E}^n), \nabla\boldsymbol{v}) + N(\boldsymbol{E}^n; \boldsymbol{u}, \boldsymbol{v}) + N(\boldsymbol{u}^n; \boldsymbol{E}^{n+1}, \boldsymbol{v}).$$

Hence, from the above equation, the inf-sup condition (1.7), (1.6), and (2.4), it follows that

$$\begin{aligned}
\beta\|e^n\|_0 \le& \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{(e^n, \operatorname{div}\boldsymbol{v})}{|\boldsymbol{v}|_1} \\
\le& \nu|\boldsymbol{E}^n|_1 + \rho^{-1}|\boldsymbol{E}^{n+1} - \boldsymbol{E}^n|_1 + \mathcal{N}|\boldsymbol{E}^n|_1|\boldsymbol{u}|_1 + \mathcal{N}|\boldsymbol{u}^n|_1|\boldsymbol{E}^{n+1}|_1 \\
\le& \nu|\boldsymbol{E}^n|_1 + \rho^{-1}|\boldsymbol{E}^{n+1}|_1 + \rho^{-1}|\boldsymbol{E}^n|_1 + \nu\Lambda|\boldsymbol{E}^n|_1 + \mathcal{N}\mathcal{F}_1|\boldsymbol{E}^{n+1}|_1 \\
=& (\nu + \nu\Lambda + \rho^{-1})|\boldsymbol{E}^n|_1 + (\mathcal{N}\mathcal{F}_1 + \rho^{-1})|\boldsymbol{E}^{n+1}|_1.
\end{aligned}$$

Squaring the above equation, applying the basic inequality: $(u+v)^2 \le 2(u^2+v^2)$ for any real numbers u and v, we arrive at

$$\beta^2\|e^n\|_0^2 \le 2(\nu + \nu\Lambda + \rho^{-1})^2|\boldsymbol{E}^n|_1^2 + 2(\mathcal{N}\mathcal{F}_1 + \rho^{-1})^2|\boldsymbol{E}^{n+1}|_1^2,$$

i.e.,

$$|\boldsymbol{E}^{n+1}|_1^2 \geq \mathcal{F}_2\|e^n\|_0^2 - \mathcal{F}_3|\boldsymbol{E}^n|_1^2, \tag{3.8}$$

where

$$\mathcal{F}_2 := \frac{1}{2}\left(\frac{\beta}{\mathcal{N}\mathcal{F}_1+\rho^{-1}}\right)^2,\ \mathcal{F}_3 := \left(\frac{\nu+\nu\Lambda+\rho^{-1}}{\mathcal{N}\mathcal{F}_1+\rho^{-1}}\right)^2.$$

Rewrite (3.6) in the form

$$\delta|\boldsymbol{E}^{n+1}|_1^2 + (2-\rho^2\alpha^{-1}-r\varepsilon-\delta)|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \leq r\varepsilon^{-1}|\boldsymbol{E}^n|_1^2 + \alpha\|e^n\|_0^2,$$

where $\delta > 0$ is a parameter to be determined. We then decrease the first term in the above inequality in view of (3.8) and reorganizing terms to get

$$(2-\rho^2\alpha^{-1}-r\varepsilon-\delta)|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \leq (r\varepsilon^{-1}+\delta\mathcal{F}_3)|\boldsymbol{E}^n|_1^2 + (\alpha-\delta\mathcal{F}_2)\|e^n\|_0^2. \tag{3.9}$$

Next, we try to find ε and δ such that

$$(2-\rho^2\alpha^{-1}-r\varepsilon-\delta)/\alpha = (r\varepsilon^{-1}+\delta\mathcal{F}_3)/(\alpha-\delta\mathcal{F}_2), \tag{3.10}$$

$\alpha - \delta\mathcal{F}_2 > 0$ and $2-\rho^2\alpha^{-1}-r\varepsilon-\delta > 0$. First of all, we have after a direct manipulation that δ is determined by the quadratic equation

$$\mathcal{F}_2\delta^2 - (\alpha+\alpha\mathcal{F}_3+\mathcal{F}_2(2-\rho^2\alpha^{-1}-r\varepsilon))\delta + \alpha(2-\rho^2\alpha^{-1}-r\varepsilon-r\varepsilon^{-1}) = 0. \tag{3.11}$$

If we suppose $2-\rho^2\alpha^{-1}-r\varepsilon > \delta > 0$, then we have by (3.11) that

$$\begin{aligned}\alpha(2-\rho^2\alpha^{-1}-r\varepsilon-r\varepsilon^{-1}) =&(\alpha+\alpha\mathcal{F}_3+\mathcal{F}_2(2-\rho^2\alpha^{-1}-r\varepsilon))\delta - \mathcal{F}_2\delta^2\\ >&\alpha\delta+\mathcal{F}_2\delta^2-\mathcal{F}_2\delta^2 = \alpha\delta > 0,\end{aligned}$$

which implies

$$2-\rho^2\alpha^{-1}-r\varepsilon-r\varepsilon^{-1} > 0. \tag{3.12}$$

This motivates us to choose ε such that (3.12) holds. Letting $\Delta = (2-\rho^2\alpha^{-1})^2 - 4r^2$, one can show $\Delta > 0$ under the condition $|1-\rho\nu| + \rho\nu\Lambda + \rho^2/(2\alpha) < 1$. Therefore, it follows from (3.12) that

$$\frac{2-\rho^2\alpha^{-1}-\sqrt{\Delta}}{2r} < \varepsilon < \frac{2-\rho^2\alpha^{-1}+\sqrt{\Delta}}{2r}.$$

Now we choose

$$\varepsilon = \varepsilon^* = \frac{2-\rho^2\alpha^{-1}}{2r} = \frac{1-\rho^2/(2\alpha)}{r} > 0.$$

In this case, the quadratic equation (3.11) becomes

$$a\delta^2 - b\delta + c = 0,$$

where

$$s := 1-\frac{\rho^2}{2\alpha},\ a := \mathcal{F}_2,\ b := \alpha+\alpha\mathcal{F}_3+\mathcal{F}_2 s,\ c := \alpha(s-\frac{r^2}{s}).$$

Observe that $a > 0, b > 0, c > 0$ and

$$b^2 - 4ac > (\alpha+\mathcal{F}_2 s)^2 - 4\alpha\mathcal{F}_2 s \geq 0.$$

So the equation (3.11) has two real roots, and we choose $\delta = \delta^* = \dfrac{b-\sqrt{b^2-4ac}}{2a} > 0$.

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With the parameter ε and δ given respectively by ε^* and δ^*, we have from (3.9) that

$$\mathcal{F}|\boldsymbol{E}^{n+1}|_1^2 + \alpha\|e^{n+1}\|_0^2 \leq \gamma(\mathcal{F}|\boldsymbol{E}^n|_1^2 + \alpha\|e^n\|_0^2),$$

where $\mathcal{F} := s - \delta^*$ and $\gamma := 1 - \alpha^{-1}\delta^*\mathcal{F}_2$.

Obviously, $\mathcal{F} < 1$ and $\gamma < 1$. Now let us show that $\mathcal{F} > 0$ and $\gamma > 0$. Consider the quadric function $f(x) = ax^2 - bx + c$. Since $s > 0$, $b > \alpha + \mathcal{F}_2 s$ and $c < \alpha s$, we have $\lim_{x\to-\infty} f(x) = \infty$ and

$$f(s) = as^2 - bs + c < \mathcal{F}_2 s^2 - (\alpha + \mathcal{F}_2 s)s + \alpha s = 0,$$

so the smaller root δ^* of $f(x)$ must belong to $(-\infty, s)$, i.e., $\mathcal{F} > 0$. In view of (3.10), we know $\gamma = (r\varepsilon^{*-1} + \delta^*\mathcal{F}_3)/\mathcal{F} > 0$. The proof is complete. ■

Remark 3.1 *Condition*

$$|1-\rho\nu| + \rho\nu\Lambda + \frac{\rho^2}{2\alpha} < 1$$

is equivalent to

$$\rho < \frac{2}{\nu(1+\Lambda)}, \quad \alpha > \begin{cases} \dfrac{\rho}{2\nu(1-\Lambda)}, & \rho \leq \dfrac{1}{\nu}, \\ \dfrac{\rho^2}{4-2\rho\nu(1+\Lambda)}, & \dfrac{1}{\nu} < \rho < \dfrac{2}{\nu(1+\Lambda)}. \end{cases}$$

Therefore, for any $\Lambda \in [0,1)$, we can find suitable ρ and α such that Algorithm 1 is convergent and converges geometrically.

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α-ψ-contractive mappings on b-metric space

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Abstract

In this paper, we introduce α-ψ-contractive self mapping in b-metric space and prove the existence of a fixed point for such mapping under some conditions. Some examples are given to illustrate the main results.

1 Introduction

Banach contraction principle is considered to be the main point of interest in fixed point theory. Most of the generalizations for metric fixed point theorems usually start from it. It is difficult to point out all the generalizations of this principle. For example, many authors recently studied this principle and its generalizations in different type metric spaces [1, 2]. Close to our interest in this article some authors studied some fixed point theorems in the so called b-metric space [6, 8, 14, 15, 16]. Samet in [7], gave a generalization of Banach's contraction principles in a metric space by introducing α-ψ-contraction. After then, some authors started to prove α-ψ-versions of of certain fixed point theorems in different type metric spaces [3, 4, 5]. In this paper, we generalize a result of Mehmet in [8], by introducing the α-ψ-contractive mapping in b-metric space.

Definition 1. *[13] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

Remark 1.1. *The class of b-metric spaces is effectively larger than the class of metric spaces, since any metric space is a special case of b-metric space when $s = 1$.*

The following examples shows that a b-metric on X need not be a metric on X see also [12], [10].

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Example 1.1. *[9] Let $X = \{x_1, x_2, x_3, x_4\}$. Define a function $d : X \times X \to [0, \infty)$ such that $d(x_1, x_2) = k \geq 2$ and $d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = 1$, $d(x_i, x_j) = d(x_j, x_i)$ for all $i, j = 1, 2, 3, 4$ and $d(x_i, x_i) = 0$ for all $i, j = 1, 2, 3, 4$. Then $d(x_i, x_j) \leq \frac{k}{2}[d(x_i, x_n) + d(x_n, x_j)]$ for $x_i \in X$ and $i, j = 1, 2, 3, 4$. Therefore, (X, d) is a b-metric space with constant $s = \frac{k}{2}$. However if $k > 2$ the ordinary triangle inequality does not hold and thus (X, d) is not a metric space.*

Example 1.2. *[16] Let $X = [0, 1]$. Define a function $d : X \times X \to [0, \infty)$ such that $d(x, y) = |x - y|^2$ For all $x, y \in X$. Then (X, d) is a complete b-metric on X with constant $s = 2$. Also, d is not a metric on X.*

Example 1.3. *[16] Let $X = [1, \infty)$. Define a function $d : X \times X \to [0, \infty)$ such that $d(x, y) = |x - y|^2$ For all $x, y \in X$. Then (X, d) is a complete b-metric on X with constant $s = 2$. Also, d is not a metric on X.*

Example 1.4. *Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{n} |x_i|^p < \infty\}$. Let $d : X \times X \to [0, \infty)$ defined by $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$, where $x = \{x_n\}$, $y = \{y_n\} \in l^p$. Then (X, d) is a b-metric space with constant $s = 2^{\frac{1}{p}} > 1$ (see in [11]) .*

Denote by Ψ_s the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. We now present this useful lemma.

Lemma 1.1. *[7] For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds: if ψ is nondecreasing, then for each $t > 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ implies that $\psi(t) < t$.*

We now introduce α-ψ-contractive self mapping on b-metric space.

Definition 2. *(see also [7]) Let (X, d) be a b-metric space $T : X \to X$ be a given mapping and We say that T is an α-ψ-contractive mapping if there exist two functions $\psi \in \Psi_s$ and $\alpha : X \times X :\to [0, \infty)$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad for \quad all \quad x, y \in X.$$

Remark 1.2. *For our purposes, for $s \geq 1$ we define $\Psi_s = \{\psi : [0, \infty) \to [0, \infty)$: ψ is nondecreasing and $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty\}$. It is clear by the help of Lemma 1.1., that if $\psi \in \Psi_s$ then $\lim_{n\to\infty} s^n\psi^n(t) = 0$, for all $t > 0$ and hence $\psi(t) < t$.*

Remark 1.3. *(see also [7]) If $T : X \to X$ satisfies the Banach contraction principle, then T is an α-ψ-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$.*

Definition 3. *(see also [7]) Let (X, d) be a b-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.*

Example 1.5. *[7] Let $X = (0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = lnx$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} 2 & if\ x \geq y, \\ 0 & if\ x < y. \end{cases}$$

Then, T is α-admissible.

Example 1.6. *[7] Let $X = [0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} e^{x-y} & if\ x \geq y, \\ 0 & if\ x < y. \end{cases}$$

Then, T is α-admissible.

2 Main result

We start this section by proving our main theorem.

Theorem 2.1. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ be an α-ψ-contractive mapping for some $\psi \in \Psi_s$. Suppose that the following conditions hold:*

(1) T is α-admissible;
(2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(3) T is continuous.

Then, T has a fixed point.

Proof. It is clear that $d(x, y) = 0$ if and only if $x = y$ is a fixed point of T. Thus we may assume that $d(x, y) > 0$ for all $x, y \in X$.
Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ (such a point exists from condition (2)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n then $x = x_n$ is a fixed point of T. So we may assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α-admissible from condition (1), we have

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we have $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$.
By applying the α-ψ-contractive condition and using the fact that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(x_n, x_{n+1}) \\ &= \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \alpha(x_{n-2}, x_{n-1}) \psi(d(x_{n-1}, x_n)) \\ &= \alpha(x_{n-2}, x_{n-1}) \psi(d(Tx_{n-2}, Tx_{n-1})) \\ &\leq \psi^2(d(x_{n-2}, x_{n-1})). \end{aligned}$$

Inductively, for all $n \in \mathbb{N}$ we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Now we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Let $m > n > 0$. Thus,

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=1}^{m-n-1} s^i d(x_{n+i-1}, x_{n+i}) + s^{m-n-1} d(x_{m-1}, x_m) \\
&\leq \sum_{i=1}^{m-n} s^i d(x_{n+i-1}, x_{n+i}) \\
&\leq \sum_{i=1}^{m-n} s^i \psi^{i+n-1}(d(x_0, x_1)) \\
&\leq \sum_{i=1}^{m-n} s^{i+n-1} \psi^{i+n-1}(d(x_0, x_1)) \\
&= \sum_{k=n}^{m-1} s^k \psi^k(d(x_0, x_1)) \\
&\leq \sum_{k=n}^{\infty} s^k \psi^k(d(x_0, x_1)).
\end{aligned}$$

But, we know that

$$\sum_{n=1}^{\infty} s^n \psi^n(d(x_0, x_1)) < \infty.$$

Hence,

$$\lim_{n\to\infty} s^n \psi^n(d(x_0, x_1)) = 0,$$

and $\psi^n(d(x_0, x_1)) \longrightarrow 0$ as $n \longrightarrow \infty$, and also, $\lim_{n\to\infty} \sum_{k=n}^{\infty} s^k \psi^k(d(x_0, x_1)) = 0$. Therefore, we obtain that $d(x_n, x_m) \longrightarrow 0$ as $m, n \longrightarrow \infty$.
This implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). It follows from the completeness of (X, d) that there exists $a \in X$ such that

$$\lim_{n\to\infty} x_n = a.$$

Since T is continuous, we obtain that

$$\lim_{n\to\infty} d(x_{n+1}, Ta) = \lim_{n\to\infty} d(Tx_n, a) = 0.$$

Thus, by the uniqueness of the limit, we get that a is a fixed point of T, that is, $Ta = a$. This completes the proof.

□

In our next theorem we omit the condition of continuity in Theorem 2.1.

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Theorem 2.2. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ be an α-ψ-contractive mapping for some $\psi \in \Psi_s$. Suppose that the following conditions hold:*

(1) T is α-admissible;
(2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(3) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n;

Then, T has a fixed point.

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, is a Cauchy sequence in the complete b-metric space (X, d). It follows from the completeness of (X, d) that there exists $a \in X$ such that $\lim_{n\to\infty} x_n = a$. On the other hand, from $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and the condition (3), that implies $\alpha(x_n, a) \geq 1$ for all $n \in \mathbb{N}$. Now, by using the α-ψ-contractive mapping condition, the triangular inequality for a b-metric space and $\alpha(x_n, a) \geq 1$, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(Ta, a) &\leq s[d(Ta, Tx_n) + d(x_{n+1}, a)] \\ &\leq \alpha(x_n, a)s(d(Ta, Tx_n)) + sd(x_{n+1}, a) \\ &\leq s(\psi(d(a, x_n))) + s(d(x_{n+1}, a)). \end{aligned}$$

Letting $n \to \infty$, since ψ is continuous at $t = 0$ we get

$$\lim_{n\to\infty} \psi(d(x_n, a)) = \lim_{n\to\infty} d(x_{n+1}, a) = 0.$$

Thus, we get that $d(Ta, a) = 0$, that is, $Ta = a$. This completes the proof. □

In our next result, we show that we can have a unique fixed point, but using extra hypothesis.

Theorem 2.3. *Let T be a self mapping on a complete b-metric space, and assume that T satisfies all the conditions of Theorem 2.1 (resp. Theorem 2.2) If for every two fixed points x, y of T there exists $z \in X$ such that*

$$\alpha(x, z) \geq 1, \quad \textit{and} \quad \alpha(y, z) \geq 1.$$

Then, T has a unique fixed point.

Proof. Let x, y be two fixed points of T, we know from the hypothesis that there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.
Since T is α-admissible and by induction on n, we obtain for all $n \in \mathbb{N}$, $\alpha(x, T^n z) \geq 1$ and $\alpha(y, T^n z) \geq 1$. Thus,

$$\begin{aligned} d(x, T^n z) &= d(Tx, T(T^{n-1}z)) \\ &\leq \alpha(x, T^{n-1}z)d(Tx, T(T^{n-1}z)) \\ &\leq \psi(d(x, T^{n-1}z)). \end{aligned}$$

So, by induction on n we get,

$$d(x, T^n z) \leq \psi^n(d(x, z)).$$

Hence, as $n \longrightarrow +\infty$ we have $T^n z \longrightarrow x$. Similarly, as $n \longrightarrow +\infty$ we have $T^n z \longrightarrow y$. By the uniqueness of the limit we obtain that $x = y$ as desired. This completes the proof.

□

Example 2.1. *Let $X = [1, \infty)$ with the functional $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$ For all $x, y \in X$. Clearly, (X, Y) is a complete b-metric space with $s = 2$. Define the mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x+6}{4} & \text{if } 1 \leq x \leq 2, \\ \frac{x^2}{2} & \text{if } x > 2. \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & if\ x, y \in [1, 2], \\ 0 & otherwise. \end{cases}$$

In this case, T is continuous, and the Banach contraction principle cannot be applied since

$$d(T4, T1) = |T4 - T1|^2 = (\frac{25}{4})^2 \nleq kd(4, 1) = k|4 - 1|^2 = k(3)^2$$

for $0 \leq k < 1$.
We will prove the following:
i) $\sum_{n=1}^{\infty} s^n \psi^n(d(x_0, Tx_0)) < \infty$.
ii) $T : X \to X$ is an $\alpha - \psi-$contractive mapping, with $\psi(t) = \dfrac{t}{3}$ for all $t \geq 0$;
iii) T is $\alpha-$admissible;
iv) there exists $x_0 = 1 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
v) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;
vi) If for every two fixed points x, y of T there exists $z \in X$ such that $\alpha(x, z) \geq 1$, and $\alpha(y, z) \geq 1$.

Proofs:
i) $\sum_{n=1}^{\infty} s^n \psi^n(d(x_0, Tx_0)) \leq \sum_{n=1}^{\infty} (\frac{2}{3})^n (d(x_0, Tx_0) < \infty$.
ii) Clearly T is α-ψ-contractive mapping with $\psi(t) = \dfrac{t}{3}$ for all $t \geq 0$, since for all $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty) = 2d(Tx, Ty) = \frac{1}{8}d(x, y) \leq \psi(d(x, y)) = \frac{1}{3}d(x, y).$$

iii) Let $(x, y) \in X \times X$ such that $\alpha(x, y) \geq 1$. From the definition of T and α we have both $Tx = \frac{x+6}{4}$,and $Ty = \frac{y+6}{4}$ are in $[1, 2]$, so we have $\alpha(Tx, Ty) = 2 \geq 1$. Then T is an α-admissible.
iv) Taking $x_0 = 1 \in X$, we have

$$\alpha(x_0, Tx_0) = \alpha(1, T1) = \alpha(1, \frac{7}{4}) = 2 \geq 1.$$

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v) let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and by the definition of α, we have $x_n \in [1, 2]$ for all $n \in \mathbb{N}$ and $x \in [1, 2]$. Then $\alpha(x_n, x) = 2 \geq 1$.
vi) Let $(x, y) \in X \times X$ such that x and y both are fixed point of T. we have $\alpha(x, 2) = \alpha(y, 2) = 2 \geq 1$.
Now, all the hypothesis of Theorem 2.1 and Theorem 2.3 are satisfied. Therefore, T has a unique fixed point 2.

Example 2.2. *Let $X = [1, \infty)$ with the functional $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, (X, d) is a complete b-metric space with $s = 2$. Define the mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x+3}{4} & \text{if } 1 \leq x \leq 2, \\ \frac{x^2}{2} & \text{if } x > 2. \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & if\ x, y \in [1, 2], \\ 0 & otherwise. \end{cases}$$

In this case, T is not continuous, and the Banach contraction principle cannot be applied since

$$d(T4, T1) = |T4 - T1|^2 = (7)^2 \not\leq kd(4, 1) = k|4 - 1|^2 = k(3)^2$$

for $0 \leq k < 1$.
We will prove the following:
i) $\sum_{n=1}^{\infty} s^n \psi^n(d(x_0, Tx_0)) < \infty$.
ii) $T : X \to X$ is an α-ψ-contractive mapping, with $\psi(t) = \dfrac{t}{3}$ for all $t \geq 0$;
iii) T is α-admissible;
iv) there exists $x_0 = 1 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
v) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;
vi) If for every two fixed points x, y of T there exists $z \in X$ such that $\alpha(x, z) \geq 1$, and $\alpha(y, z) \geq 1$.
Proofs:
i) $\sum_{n=1}^{\infty} s^n \psi^n(d(x_0, Tx_0)) \leq \sum_{n=1}^{\infty} (\frac{2}{3})^n (d(x_0, Tx_0) < \infty$.
ii) Clearly T is α-ψ-contractive mapping with $\psi(t) = \dfrac{t}{3}$ for all $t \geq 0$, since for all $x, y \in X$.

$$\alpha(x, y)d(Tx, Ty) = 2d(Tx, Ty) = \frac{1}{8}d(x, y) \leq \psi(d(x, y)) = \frac{1}{3}d(x, y).$$

iii) Let $(x, y) \in X \times X$ such that $\alpha(x, y) \geq 1$. From the definition of T and α we have both $Tx = \frac{x+3}{4}$,and $Ty = \frac{y+3}{4}$ are in $[1, 2]$, so we have $\alpha(Tx, Ty) = 2 \geq 1$. Then T is an α-admissible.
iv) Taking $x_0 = 1 \in X$, we have

$$\alpha(x_0, Tx_0) = \alpha(1, T1) = \alpha(1, \frac{7}{4}) = 2 \geq 1.$$

v) let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and by the definition of α, we have $x_n \in$[1,2] for all $n \in \mathbb{N}$ and $x \in [1, 2]$. Then $\alpha(x_n, x) = 2 \geq 1$.
vi) Let $(x, y) \in X \times X$ such that x and y both are fixed point of T. we have $\alpha(x, 2) = \alpha(y, 2) = 2 \geq 1$.
Now, all the hypothesis of Theorem 2.2 and Theorem 2.3 are satisfied. Therefore, T has a unique fixed point 1.

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Refinement of some inequalities of $q-$gamma function

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Abstract

We prove the following theorem for the $q-$gamma function $\Gamma_q(x)$:

$$(q;q)_\infty(1-q)^{1-x}e^{\sum_{i=0}^{\infty}f(q^{x+i})} < \Gamma_q(x) < (q;q)_\infty(1-q)^{1-x}e^{\sum_{i=0}^{\infty}g(q^{x+i})},$$

where $x>0$, $0<q<1$ and $f(x)$, $g(x)$ are two positive functions pass through the point of origin such that $0<f'(x)<\frac{1}{1-x}<g'(x)$; for $0\le x<1$. As consequences of this result we will refine some recent results and we will present the following double inequality of $\Gamma_q(x)$ in terms of the $q-$digamma function $\psi_q(x)$ for $x>0$ and $0<q<1$:

$$(q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}\left[\psi_q\left(x+\frac{\log(3/4)}{\log q}\right)+\log(1-q)\right]} < \Gamma_q(x) < (q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}[\psi_q(x)+\log(1-q)]}.$$

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Key Words: $q-$gamma function, $q-$factorial, $q-$digamma function, Inequalities, Monotonicity.

1 Introduction.

The $q-$gamma function $\Gamma_q(x)$ is defined by the infinite product [9]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}; \qquad x \neq 0,-1,-2,..., \tag{1}$$

where q is a fixed real number $0<q<1$. Here we use the following notation [12]:
$(a;q)_0 = 1$,
$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j); \qquad k=1,2,\dots$.
This function is a $q-$analogue of the gamma function since we have

$$\lim_{q\to 1}\Gamma_q(x) = \Gamma(x).$$

Also, it satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x); \quad \Gamma_q(1) = 1, \tag{2}$$

which is a $q-$extension of the well-known functional equation

$$\Gamma(x+1) = x\Gamma(x); \quad \Gamma(1) = 1,$$

where $[x]_q = \frac{1-q^x}{1-q}$ is the $q-$number of x and $\lim_{q\to 1}[x]_q = x$.

The $q-$factorial is defined by [11]

$$[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q = \Gamma_q(n+1), \tag{3}$$

which is the $q-$analog of the relation $n! = \Gamma(n+1)$ where $\lim_{q\to 1}[n]_q! = n!$. In [13], Mansour and et al obtained the following upper and lower bounds for $q-$factorial $[n]_q!$

$$(q;q)_\infty (1-q)^{-n} e^{f_q(n+1)} < [n]_q! < (q;q)_\infty (1-q)^{-n} e^{g_q(n+1)}, \tag{4}$$

where $n \geq 1$, $0 < q < 1$ and the two sequences $f_q(n)$ and $g_q(n)$ tends to zero through positive values. Also, they presented two special cases of (4), one of them is

$$(q;q)_\infty (1-q)^{-n} e^{\frac{q^{n+1}}{1-q}} < [n]_q! < (q;q)_\infty (1-q)^{-n} e^{\frac{q^{n+1}}{(1-q)(1-q^{n+1})}}, \quad n \geq 1; \; 0 < q < 1. \tag{5}$$

The q-digamma ($q-$psi) function is defined for the real variable $x > 0$ as

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}, \qquad 0 < q < 1. \tag{6}$$

It is also given by the sums

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n=1}^{\infty} \frac{q^{xn}}{1-q^n} = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{x+n}}{1-q^{x+n}}. \tag{7}$$

Using the first representation of $\psi_q(x)$ given in (7), it can be shown that

$$\psi_q^{(k)}(x) = (\log q)^{k+1} \sum_{n=1}^{\infty} \frac{n^k q^{xn}}{1-q^n}, \qquad 0 < q < 1 \tag{8}$$

and hence $(-1)^k(\psi_q'(x))^{(k)} > 0$ with $x > 0$, $0 < q < 1$ and for all $k \geq 0$.

In [1], Alzer and Batir investigated the monotonicity property for a function involving the gamma function and the digamma function and they determined the the best possible nonnegative constants α and β in the two-sided inequality

$$\sqrt{2\pi} x^x e^{-x-\frac{1}{2}\psi(x+\alpha)} < \Gamma(x) < \sqrt{2\pi} x^x e^{-x-\frac{1}{2}\psi(x+\beta)}; \qquad x > 0 \tag{9}$$

by $\alpha = 1/3$ and $\beta = 0$. Şevli and Batir [22] established the following new upper and lower bounds of $\Gamma(x)$ in terms of the $\psi'-$function :

$$\sqrt{2\pi}x^{x+1/2}e^{-x+\frac{1}{12}\psi'(x+1/2)} < \Gamma(x) < \sqrt{2\pi}x^{x+1/2}e^{-x+\frac{1}{12}\psi'(x)}; \qquad x > 0. \tag{10}$$

Also, Alzer and Grinshpan [2] presented several sharp inequalities for the gamma and $q-$gamma functions and some inequalities involve the psi and $q-$psi functions. In [21], Salem generalized the results of Alzer and Batir [1] and presented a $q-$analogue the two-sided inequality (9) by

$$\sqrt{2\pi}S_{\hat{q}}[x]_q^x q^{\frac{1}{2}\lambda H(q-1)} \exp\left(\frac{Li_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x+\alpha)\right) < \Gamma_q(x)$$

$$< \sqrt{2\pi}S_{\hat{q}}[x]_q^x q^{\frac{1}{2}\lambda H(q-1)} \exp\left(\frac{Li_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x+\beta)\right); x > 0,\ \alpha,\beta \in \mathbb{R} \tag{11}$$

with the best possible constants $\alpha = \frac{\log 2 + \log(\hat{q} - \log\hat{q} - 1) - \log(\log^2\hat{q})}{\log\hat{q}}$ and $\beta = 0$ when $0 < q \leq 1$ and $\lambda = a + 1/2$ with $a \geq 0$ or $\alpha = 1/2$ and $\beta = 0$ when $q > 1$ and $\lambda = 1$, where $S_q = q^{\frac{-1}{24}}\sqrt{\frac{q-1}{\log q}}\sum_{m=-\infty}^{\infty}\left(e^{(4\pi^2/\log q)m(6m+1)} - e^{(4\pi^2/\log q)(2m+1)(3m+1)}\right)$, $\hat{q} = q$ if $0 < q \leq 1$ and $\hat{q} = q^{-1}$ if $q \geq 1$, $H(z)$ is the Heaviside step function and $Li_2(z)$ is the dilogarithm function.

For the developments in the area of inequalities for the gamma, psi, $q-$gamma and $q-$psi functions see [3]-[8], [10], [14]-[16], [18]-[20] and the references therein. Most of these inequalities follow from the studying of the monotonicity properties of functions which are closely related these functions.

In this paper, we will find the formula of the double inequality (4) in the case of $q-$gamma function $\Gamma_q(x)$ and we will deduce the double inequality (5) for $\Gamma_q(x)$ as a special case and some of its refinements. Also, we will present a new inequality of the function $\Gamma_q(x)$ in terms of the function $\psi_q(x)$, which refines the upper bound of the inequality (11) in the domain $0 < q < e^{-2}$.

2 Main results.

From the definition of the function $\Gamma_q(x)$, we have

$$\Gamma_q(x) = (q;q)_\infty(1-q)^{1-x}e^{\mu_q(x)}; \qquad x \neq 0, -1, -2, \dots, \tag{12}$$

where

$$\mu_q(x) = -\sum_{i=0}^{\infty}\ln(1-q^{x+i}). \tag{13}$$

Now let $f(x)$ be a positive function passes through the point of origin such that $0 < f'(x) < \frac{1}{1-x}$; for $0 \leq x < 1$. Then the function

$$F(x) = -\ln(1-x) - f(x), \qquad 0 \leq x < 1$$

is monotonically increasing function and hence

$$F(x) > f(0) = 0.$$

Then
$$-\ln(1-x) > f(x) \qquad 0 \le x < 1.$$
Similarly, let $g(x)$ be a positive function passes through the point of origin such that $g'(x) > \frac{1}{1-x}$; for $0 \le x < 1$. Then
$$-\ln(1-x) < g(x) \qquad 0 \le x < 1.$$
Now we get
$$f(x) < -\ln(1-x) < g(x) \qquad 0 \le x < 1.$$
The increasing property of each of $f(x)$ and $g(x)$ will guarantee the absolutely convergence of the two series $\sum_{i=0}^{\infty} f(q^{x+i})$ and $\sum_{i=0}^{\infty} g(q^{x+i})$ for $x > 0$. Then
$$\sum_{i=0}^{\infty} f(q^{x+i}) < \mu_q(x) < \sum_{i=0}^{\infty} g(q^{x+i}).$$

So we get the following result:

Theorem 1. *The $q-$gamma function satisfies the double inequality*
$$(q;q)_\infty (1-q)^{1-x} e^{\sum_{i=0}^{\infty} f(q^{x+i})} < \Gamma_q(x) < (q;q)_\infty (1-q)^{1-x} e^{\sum_{i=0}^{\infty} g(q^{x+i})}; \quad x > 0; \ 0 < q < 1, \tag{14}$$
where $f(x)$ and $g(x)$ are two positive functions pass through the point of origin such that $0 < f'(x) < \frac{1}{1-x} < g'(x)$; for $0 \le x < 1$.

The double inequality (14) is a generalization of the inequality (4) in the case of $\Gamma_q(x)$ for $x > 0$.

As a special case, if we put $f_1(x) = x$ and $g_1(x) = \frac{x}{1-x}$, we obtain
$$\sum_{i=0}^{\infty} q^{x+i} < \mu_q(x) < \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}},$$
$$\sum_{i=0}^{\infty} q^{x+i} < \mu_q(x) < \frac{1}{1-q^x} \sum_{i=0}^{\infty} q^{x+i}$$
and
$$\frac{q^x}{1-q} < \mu_q(x) < \frac{q^x}{(1-q)(1-q^x)}.$$
Then we obtain the following result:

Corollary 2.1. *The $q-$gamma function satisfies the double inequality*
$$(q;q)_\infty (1-q)^{1-x} e^{\frac{q^x}{1-q}} < \Gamma_q(x) < (q;q)_\infty (1-q)^{1-x} e^{\frac{q^x}{(1-q)(1-q^x)}}; \quad x > 0; \ 0 < q < 1. \tag{15}$$

The double inequality (15) is a generalization of the double inequality (5) for $x > 0$.

3 New double inequality of $\Gamma_q(x)$.

In this section, we will refinement the inequality (15). Let

$$f_2(x) = x + x^3, \qquad 0 \le x < 1.$$

Then $f_2(0) = 0$ and $f_2'(x) = 1 + 3x^2 < \frac{1}{1-x}$ for $0 \le x < 1$. Using Theorem (1)

$$\mu_q(x) > \sum_{i=0}^{\infty} \left[q^{x+i} + q^{3(x+i)}\right]$$

and hence

$$\mu_q(x) > \frac{q^x}{1-q} + \frac{q^{3x}}{1-q^3}. \tag{16}$$

Now let

$$g_2(x) = \frac{x}{\sqrt{1-x}}, \qquad 0 \le x < 1$$

$g_2(0) = 0$ and $g_2'(x) = \frac{2-x}{2(1-x)^{3/2}}$. The function $\frac{1}{(1-x)g_2'(x)}$ is monotonically decreasing function for $0 \le x < 1$ since $\frac{d}{dx}\frac{1}{(1-x)g_2'(x)} = \frac{-x}{\sqrt{1-x}\,(x-2)^2}$. Then $\frac{1}{(1-x)g_2'(x)} < \frac{1}{g_2'(0)} = 1$ and hence

$$g_2'(x) > \frac{1}{1-x}.$$

Using Theorem (1)

$$\mu_q(x) < \sum_{i=0}^{\infty} \frac{q^{x+i}}{\sqrt{1-q^{x+i}}}$$

and hence

$$\mu_q(x) < \frac{q^x}{(1-q)\sqrt{1-q^x}}. \tag{17}$$

Then we get the following result:

Corollary 3.1. *The $q-$gamma function satisfies the double inequality*

$$(q;q)_\infty (1-q)^{1-x} e^{\frac{q^x}{1-q} + \frac{q^{3x}}{1-q^3}} < \Gamma_q(x) < (q;q)_\infty (1-q)^{1-x} e^{\frac{q^x}{(1-q)\sqrt{1-q^x}}}; \;\; x > 0; \; 0 < q < 1. \tag{18}$$

Now let

$$f_3(x) = x + ax^3; \qquad 0 \le x < 1, \; 0 < a < 1.$$

Then $f_3(0) = 0$ and $f_3'(x) - \frac{1}{1-x} = \frac{x(3ax^2 - 3ax + 1)}{x-1}$. The function $M(x) = 3ax^2 - 3ax$, for $0 \le x \le 1$ is convex function and its minimum value at $x = 1/2$. So,

$$M(x) > -\frac{3a}{4}.$$

Hence $M(x)+1>1-\frac{3a}{4}$ and $f_3'(x)-\frac{1}{1-x}>0$ if $a\le\frac{4}{3}$. The best formula of the function $f_3(x)$ which satisfies Theorem (1) will be at $a=\frac{4}{3}$. Using Theorem (1)

$$\mu_q(x)>\sum_{i=0}^{\infty}\left[q^{x+i}+\frac{4}{3}q^{3(x+i)}\right]$$

and hence

$$\mu_q(x)>\frac{q^x}{1-q}+\frac{4}{3}\frac{q^{3x}}{1-q^3}. \tag{19}$$

Now let

$$g_3(x)=\frac{bx}{\sqrt{1-x}};\qquad 0\le x<1,\ 0<b<1.$$

$g_3(0)=0$ and $g_3'(x)-\frac{1}{1-x}=\frac{-2b+2\sqrt{1-x}+bx}{2\sqrt{1-x}(x-1)}$. The function $N(x)=-2b+2\sqrt{1-x}+bx$ is concave function for $0\le x<1$ since $N''(x)=\frac{-1}{(1-x)^{3/2}}$. Then $N(x)\le N(1-\frac{1}{b^2})=\frac{1-b^2}{b}<0$ if $b>1$. The best formula of the function $g_3(x)$ which satisfies Theorem (1) will be at $b=1$. Hence, we obtain the following result:

Corollary 3.2. *The $q-$gamma function satisfies the double inequality*

$$(q;q)_\infty(1-q)^{1-x}e^{\frac{q^x}{1-q}+\frac{4q^{3x}}{3(1-q^3)}}<\Gamma_q(x)<(q;q)_\infty(1-q)^{1-x}e^{\frac{q^x}{(1-q)\sqrt{1-q^x}}};\ \ x>0;\ 0<q<1. \tag{20}$$

The double inequality (20) is a refinement of each of the inequalities (15) and (18).

4 Estimating $q-$gamma function by $q-$digamma function.

In this section, we will concern the problem of approximation of the $q-$gamma function in terms of the $q-$digamma function. In fact, we will present a new double inequality similar to the inequality (9) in the case of $\Gamma_q(x)$ and $\psi_q(x)$ using our main result of Theorem (1). Let

$$H(x)=\frac{hx}{1-hx},\qquad 0\le x<1; h>0.$$

Then $H(0)=0$ and $H'(x)-\frac{1}{1-x}=\frac{h^2x^2-hx+1-h}{(x-1)(hx-1)^2}$. The parabola $y(x)=h^2x^2-hx+1-h$ will be greater than or equal zero for all $0\le x<1$ iff $h\le\frac{3}{4}$. Also, using $y(0)=1-h$ and $y(1)=(1-h)^2$, we get that the parabola will be less than zero for all $0\le x\le 1$ iff $h=1$. So, if we put $f_4(x)=\frac{hx}{1-hx}$ with $0<h\le\frac{3}{4}$ and $g_1(x)=\frac{x}{1-x}$, Theorem (1) will give us that

$$\sum_{i=0}^{\infty}\frac{q^{x+\alpha+i}}{1-q^{x+\alpha+i}}<\mu_q(x)<\sum_{i=0}^{\infty}\frac{q^{x+i}}{1-q^{x+i}},\qquad 0<q^\alpha\le\frac{3}{4}.$$

Using the second sum of equation (7)

$$\sum_{n=0}^{\infty}\frac{q^{x+n}}{1-q^{x+n}}=\frac{1}{\log q}\left[\psi_q(x)+\log(1-q)\right],$$

we get the following result:

Lemma 4.1. *The $q-$gamma function satisfies the double inequality*

$$(q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}[\psi_q(x+\alpha)+\log(1-q)]} < \Gamma_q(x) < (q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}[\psi_q(x)+\log(1-q)]}, \qquad (21)$$

where $x>0$; $0<q<1$ and $0<q^\alpha\leq\frac{3}{4}$.

Now, we will obtain the best value of the constant α in (21). Since $\psi_q(x)$ is strictly increasing on $(0,\infty)$, we get

$$\psi_q(x+A) > \psi_q(x+B), \qquad A>B>0;\ x>0;\ 0<q<1$$

and

$$\frac{1}{\log q}\left[\psi_q(x+A)+\log(1-q)\right] < \frac{1}{\log q}\left[\psi_q(x+B)+\log(1-q)\right], \qquad A>B>0;\ x>0;\ 0<q<1.$$

Then

$$\frac{1}{\log q}\left[\psi_q(x+\alpha)+\log(1-q)\right] < \frac{1}{\log q}\left[\psi_q\left(x+\frac{\log(3/4)}{\log q}\right)+\log(1-q)\right],$$

where $0<q^\alpha\leq\frac{3}{4}$, $x>0$ and $0<q<1$.

Then we get the following result:

Lemma 4.2. *The $q-$gamma function satisfies the double inequality*

$$(q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}\left[\psi_q\left(x+\frac{\log(3/4)}{\log q}\right)+\log(1-q)\right]} < \Gamma_q(x) < (q;q)_\infty(1-q)^{1-x}e^{\frac{1}{\log q}[\psi_q(x)+\log(1-q)]}, \quad (22)$$

where $x>0$ and $0<q<1$.

Now we will prove that our upper bound in the inequality (22) is a refinement of the upper bound of the inequality (11) in the domain $0<q<e^{-2}$. Using the relation [17]

$$(q;q)_\infty = e^{\frac{\pi^2}{6\log q}}q^{\frac{-1}{24}}\sqrt{\frac{2\pi}{-\log q}}\sum_{m=-\infty}^{\infty}\left(e^{(4\pi^2/\log q)m(6m+1)} - e^{(4\pi^2/\log q)(2m+1)(3m+1)}\right),$$

we get

$$\sqrt{2\pi}S_q = e^{-\frac{\pi^2}{6\log q}}(q;q)_\infty\sqrt{1-q}.$$

Hence for $0<q<1$, the upper bound of the inequality of (11) will be

$$\begin{aligned}
&\sqrt{2\pi}S_{\hat{q}}[x]_q^x q^{\frac{1}{2}\lambda H(q-1)}e^{\frac{Li_2(1-q^x)}{\log q}-\frac{1}{2}\psi_q(x)} \\
=\ & e^{-\frac{\pi^2}{6\log q}}(q;q)_\infty\sqrt{1-q}[x]_q^x e^{\frac{Li_2(1-q^x)}{\log q}-\frac{1}{2}\psi_q(x)} \\
=\ & \frac{e^{-\frac{\pi^2}{6\log q}}}{\sqrt{1-q}}(1-q^x)^x(q;q)_\infty(1-q)^{1-x}e^{\frac{Li_2(1-q^x)}{\log q}-\frac{1}{2}\psi_q(x)} \\
=\ & (q;q)_\infty(1-q)^{1-x}e^{-\frac{\pi^2}{6\log q}-\frac{1}{2}\log(1-q)+x\log(1-q^x)+\frac{Li_2(1-q^x)}{\log q}-\frac{1}{2}\psi_q(x)}.
\end{aligned}$$

Let

$$U_1(x;q) = -\frac{\pi^2}{6\log q} - \frac{1}{2}\log(1-q) + x\log(1-q^x) + \frac{Li_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x),$$

$$U_2(x;q) = \frac{1}{\log q}\left[\psi_q(x) + \log(1-q)\right]$$

and

$$K(x;q) = U_1(x;q) - U_2(x;q).$$

Then

$$\frac{\partial}{\partial x}K(x;q) = \ln(1-q^x) - \sum_{k=1}^{\infty} \frac{kq^{kx}}{1-q^k}(\ln q)\left(1 + \frac{\ln q}{2}\right)$$

and hence

$$\frac{\partial}{\partial x}K(x;q) < 0 \qquad \text{for } \ 0 < q < e^{-2}.$$

The function $K(x;q)$ is monotonically decreasing function for $0 < q < e^{-2}$ and $\lim_{x\to\infty} K(x;q) = 0$, so

$$K(x;q) > 0 \qquad \text{for } \ 0 < q < e^{-2}.$$

Hence

$$U_1(x;q) > U_2(x;q) \qquad \text{for } \ 0 < q < e^{-2}$$

and then our upper bound in the inequality (22) refined the upper bound of the inequality (11) in the domain $0 < q < e^{-2}$.

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SOME NEW HERMITE-HADAMARD'S TYPE FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. In this paper, we study some new inequalities of Hermite-Hadamard's type fractional integral inequalities for functions whose absolute values of second derivatives are convex and concave . The obtained results generalize the existing Hermite-Hadamard type integral inequalities and their application in special means and numerical integration.

1. Introduction and Definitions

[5] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave. It is well known that the Hermite−Hadamard inequality plays an important role in nonlinear analysis. In the recent years, this classical inequality has been improved and generalized in a number of ways and a large number of research papers have been written on this inequality, (see, [1]−[5] and [7]−[10]) and the references therein.

In recent paper, [10] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

Remark 1. *For $\alpha = 1$, inequality 2 reduces to inequality 1.*

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $f \in L[a, b]$, the Reimann-Liouville integrals $J^\alpha_{a^+}$ and $J^\alpha_{b^-}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a^+} = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > \alpha$$

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and

$$J_{b^-}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < \alpha$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$ *is the Gamma function and* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in [6]. In this paper, we establish some new estimates of right Hermite−Hadamard inequality in the form of fractional integrals for functions whose absolute values of second derivatives are convex and concave.

2. Main Results

In order to prove our main results, we modified [7, Lemma 2]:

Lemma 1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I°*,the interior of* I*. Assume that* $a, b \in I^\circ$ *with* $a < b$ *and* $f'' \in L[a, b]$*, then the following identity for fractional integral with* $\alpha > 0$ *holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\ &\qquad = \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha) \left[f''(ta+(1-t)b) + f''((1-t)a+tb)\right] dt, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$.

Proof. Integrating by parts, we can state

$$\begin{aligned} I_1 &= \int_0^1 \left(t - t^{\alpha+1}\right) f''(ta+(1-t)b)dt \\ &= \frac{1}{b-a} \int_0^1 \left[1-(\alpha+1)t^\alpha\right] f'(ta+(1-t)b)dt \\ &= \frac{\alpha f(a)+f(b)}{(b-a)^2} - \frac{\alpha(\alpha+1)}{(b-a)^2} \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt, \end{aligned}$$

now making substitution $u = ta + (1-t)b$ and using the reduction formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $(\alpha > 0)$ for Euler gamma function,we have

$$I_1 = \frac{\alpha f(a)+f(b)}{(b-a)^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} J_{a^+}^{\alpha} f(b),$$

analogously:

$$I_2 = \frac{f(a)+\alpha f(b)}{(b-a)^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} J_{b^-}^{\alpha} f(a),$$

we obtain the desired result. □

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Using this lemma, we can obtain the following fractional integral inequalities.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''|$ is a convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \right| \\ \leq \frac{(b-a)^2}{\alpha+1} \beta(2, \alpha+1) \left[\frac{|f''(a)| + |f''(b)|}{2}\right], \end{aligned} \tag{3}$$

where β is Euler Beta function.

Proof. From lemma 1, using the convexity of $|f''|$ with properties of modulus, we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \right| \\ &\leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| \left(|f''(ta+(1-t)b)| + |f''((1-t)a+tb)|\right) dt \\ &\leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)} \left[\frac{|f''(a)| + |f''(b)|}{2}\right], \end{aligned}$$

where we have used the fact that

$$\int_0^1 t^2(1-t^\alpha)dt = \frac{1}{3} - \frac{1}{\alpha+3} \quad \text{and} \quad \int_0^1 t(1-t)(1-t^\alpha)dt = \frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}.$$

To prove the second inequality we used the fact that

$$|{t_1}^\alpha + {t_2}^\alpha| \leq |t_1 + t_2|^\alpha, \quad \text{for } \alpha \in [0,1] \text{ and } \forall\, t_1, t_2 \in [0,1],$$

and the Beta function,

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \qquad p, q > 0,$$

we obtaine

$$\int_0^1 t^2(1-t^\alpha)dt + \int_0^1 t(1-t)(1-t^\alpha)dt \leq \int_0^1 t(1-t)^\alpha dt = \beta(2, \alpha+1)$$

□

The corresponding versions for powers of the absolute value of the second derivative is incorporated in the following theorems.

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°. Assume that $p \in \mathbf{R}, p > 1$ such that $|f''|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \right| \\ \leq \frac{(b-a)^2}{\alpha+1} \beta^{1/p}(p+1, \alpha p+1) \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{1/q}, \end{aligned} \tag{4}$$

where β is Euler,s Beta function.

Proof. From lemma 1,using the convexity of $|f''|^q$ and the Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right)dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^\alpha)^p dt\right)^{1/p} \\
&\times\left[\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{1/q}+\left(\int_0^1 |f''((1-t)a+tb)|^q dt\right)^{1/q}\right] \\
&\leq \frac{(b-a)^2}{\alpha+1}\beta^{1/p}(p+1,\alpha p+1)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{1/q},
\end{aligned}$$

where $p^{-1}+q^{-1}=1$.

□

Theorem 4. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°. Assume that $q\geq 1$ such that $|f''|^q$ is convex function on I. Suppose that $a,b\in I^\circ$ with $a<b$ and $f''\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q\right.\right. \\
&\left.\left.+\frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{1/q}+\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q+\frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right]. \qquad (5)
\end{aligned}$$

Proof. Suppose that $a,b\in I^\circ$. From lemma 1and using the well-known power mean integral inequality with convexity of $|f''|^q$ for $q>1$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right)dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t(1-t^\alpha)dt\right)^{1-1/q}\left[\left(\int_0^1 t(1-t^\alpha)|f''(ta+(1-t)b)|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 t(1-t^\alpha)|f''((1-t)a+tb)|^q dt\right)^{1/q}\right] \\
&\leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q+\frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right. \\
&\left.+\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q+\frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right],
\end{aligned}$$

which completes the proof. □

Other similar results for concave functions may be extended in the following theorems.

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbf{R}, p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le \frac{(b-a)^2}{\alpha+1}\beta^{1/p}(1+p, 1+\alpha p)\left|f''\left(\frac{a+b}{2}\right)\right|, \tag{6}$$

where β is Euler,s Beta function.

Proof. By assumption, lemma 1,and the Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\
&\le \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)| + |f''((1-t)a+tb)|\right) dt \\
&\le \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^\alpha)^p dt\right)^{1/p} \\
&\times\left[\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{1/q} + \left(\int_0^1 |f''((1-t)a+tb)|^q dt\right)^{1/q}\right],
\end{aligned}$$

Since $|f''|^q$ is concave on $[a, b]$; we can use the integral Jensen's inequality to obtain

$$\begin{aligned}
\int_0^1 |f''(ta+(1-t)b)|^q dt &= \int_0^1 t^0 |f''(ta+(1-t)b)|^q dt \\
&\le \left(\int_0^1 t^0 dt\right)\left|f''\left(\frac{\int_0^1 t^0(ta+(1-t)b)dt}{\int_0^1 t^0 dt}\right)\right|^q = \left|f''\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}$$

Analogously:

$$\int_0^1 |f''((1-t)a+tb)|^q dt \le \left|f''\left(\frac{a+b}{2}\right)\right|,$$

which completes the proof. □

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°. Assume that $p \ge 1$ with $q = \frac{p}{p-1}$, such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)} \\
&\left[\left|f''\left(\frac{2\alpha+4}{3\alpha+9}a + \frac{\alpha+5}{3\alpha+9}b\right)\right| + \left|f''\left(\frac{\alpha+5}{3\alpha+9}a + \frac{2\alpha+4}{3\alpha+9}b\right)\right|\right].
\end{aligned} \tag{7}$$

Proof. Using the concavity of $|f''|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}
|f''(tx+(1-t)y)|^q &> t|f''(x)|^q + (1-t)|f''(y)|^q \\
&\ge (t|f''(x)| + (1-t)|f''(y)|)^q.
\end{aligned}$$

Hence

$$|f''(tx + (1-t)y)| \geq t|f''(x)| + (1-t)|f''(y)|,$$

so, $|f''|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned}
&\left|\int_0^1 t(1-t^\alpha) f''(ta + (1-t)b)dt\right| \\
&\leq \left(\int_0^1 t(1-t^\alpha)dt\right) \left|f''\left(\frac{\int_0^1 t(1-t^\alpha)(ta + (1-t)b)dt}{\int_0^1 t(1-t^\alpha)dt}\right)\right|^q \\
&= \frac{\alpha}{2(\alpha+2)} \left|f''\left(\frac{2(\alpha+2)a + (\alpha+5)b}{3(\alpha+3)}\right)\right|^q .
\end{aligned}$$

Analogously:

$$\left|\int_0^1 t(1-t^\alpha) f''((1-t)a + tb)dt\right| \leq \frac{\alpha}{2(\alpha+2)} \left|f''\left(\frac{(\alpha+5)a + 2(\alpha+2)b}{3(\alpha+3)}\right)\right|^q ,$$

which completes the proof. □

3. Applications

For $\alpha = 1$, Theorem 2 reduces to [9, Theorem 2] and Theorem 5 reduces to [7, Theorem 9].

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A note on a famous theorem of Pang and Zalcman

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Abstract

In this paper, by studying the famous theorem of Pang and Zalcman, we find a normal family and obtain a result, which is an improvement of Pang and Zalcman's theorem in some sense. Meanwhile, several examples are provided to show that our result's conditions are necessary.

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1 Introduction

Let D be a domain in $\mathbb{C}$, let f be a meromorphic function on D, and let S be a set with the finite elements. Set

$$\overline{E}_f(S) = f^{-1}(\{S\}) \cap D = \{z \in D : f(z) \in S\}.$$

In this paper, we assume that $f,\ g$ are two meromorphic functions on D and $S_1,\ S_2$ are two sets. We denote $\overline{E}_f(S_1) \subset \overline{E}_g(S_2)$ by $f(z) \in S_1 \Rightarrow g(z) \in S_2$. If $\overline{E}_f(S_1) = \overline{E}_g(S_2)$, we denote this condition by $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$. If the set S has only one element, say a, we denote $f(z) \in S$ by $f(z) = a$ (see [16]).

Now, let $\mathcal{F}$ be a family of meromorphic functions on a domain D. We say that $\mathcal{F}$ is normal in D if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence which converges to a meromorphic function f uniformly on each compact subset of D or a subsequence which converges to ∞ uniformly on each compact subset of D(see. [12]).

According to Bloch's principle, a lot of normality criteria have been obtained by starting from Picard type theorems. On the other hand, by Nevanlinnas famous five point theorem and Montel's theorem, it is interesting to establish normality criteria by using conditions known from a uniqueness theorem. A first attempt to this was made by W. Schwick (see. [13]).

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Up to now, many normality criteria have been obtained in this direction.(see. [1, 2, 3, 4, 6, 7, 8, 9, 11, 14, 15]). In 2000, Pang and Zalcman [11] proved a famous theorem.

Theorem A. *Let $\mathcal{F}$ be a family of functions meromorphic on a domain, all of whose zeros are of multiplicity (at least) k. If there exist $b \neq 0$ and $h > 0$ such that for every $f \in \mathcal{F}$, $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ and $0 < |f^{(k+1)}(z)| \leq h$ whenever $z \in \overline{E}_f(0)$, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Theorem A still holds if the condition $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ is replaced by $\overline{E}_f(0) \subset \overline{E}_{f^{(k)}}(b)$. Unfortunately, we neither give a negative example nor prove it true. This problem is very difficult even for the family of holomorphic functions(see. [1, 2, 15]). In this note, we study the special case that $k = 2$ and obtain the following result.

Theorem 1. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D, all of whose zeros are of multiplicity (at least) 2. If there exist a non-zero constant b and a positive constant M such that for every $f \in \mathcal{F}$,*
(1) $f(z) = 0 \Rightarrow f''(z) = b$,
(2) $f''(z) = b \Rightarrow 0 < |f'''(z)| \leq M$ and
(3) $f'^2(z) = Bf(z)$ whenever $z \in \overline{E}_{f''}(b)$,
where B is a non-constant, then $\mathcal{F}$ is normal in D.

Remark 1. *Here, if f omits a constant b, we can say that all the zeros of $f - b$ are of multiplicity ∞.*

Remark 2. *For the special cases that $\mathcal{F}$ is holomorphic functions and $k = 2$ of Theorem A, from $\overline{E}_f(0) = \overline{E}_{f''}(b)$, it is easy to deduce $\mathcal{F}$ satisfies the condition (3) of Theorem 1. Thus, in some sense, our result is an improvement of Theorem A. Meanwhile, we know that the condition $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ is not necessary for holomorphic functions in Theorem A.*

Remark 3. *We give an example to show that there exists a normal family $\mathcal{F}$ satisfying the conditions of Theorem 1.*

Consider the family $\mathcal{F} = \{f_n,\ n = 1,\ 2, \ldots\}$ on the unit disc, where

$$f_n(z) = e^{\frac{z}{n}},$$

so that

$$f'_n(z) = \frac{1}{n}\, e^{\frac{z}{n}} \quad \text{and} \quad f''_n(z) = \frac{1}{n^2}\, e^{\frac{z}{n}}.$$

Let b be a non-zero constant and $B = b$. Then, it is easy to see the family $\mathcal{F}$ satisfies the conditions of Theorem 1 and $\mathcal{F}$ is normal on the unit disc.

Remark 4. *The assumption $0 < |f''(z)| \leq M$ cannot be replaced by $|f''(z)| \leq M$. We have a counter-example [11] to show it.*

Consider the family $\mathcal{F} = \{f_n,\ n = 1,\ 2, \ldots\}$ on the unit disc, where

$$f_n(z) = \frac{1}{n^2}(e^{nz} + e^{-nz} - 2) = \frac{1}{n^2}e^{-nz}(e^{nz} - 1)^2,$$

so that

$$f_n^{(j)}(z) = n^{(j-2)}[e^{nz} + (-1)^j e^{-nz}], \quad j = 1,\ 2, \ldots$$

It is easy to see all the zeros of f_n are of multiplicity 2 and

$$f_n(z) = 0 \Leftrightarrow f_n''(z) = 2 \Rightarrow f_n'''(z) = 0.$$

While the family $\mathcal{F}$ is not normal on the unit disc.

2 Some Lemmas

In order to prove our theorems, we need several lemmas. For the convenience of the reader, we recall these lemmas here.

The following result is due to Pang and Zalcman, see [11].

Lemma 1. *Let $\mathcal{F}$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, if $\mathcal{F}$ is not normal, then there exist, for each $0 \leq \alpha \leq k$,*

(a) a number $0 < r < 1$;

(b) points z_n, $z_n < r$;

(c) functions $f_n \in \mathcal{F}$, and

(d) positive number $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant holomorphic function on $\mathbb{C}$, whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$ and $\rho(g) \leq 1$.

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative and $\rho(g)$ is the order of g.

Next, we need to introduce a result, see [5, Theorem 4.1] or [10], which plays an important part in the proof of our Theorem.

Lemma 2. *Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m(r, \frac{f^{(k)}}{f}) = o(\log\ r), \quad \text{as} \quad r \to \infty.$$

Finally, we recall the theorem of Chang, Fang and Zalcman, see [3], which is crucial to the proof of our theorem.

Lemma 3. *Let g be a non-constant entire function with $\rho(g) \leq 1$, let $k \geq 2$ be an integer, and let a be a non-zero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then*

$$g(z) = a(z - z_0),$$

where z_0 is a constant.

3 Proof of Theorem 1

Now, we prove Theorem 1. For every $f \in \mathcal{F}$, it follows from the assumption (1) that all the zeros of f have multiplicity 2. Noting that f is holomorphic in D, we can set

$$f = h^2, \tag{3.1}$$

where h is holomorphic in D. Differentiating (3.1) yields

$$f' = 2hh', \quad f'' = 2(h'^2 + hh'') \text{ and } f''' = 6h'h'' + 2hh'''. \tag{3.2}$$

We know that if $\mathcal{H} = \{h\}$ is normal in D, then $\mathcal{F}$ is normal in D. Thus, we need only to prove that $\mathcal{H}$ is normal in D. Suppose, to the contrary, that $\mathcal{H}$ is not normal in D.

It is clear from (3.1), the middle function of (3.2) and the condition (1) that

$$h = 0 \Rightarrow h' \in \{a, \ -a\} \tag{3.3}$$

where $2a^2 = b$. Combining the condition (2) and the last two functions of (3.2) yields

$$2(h'^2 + hh'') = b \Rightarrow 0 < |6h'h'' + 2hh'''| \leq M.$$

By Lemma 1, we can find $|z_n| < 1$, $\rho_n \to 0$ and $h_n \in \mathcal{H}$ such that

$$g_n(\xi) = \rho_n^{-1} h_n(z_n + \rho_n \xi) \to g(\xi) \tag{3.4}$$

locally uniformly on $\mathbb{C}$, where g is a non-constant entire function such that $g^\sharp(\xi) \leq g^\sharp(0) = M_1 = |a| + 1$. In particular $\rho(g) \leq 1$.

From (3.4), it is easy to obtain that

$$g_n'(\xi) = h_n'(z_n + \rho_n \xi) \to g'(\xi) \tag{3.5}$$

and

$$g_n''(\xi) = \rho_n h_n''(z_n + \rho_n \xi) \to g''(\xi)$$

locally uniformly on $\mathbb{C}$. Let

$$H_n(\xi) = 2[(g_n'(\xi))^2 + g_n(\xi) g_n''(\xi)].$$

Then, a routine calculation leads to

$$H_n(\xi) = f_n''(z_n + \rho_n \xi).$$

Set

$$G = 2(g'^2 + gg''). \tag{3.6}$$

Thus, we can deduce that

$$H_n(\xi) = 2[(g_n'(\xi))^2 + g_n(\xi) g_n''(\xi)] = f_n''(z_n + \rho_n \xi) \to 2[g'^2(\xi) + g(\xi) g''(\xi)] = G(\xi) \tag{3.7}$$

locally uniformly on $\mathbb{C}$.

We claim that

(I) $g(\xi)=0 \Rightarrow g'(\xi)\in\{a,\ -a\}$,
(II) $g(\xi)=0 \Rightarrow G(\xi)=b$ and
(III) $G(\xi)=b \Rightarrow G'(\xi)=0$.

First we prove (I).

Suppose that $g(\xi_0)=0$, then by Hurwitz's theorem and (3.4), there exist a sequence $\{\xi_n\}$ such that $\xi_n\to\xi_0$ and (for n sufficiently large)

$$g_n(\xi_n)=\rho_n^{-1}h_n(z_n+\rho_n\xi_n)=0.$$

Thus $h_n(z_n+\rho_n\xi_n)=0$. It is clear from (3.3) that

$$h_n'(z_n+\rho_n\xi_n)\in\{a,\ -a\}.$$

By (3.5), we obtain

$$g'(\xi_0)=\lim_{n\to\infty}h_n'(z_n+\rho_n\xi_n)\in\{a,\ -a\},$$

which implies $g(\xi)=0\Rightarrow g'(\xi)\in\{a,\ -a\}$. It is (I).

Similarly as above, we can get (II).

We prove (III) as follows.

We affirm that $G\neq b$. Otherwise, suppose that $G=b$. That is

$$2(g'^2+gg'')=b.$$

Integrating the above differential equation yields $2gg'=bz+c$, where c is a constant.

If g is a polynomial, then the equation $2gg'=bz+c$ implies that $\deg(g)=1$. From (I), we get $g'=a$ or $-a$. Then

$$|a|+1=g^\sharp(0)\leq|g'(0)|=|a|<|a|+1,$$

a contradiction.

If g is a transcendental entire function, then g' is also a transcendental entire function. By the lemma of logarithmic derivative, we have

$$\begin{aligned}2T(r,g')=T(r,g'^2)=m(r,g'^2)&\leq m(r,\frac{g'^2}{gg'})+m(r,gg')\\&=m(r,\frac{g'}{g})+m(r,(bz+c)/2)=S(r,g)=S(r,g'),\end{aligned}$$

which is a contradiction. Thus, we finish the proof of $G\neq b$.

Now, we return to the proof of (III).

Suppose that $G(\zeta_0)=b$. By Hurwitz's theorem and (3.7), there exist a sequence $\{\zeta_n\}$ such that $\zeta_n\to\zeta_0$ and (for n sufficiently large)

$$H_n(\zeta_n)=f_n''(z_n+\rho_n\zeta_n)=b.$$

It follows from the assumption (2) that

$$0<|f_n'''(z_n+\rho_n\xi_n)|\leq M.$$

With (3.7), we deduce

$$H_n'(\xi) = \rho_n f_n'''(z_n + \rho_n \xi) \to G'(\xi)$$

locally uniformly on $\mathbb{C}$. Thus, it is not difficult to deduce that

$$G'(\zeta_0) = \lim_{n\to\infty} \rho_n f_n'''(z_n + \rho_n \zeta_n) = 0,$$

which implies (III).

Now, we continue to prove our theorem.

Suppose that η_0 is a zero of g. That is $g(\eta_0) = 0$. By the claim (I) and (II), we get $g'(\eta_0) = a$ or $-a$ and $G(\eta_0) = b$. Differentiating (3.6) yields that

$$G' = 6g'g'' + 2gg'''. \tag{3.8}$$

It is clear from (III) and (3.8) that

$$G'(\eta_0) = 6g'(\eta_0)g''(\eta_0) + 2g(\eta_0)g'''(\eta_0) = 0.$$

Then, we obtain $g''(\eta_0) = 0$, which implies that

$$g(\xi) = 0 \Rightarrow g''(\xi) = 0.$$

Suppose that g is a polynomial with $\deg g = n$. Noting that (I), we know that g has only simple zeros. Thus, g has n distinct zeros z_m $(m = 1,\ 2, \dots, n)$. By (I), we get $g'(z_m) = a$ or $-a$ $(m = 1,\ 2, \dots, n)$. Thus, either $g' - a$ or $g' + a$ has at least p distinct zeros, here $p = \frac{n}{2}$ if n is an even number, $p = \frac{n+1}{2}$ if n is an odd number. Without loss of generality, we assume that $g'(z_m) - a = 0$ $(m = 1,\ 2, \dots, p)$. Obviously, $g''(z_m) = 0$ $(m = 1,\ 2, \dots, p)$. It implies that each z_m $(m = 1,\ 2, \dots, p)$ is a multiple zero of $g' - a$. Furthermore, it is easy to deduce that

$$n - 1 = \deg(g') = \deg(g' - a) \geq 2p \geq n,$$

a contradiction.

All the foregoing discussion shows that g is a transcendental entire function. Set

$$\phi = \frac{g''}{g}. \tag{3.9}$$

We find that ϕ is an entire function and $\rho(\phi) \leq \rho(g) \leq 1$. Combining Lemma 2 and the lemma of logarithmic derivative yields

$$T(r, \phi) = m(r, \phi) = m(r, \frac{g'}{g}) = o(\log\ r),$$

which implies ϕ is a non-zero constant. By solving the differential equation (3.9), we have

$$g = c_1 e^{\lambda\xi} + c_2 e^{-\lambda\xi}, \tag{3.10}$$

where $c_1,\ c_2$ are two constants and $\lambda^2 = \phi$.

Next, we prove that neither c_1 nor c_2 is zero. Otherwise, without loss of generality, suppose that $c_2 = 0$. Combining (3.6) and (3.10) yields

$$G(\xi) = 4c_1^2\lambda^2 e^{2\lambda\xi}$$

and

$$G'(\xi) = 8c_1^2\lambda^3 e^{2\lambda\xi}.$$

From (III) and the above two functions, it is easy to deduce a contradiction. Thus, we finish the proof of that $c_1,\ c_2$ are two non-zero constants.

Differentiating the function g yields

$$g'(\xi) = \lambda[c_1 e^{\lambda\xi} - c_2 e^{-\lambda\xi}] \tag{3.11}$$

and

$$g''(\xi) = \lambda^2[c_1 e^{\lambda\xi} + c_2 e^{-\lambda\xi}]. \tag{3.12}$$

From (3.9), it is obvious that

$$g(\xi) = 0 \Leftrightarrow g''(\xi) = 0. \tag{3.13}$$

By (3.10), we get

$$g(\xi) = 0 \Leftrightarrow e^{\lambda\xi} \in \{A,\ -A\},$$

here $A = \sqrt{-\frac{c_2}{c_1}}$. From (I), we can see that

$$e^{\lambda\xi} = A \Rightarrow g'(\xi) \in \{a,\ -a\}.$$

Noting that the form of g', without loss of generality, we can assume that

$$e^{\lambda\xi} = A \Rightarrow g'(\xi) = a.$$

Thus, we have

$$g'(\xi) - a = e^{-\lambda\xi}[c_1\lambda e^{2\lambda\xi} - a e^{\lambda\xi} - c_2\lambda] = A_1 e^{-\lambda\xi}[e^{\lambda\xi} - A][e^{\lambda\xi} - A_2], \tag{3.14}$$

where A_1 and A_2 are two non-zero constants. Observing that (3.13), we get

$$e^{\lambda\xi} = A \Rightarrow g''(\xi) = 0,$$

which implies that all the zeros of $e^{\lambda\xi} - A$ are multiple zeros of $g' - a$. Therefore, we deduce that $A_2 = A$. Rewriting (3.14) as

$$g'(\xi) - a = A_1 e^{-\lambda\xi}[e^{\lambda\xi} - A]^2.$$

It indicates that $g'(\xi) = a \Leftrightarrow e^{\lambda\xi} = A$. Meanwhile, with the same argument, we can deduce that $g'(\xi) = -a \Leftrightarrow e^{\lambda\xi} = -A$. Combining the two cases yields that $g'(\xi) \in \{a,\ -a\} \Leftrightarrow e^{\lambda\xi} \in \{A,\ -A\}$. Thus, we have

$$g(\xi) = 0 \Leftrightarrow g'(\xi) \in \{a,\ -a\}.$$

Furthermore, we obtain

$$g = 0 \Leftrightarrow g' \in \{a, \ -a\} \Leftrightarrow g'' = 0 \Rightarrow G = b. \tag{3.15}$$

Noting that (3.11), we know $g' - a$ has multiple zeros. Differentiating (3.12) yields

$$g''' = \lambda^3[c_1 e^{\lambda\xi} - c_2 e^{-\lambda\xi}].$$

From the above function, it is not difficult to deduce that $g' - a$ has zeros with multiplicity 2.

Suppose $g'(\alpha_0) = a$. By (3.15) we get $g(\alpha_0) = 0$ and $G(\alpha_0) = b$. From (III), we find that α_0 is a multiple zero of $G - b$. Noting that $G \neq b$, then there exists $\delta > 0$ such that

$$g(\xi) \neq 0, \ G(\xi) - b \neq 0,$$

in $D'(\alpha_0, \delta) = \{\xi : 0 < |\xi - \alpha_0| < \delta\}$. By (3.7), there exists $\varepsilon_0 > 0$ such that, for each $0 < \delta' < \delta$ and sufficiently large n,

$$|f_n''(z_n + \rho_n\xi) - b - (G(\xi) - b)| < \varepsilon_0 < |G(\xi) - b|$$

on the circle $C(\alpha_0, \delta') = \{\xi : |\xi - \alpha_0| = \delta'\}$. By Rouché theorem, there exist $\{\alpha_{n,j}\}$ $(j = 1, 2)$ tending to α_0, such that, for each large n

$$H_n(\alpha_{n,j}) = f_n''(z_n + \rho_n\alpha_{n,j}) = b \ \ (j = 1, 2). \tag{3.16}$$

And the assumption (2) implies that $\alpha_{n,1} \neq \alpha_{n,2}$. Then, for $j = 1, \ 2$, it follows from the assumption (3) that

$$f_n'(z_n + \rho_n\alpha_{n,j})^2 = Bf_n(z_n + \rho_n\alpha_{n,j}). \tag{3.17}$$

We distinguish the following three cases.

Case 1. For $j = 1, \ 2$, there exist infinitely many n_t satisfying

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0.$$

Then we get $h_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0$ $(j = 1, \ 2)$. It follows from (3.4) and Rouché theorem that α_0 is a zero of g with multiplicity at least 2. But g has only simple zeros, a contradiction.

Case 2. For $j = 1, \ 2$, there exist infinitely many n_t satisfying

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) \neq 0.$$

We claim that there exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h_{n_t}'(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = a \ \ (j = 1, \ 2). \tag{3.18}$$

Without loss of generality, we need only to prove it holds for $j = 1$. By (3.1), the first item of (3.2) and (3.17), it is not difficult to deduce

$$h_{n_t}'(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) \in \{d, \ -d\},$$

where $d = \frac{\sqrt{B}}{2}$ is a constant. It is clear from the assumption $f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) \neq 0$ that d is a non-zero constant.

Then, there must exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = e, \tag{3.19}$$

here $e \in \{d, \ -d\}$ is a non-zero constant. Then

$$g'(\alpha_0) = \lim_{n\to\infty} h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = e.$$

Noting that $g'(\alpha_0) = a$, we get $e = a$. With (3.19), we prove the claim.

On the other hand, by the middle item of (3.2), (3.16), (3.18) and the assumption of Case 2, we can deduce $h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0$ for $j = 1, 2$.

Observing that $h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0$ for $j = 1, 2$, so each $\alpha_{n_t,j}$ $(j = 1, 2)$ is a multiple zero of $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$. It follows from (3.5) and Rouché theorem that α_0 is a zero of $g' - a$ with multiplicity at least 4, a contradiction.

Case 3. There exist infinitely many n_t satisfying either

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = 0, \quad f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) \neq 0$$

or

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) \neq 0, \quad f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = 0.$$

Without loss of generality, suppose that

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = 0 \ \text{ and } \ f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) \neq 0.$$

Similarly as Case 2, there exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = a,$$

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = a \ \text{ and } \ h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = 0.$$

That means $\alpha_{n,2}$ is a multiple zero of $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$. Meanwhile, $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$ has another zero $\alpha_{n,1}$. Then, it follows from (3.6) and Rouché theorem that α_0 is a zero of $g' - a$ with multiplicity at least 3, a contradiction.

Thus, we get $g'(\alpha_0) \neq a$, which is a contradiction.

All the above discussion yields $\mathcal{H}$ is normal in D, so $\mathcal{F}$ is also normal in D.

Hence, we complete the proof of Theorem 1.

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Asymptotic representations in Stochastic Process Approximations

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Abstract. Fourier cosine transforms are applied widely in applied mathematics, PDEs, and signal processing. In this paper we give some asymptotic representations in stochastic Fourier cosine analyses based on our decomposition. More importantly, we propose a stochastic Fourier cosine expansion with a polynomial term. In this expansion, Fourier cosine coefficients decay fast such that we can reconstruct stochastic processes by using the least coefficients. Although our research is in the setting of stochastic processes, our results are also new for deterministic functions.

Key words: stochastic process, decomposition, Fourier cosine coefficient, asymptotic representation

1. Introduction

Fourier cosine transforms are applied widely in applied mathematics, PDEs, and signal processing. In this paper, we will deeply study stochastic Fourier cosine analyses. Although our research is in the setting of stochastic processes, our results are also new for deterministic functions.

We will decompose a stochastic process on $[0,1]$ into two parts, the first part is a stochastic polynomial and the residual is such that its odd-order derivatives vanish at endpoints of $[0,1]$. Based on this decomposition, for stochastic processes, we will give some asymptotic representations in Fourier cosine analyses which includes asymptotic representations of Fourier cosine coefficients; asymptotic representations of the expectations and variances of Fourier cosine coefficients; and asymptotic representations of the mean square error of partial sums of Fourier cosine series.

More importantly, in order to reconstruct the stochastic process by using the least Fourier cosine coefficients, based on our decomposition, we propose a stochastic Fourier cosine expansion with a polynomial term, where the polynomial is determined by odd-order derivatives of the stochastic process at endpoints of $[0,1]$. Since Fourier cosine coefficients in this expansion decay fast, this expansion provides a good approximation tool which can attain the best approximation order.

This paper is organized as follows. In Section 2 we recall calculus of stochastic processes. In Section 3 we give decompositions of stochastic processes. In Sections 4 we give the asymptotic representations of Fourier cosine coefficients for stochastic processes. In Section 5 we give the approximation of partial sums for stochastic processes. In Section 6 we propose stochastic Fourier cosine expansions with a polynomial term. In Section 7, we give the corresponding results for deterministic functions.

We need some notations. The notation $[\lambda]$ is the integral part of a positive real number λ. For sequences $\{a_n\}$ and $\{b_n\}$, the notation $a_n = O(b_n)$ means that $|a_n| \le K|b_n|$ for any n, where K is a constant; for $a_n \to 0$ and $b_n \to 0$, the notation $a_n = o(b_n)$ mean that $\frac{a_n}{b_n} \to 0$. The notation $C_s([0,1])$ is the set of continuous stochastic processes on $[0,1]$, here the letter s implies that it is a family of stochastic processes. Denote the expectation, second-order moment, and variance of a stochastic process ξ by $E[\,\xi\,]$, $E[\,\xi^2\,]$ and $\mathrm{Var}(\xi)$, respectively. Denote the covariance of stochastic processes ξ and η by $\mathrm{Cov}(\xi,\eta)$.

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2. Calculus of stochastic processes

We recall some concepts in calculus of stochastic processes [1,2].

Let $\{\xi_n\}_1^\infty$ be a sequence of stochastic variables and ξ be a stochastic variable. If $\lim\limits_{n\to\infty} E\left[|\xi_n-\xi|^2\right]=0$, we say $\{\xi_n\}$ converges to ξ in the mean square sense.

Stochastic processes are a generalization of deterministic functions. If for each fixed $t\in[a,b]$, $X(t)$ is a stochastic variable , then we say $X(t)$ is a stochastic process on $[a,b]$. In this paper, we always assume that a stochastic process $X(t)$ is real-valued and satisfies $E[\,X^2(t)\,]<\infty$ for each t.

Let $X=X(t)\,(t\in[a,b])$ be a stochastic process. If

$$\lim_{s\to t_0} E[(X(s)-Y)^2]=0,$$

we say the stochastic process X has the limit Y at t_0 in the mean square sense, denoted by $\lim\limits_{s\to t_0} X(s)=Y\ (m.s.)$. If

$$\lim_{s\to t_0} E[(X(s)-X(t_0))^2]=0,$$

we say the process X is continuous at $t_0\in[a,b]$ in the mean square sense. If

$$\lim_{s\to t_0}\frac{X(s)-X(t_0)}{s-t_0}=Y\ (m.s.),$$

we say the process X has derivative Y in the mean square sense and denoted by $X'(t_0)=Y$.

Let $X=X(t)\,(a\le t\le b)$ be a stochastic process, for given a partition of $[a,b]$:

$$a=t_0<t_1<\cdots<t_n=b.$$

Denote $\delta=\max\limits_k|t_k-t_{k-1}|$. Arbitrarily take $\zeta_k\in[t_k,t_{k+1}]$ $(k=1,...,n)$. If the following limit exists and

$$\lim_{\delta\to 0}\sum_{k=1}^n X(\zeta_k)(t_k-t_{k-1})=I\ (m.s.),$$

then we say the integral $\int_a^b X(t)\,\mathrm{d}t=I$.

For convenience, through this paper, we often omit the notation $m.s.$.

If a stochastic process X is differentiable, then

$$E[\,X'(t)\,]=(E[\,X(t)\,])',$$

i.e., the expectation of the derivative of a stochastic process is equal to the derivative of its expectation.

If a stochastic process $X(t)$ is integrable over $[a,b]$, then

$$E\left[\int_a^b X(t)\,\mathrm{d}t\right]=\int_a^b E[\,X(t)\,]\,\mathrm{d}t,$$

i.e., the expectation of an integral of a stochastic process is equal to the integral of its expectation.

If X is a differentiable stochastic process and f is a differentiable deterministic function, then

$$(Xf)'=X'f+Xf'.$$

If X is a continuously, differentiable stochastic process on $[a,b]$, then

$$X(b)-X(a)=\int_a^b X'(t)\,\mathrm{d}t.$$

If X is a continuously differentiable stochastic process and f is a continuously, differentiable deterministic function, then

$$\int_a^b X' f \,\mathrm{d}t = X f|_a^b - \int_a^b X f' \,\mathrm{d}t.$$

i.e., the integration formula by parts holds.

If a stochastic process $\xi(t)$ on $[0,1]$ satisfies $E\left[\int_0^1 |\xi(t)|^2 \mathrm{d}t\right] < \infty$, then it can be expanded into the stochastic Fourier cosine series

$$\xi(t) = \sum_{n=0}^{\infty} c_n(\xi)\,\cos(\pi n t)$$

in the mean square sense, where

$$c_0 = \int_0^1 \xi(t)\,\mathrm{d}t, \qquad c_n(\xi) = 2\int_0^1 \xi(t)\,\cos(\pi n t)\,\mathrm{d}t \quad (n = 1, 2, ...)$$

and the Parseval identity holds:

$$E[\,\|\,\xi\,\|_2^2\,] = E[\,c_0^2(\xi)\,] + \frac{1}{2}\sum_{n=1}^{\infty} E[\,c_n^2(\xi)\,],$$

where $\|\,\xi\,\|_2^2 = \int_0^1 \xi^2(t)\,\mathrm{d}t$.

3. Decomposition of stichastic processes

At first we give a sequence of fundamental polynomials. Based on these polynomials, we give decompositions of stochastic processes to satisfy the need of studying stochastic Fourier cosine analyses. Although this decomposition is given in the setting of stochastic processes, it is also new for deterministic functions.

Define fundamental polynomials as follows:

$$p_0(t) = \frac{1}{2}t^2 - \frac{1}{6},$$

$$p_n(t) = \frac{1}{(2n+2)!}\left(t^{2n+2} - \frac{1}{2n+3}\right) + \sum_{k=0}^{n-1} \tau_{k,n}\left(t^{2k+2} - \frac{1}{2k+3}\right) \ (n = 1, 2, ...),$$

where the coefficients $\tau_{k,n}$ satisfy

$$\sum_{k=j}^{n-1} \frac{(2k+2)!}{(2k-2j+1)!}\tau_{k,n} = -\frac{1}{(2n-2j+1)!} \qquad (j = 0, 1, ..., n-1).$$

Especially, when $n = 1, 2$, the fundamental polynomials are

$$p_1(t) = \frac{1}{24}t^4 - \frac{1}{12}t^2 + \frac{7}{360},$$

$$p_2(t) = \frac{1}{720}\left(t^6 - 5t^4 + 7t^2 - \frac{31}{21}\right).$$

We can directly check the following proposition.

Proposition 3.1. For $n, j = 0, 1, ...,$ the fundamental polynomials $p_n(t)$ satisfy

$$\int_0^1 p_n(t)\,\mathrm{d}t = 0, \qquad p_n^{(2j+1)}(0) = 0, \qquad p_n^{(2j+1)}(1) = \begin{cases} 0, & n \neq j, \\ 1, & n = j. \end{cases}$$

Let ξ be a stochastic process on $[0,1]$ and $\xi^{(l)} \in C([0,1])\,(l \geq 2)$. We construct a stochastic polynomial:

$$g(t) = \sum_{k=0}^{[\frac{l}{2}]-1} \left(-\xi^{(2k+1)}(0)\,p_k(1-t) + \xi^{(2k+1)}(1)\,p_k(t) \right), \tag{3.1}$$

where p_k are fundamental polynomials stated as above. We call g a stochastic polynomial associated with the stochastic process ξ. From (3.1), we see that g is uniquely determined by odd-order derivatives of ξ at endpoints of $[0,1]$ and its degree $\leq l$.

Proposition 3.2. Let ξ be a stochastic process on $[0,1]$ and $\xi^{(l)} \in C([0,1])$ $(l \geq 2)$ and g be stated in (3.1). Then the residual $h = \xi - g$ satisfies

$$h^{(l)} \in C([0,1]), \qquad h^{(2j+1)}(0) = h^{(2j+1)}(1) = 0 \quad (j = 0, 1, ..., [l/2]-1). \tag{3.2}$$

Proof. From $\xi^{(l)} \in C([0,1])$ and g is a stochastic polynomial, we know that $h^{(l)} = \xi^{(l)} - g^{(l)}$ is a continuous stochastic process on $[0,1]$. Differentiating both sides of (3.1), we get

$$g^{(2j+1)}(t) = \sum_{k=0}^{[\frac{l}{2}]-1} \left(\xi^{(2k+1)}(0)p_k^{(2j+1)}(1-t) + \xi^{(2k+1)}(1)p_k^{(2j+1)}(t) \right).$$

Again, by Proposition 3.1, we deduce that

$$g^{(2j+1)}(0) = \xi^{(2j+1)}(0), \qquad g^{(2j+1)}(1) = \xi^{(2j+1)}(1) \quad (j = 0, ..., [l/2]-1).$$

From this and $h = \xi - g$, we get Proposition 3.2. □

Let ξ be a stochastic process and $\xi^{(l)} \in C([0,1])$ $(l \geq 2)$. From Proposition 3.2, we get a decomposition formula:

$$\xi = g + h \tag{3.3}$$

where g is a stochastic polynomial associated with ξ stated in (3.1) and the residual h satisfies (3.2).

For the residual h on $[0,1]$, we first do an even extension and then do a 2−periodic extension to the whole real axis. The obtained stochastic process is denoted by $\widetilde{h}$, i.e.,

$$\widetilde{h}(t) = h(t) \quad (t \in [0,1]), \qquad \widetilde{h}(-t) = \widetilde{h}(t), \quad \widetilde{h}(t+2) = \widetilde{h}(t) \quad (t \in \mathbb{R}).$$

Definition 3.3. For a stochastic process ξ on $[0,1]$, if, there exists a constant $K > 0$ and $0 \leq \alpha \leq 1$ such that ξ satisfies

$$\begin{aligned} &E[\,|\xi(t) - \xi(s)|\,] \leq K|t-s|^{\alpha}, \\ &E[\,|\xi(t) - \xi(s)|^2\,] \leq K|t-s|^{2\alpha} \quad (t,\, s \in [0,1]), \end{aligned} \tag{3.4}$$

then we say $\xi \in H_s^{\alpha}([0,1])$; if the $l-$order derivative $\xi^{(l)} \in H_s^{\alpha}([0,1])$, then we say $\xi \in W^l\,H_s^{\alpha}([0,1])$; if $\xi^{(l)} \in C_s([0,1])$, then we say $\xi \in W_s^l$. For convenience, we denote $W^l\,H_s^0([0,1]) = W_s^l([0,1])$.

Similarly, we can define the stochastic processes family $W^l\,H_s^{\alpha}(\mathbb{R})$ where $\mathbb{R}$ is the whole real axis.

Proposition 3.4. Let ξ be a stochastic process and $\xi \in W^l\,H_s^{\alpha}([0,1])$ $(0 \leq \alpha \leq 1$ and l is even numbers), and $\xi = g + h$, where g is the stochastic polynomial associated with ξ stated in (3.1), and let $\widetilde{h}$ be a 2−periodic even extension of the residual h to whole real axis $\mathbb{R}$. Then $\widetilde{h} \in W^l\,H_s^{\alpha}(\mathbb{R})$.

Proof. In order to $\widetilde{h}^{(l)} \in C(\mathbb{R})$, since $\widetilde{h}$ is the 2−periodic even extension of h and $h^{(l)} \in C([0,1])$, we only need prove that $\widetilde{h}^{(l)}$ exist at endpoints of $[0,1]$.

Let l be odd. By Proposition 3.2, we deduce that the right derivatives $\widetilde{h}^{(l)}(0+) = h^{(l)}(0+) = 0$ and the left derivative $\widetilde{h}^{(l)}(1-) = h^{(l)}(1-) = 0$. Since l is odd, we deduce that the left derivative $\widetilde{h}^{(l)}(0-) = 0$. Noticing that $\widetilde{h}$ is 2−periodic, we obtain the right derivative $\widetilde{h}^{(l)}(1+) = \widetilde{h}^{(l)}(-1+) = 0$. So the derivatives $\widetilde{h}^{(l)}$ exist at endpoints of $[0,1]$.

Let l be even. Then

$$\widetilde{h}^{(l)}(0+) = \widetilde{h}^{(l)}(0-), \qquad \widetilde{h}^{(l)}(1-) = \widetilde{h}^{(l)}(-1-) = \widetilde{h}^{(l)}(1+),$$

i.e., the derivatives $\widetilde{h}^{(l)}$ also exist at endpoints of $[0,1]$.

Now we prove from the assumption $h^{(l)} \in H_s^\alpha([0,1])$ $(0 < \alpha \le 1)$ that $\widetilde{h}^{(l)} \in H_s^\alpha(\mathbb{R})$.

Let $t \in (0,1)$ and $s \in (-1,0)$. If l is even, then $\widetilde{h}^{(l)}(s) = \widetilde{h}^{(l)}(-s) = h^{(l)}(-s)$ and $\widetilde{h}^{(l)}(t) = h^{(l)}(t)$, and so

$$E[\,|\widetilde{h}^{(l)}(t) - \widetilde{h}^{(l)}(s)|\,] = E[\,|h^{(l)}(t) - h^{(l)}(0)|\,] + E[\,|h^{(l)}(0) - h^{(l)}(-s)|\,] \le 2K|t-s|^\alpha, \tag{3.5}$$

$$E[\,|\widetilde{h}^{(l)}(t) - \widetilde{h}^{(l)}(s)|^2\,] \le 2E[\,|h^{(l)}(t) - h^{(l)}(0)|^2\,] + 2E[\,|h^{(l)}(0) - h^{(l)}(-s)|^2\,] \le 4K|t-s|^{2\alpha}. \tag{3.6}$$

If l is odd, then $h^{(l)}(0) = 0$ and $\widetilde{h}^{(l)}(-s) = -\widetilde{h}^{(l)}(s)$, and so (3.5) and (3.6) also hold. For arbitrary two points $t,\ s \in \mathbb{R}$, it is easy to deduce that (3.5) and (3.6) hold. Therefore, by Definition 3.3, we have $\widetilde{h} \in W^l\, H_s^\alpha(\mathbb{R})$. Proposition 3.4 is proved. □

4. Asymptotic representations of Fourier cosine coefficients

Let ξ be a stochastic process on $[0,1]$ and $\xi^{(l)} \in W^l\, H_s^\alpha([0,1])$ $(l \ge 2,\ 0 \le \alpha \le 1)$. We expand it into Fourier cosine series,

$$\xi(t) = \sum_{n=0}^{\infty} c_n(\xi)\, \cos(\pi n t),$$

where

$$c_0(\xi) = \int_0^1 \xi(t)\,\mathrm{d}t, \qquad c_n(\xi) = 2\int_0^1 \xi(t)\, \cos(\pi n t)\,\mathrm{d}t \ \ (n = 1,2,...).$$

We give the asymptotic representations of its Fourier cosine coefficients. By using the decomposition formula (3.3) and the linear property of Fourier coefficients, we have

$$c_n(\xi) = c_n(g) + c_n(h) \qquad (n \ge 0).$$

Again, by (3.1), we get

$$c_n(g) = \sum_{i=0}^{[\frac{l}{2}]-1} \left(-\xi^{(2i+1)}(0)\, c_n(p_i(1-\cdot)) + \xi^{(2i+1)}(1)\, c_n(p_i) \right).$$

Noticing that Proposition 3.1 gives values of odd-order derivatives of fundamental polynomials p_i at endpoints of the interval $[0,1]$, using integration by parts, we deduce that the Fourier cosine coefficients $c_n(p_i)$ satisfy

$$c_n(p_i) = 2\int_0^1 p_i(t)\, \cos(\pi n t)\,\mathrm{d}t = \frac{2(-1)^{i+1}}{n^{2i+1}\pi^{2i+1}} \int_0^1 p_i^{(2i+1)}(t)\, \sin(\pi n t)\,\mathrm{d}t = \frac{2(-1)^{n+i}}{\pi^{2i+2} n^{2i+2}} \quad (n = 1,2,...;\ i = 0,1,...).$$

From this, we get

$$c_n(p_i(1-\cdot)) = (-1)^n c_n(p_i) = \frac{2(-1)^i}{\pi^{2i+2} n^{2i+2}}.$$

For $n=0$, by $\int_0^1 p_i(t)\,\mathrm{d}t = 0$, we have $c_0(p_i) = c_0(p_i(1-\cdot)) = 0$. Therefore, $c_0(g)=0$ and for $n \neq 0$,

$$c_n(g) = -\frac{2}{(n\pi)^2} \sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}} \left(\xi^{(2i+1)}(0) - (-1)^n \xi^{(2i+1)}(1) \right), \tag{4.1}$$

and so

$$c_n(\xi) = -\frac{2}{(n\pi)^2} \sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}} \left(\xi^{(2i+1)}(0) - (-1)^n \xi^{(2i+1)}(1) \right) + c_n(h).$$

In the decomposition (3.3), g is a stochastic polynomial. Therefore, from $\xi^{(l)} \in C_s([0,1])$, we deduce that $h^{(l)} \in C_s([0,1])$. Since h is a $l-$order differentiable stochastic process and $\cos(\pi n t)$ is a deterministic function, using the integration by parts l times, by Proposition 3.2, we deduce that if l is an even number, for $n \neq 0$,

$$c_n(h) = 2\int_0^1 h(t)\,\cos(\pi n t)\,\mathrm{d}t = \frac{2(-1)^{\frac{l}{2}}}{(n\pi)^l} \int_0^1 h^{(l)}(t)\,\cos(\pi n t)\,\mathrm{d}t, \tag{4.2}$$

Since the expectation and the integral can be exchanged, we have

$$E[\,c_n(h)\,] = \frac{2(-1)^{\frac{l}{2}}}{(n\pi)^l} \int_0^1 E[\,h^{(l)}(t)\,]\,\cos(\pi n t)\,\mathrm{d}t, \tag{4.3}$$

where $E[\,h^{(l)}(t)\,]$ is a deterministic function. From $h^l \in C_s([0,1])$, we deduce that $E[\,h^{(l)}\,] \in C([0,1])$. By the Riemann-Lebesgue lemma [6], we have $E[\,c_n(h)\,] = o\left(\frac{1}{n^l}\right)$.

For $\alpha > 0$, by the assumption $h \in W^l\, H_s^\alpha([0,1])$, using Proposition 3.4, we have $h^{(l)} \in H_s^\alpha([0,1])$, and so

$$|E[\,h^{(l)}(t)\,] - E[\,h^{(l)}(s)\,]| = |E[\,h^{(l)}(t) - h^{(l)}(s)\,]| \le E[\,|h^{(l)}(t) - h^{(l)}(s)|\,] \le K|t-s|^\alpha.$$

This means that the deterministic function $E[\,h^{(l)}\,] \in \mathrm{Lip}\,\alpha$ on $[0,1]$. By a well-known result in Fourier analyses [6], we have

$$2\int_0^1 E[\,h^{(l)}(t)\,]\,\cos(\pi n t)\,\mathrm{d}t = O\left(\frac{1}{n^{l+\alpha}}\right).$$

From this and (4.3), we have $E[\,c_n(h)\,] = O\left(\frac{1}{n^{l+\alpha}}\right)$.

By (4.2), we deduce that

$$c_n^2(h) = \frac{4}{(n\pi)^{2l}} \int_0^1 h^{(l)}(t)\,\cos(\pi n t)\,\mathrm{d}t \int_0^1 h^{(l)}(s)\,\cos(\pi n s)\,\mathrm{d}s.$$

It can be written a double integral:

$$c_n^2(h) = \frac{4}{(n\pi)^{2l}} \int_{[0,1]^2} h^{(l)}(t)\,h^{(l)}(s)\,\cos(\pi n t)\,\cos(\pi n s)\,\mathrm{d}t\,\mathrm{d}s,$$

and so

$$E[\,c_n^2(h)\,] = \frac{4}{(n\pi)^{2l}} \int_{[0,1]^2} E[\,h^{(l)}(t) h^{(l)}(s)\,]\,\cos(\pi n t)\,\cos(\pi n s)\,\mathrm{d}t\,\mathrm{d}s.$$

Let $\widetilde{h}$ be the 2−periodic even extension of h. Then

$$E[\,c_n^2(h)\,] = \frac{2}{(n\pi)^{2l}} \int\limits_{[-1,1]\times[0,1]} E[\,\widetilde{h}^{(l)}(t)h^{(l)}(s)\,]\,\cos(\pi nt)\,\cos(\pi ns)\,\mathrm{d}t\mathrm{d}s, \tag{4.4}$$

where $E[\,\widetilde{h}^{(l)}(t)h^{(l)}(s)\,]$ is a bivariate deterministic function and $E[\,\widetilde{h}^{(l)}(t)h^{(l)}(s)\,] \in C([0,1]^2)$. By the bivariate Riemann-Lebesgue lemma [4], we have $E[\,c_n^2(h)\,] = o\left(\frac{1}{n^{2l}}\right)$.

For $\alpha > 0$, doing transform of variable in (4.4), we have

$$E[\,c_n^2(h)\,] = -\frac{2}{(n\pi)^{2l}} \int\limits_{[-1-\frac{1}{n},1-\frac{1}{n}]\times[0,1]} E[\,\widetilde{h}^{(l)}(t+\frac{1}{n})\,\widetilde{h}^{(l)}(s)\,]\,\cos(\pi nt)\,\cos(\pi ns)\,\mathrm{d}t\mathrm{d}s.$$

Since $\widetilde{h}^{(l)}$ is 2-periodic, we have

$$E[\,c_n^2(h)\,] = -\frac{2}{(n\pi)^{2l}} \int\limits_{[-1,1]\times[0,1]} E[\,\widetilde{h}^{(l)}(t+\frac{1}{n})\,\widetilde{h}^{(l)}(s)\,]\,\cos(\pi nt)\,\cos(\pi ns)\,\mathrm{d}t\mathrm{d}s.$$

Adding this and (4.4), we get

$$2E[\,c_n^2(h)\,] = \frac{2}{(n\pi)^{2l}} \int\limits_{[-1,1]\times[0,1]} \left(E[\,\widetilde{h}^{(l)}(t)\,\widetilde{h}^{(l)}(s)\,] - E[\,\widetilde{h}^{(l)}(t+\frac{1}{n})\widetilde{h}^{(l)}(s)\,] \right) \cos(\pi nt)\,\cos(\pi ns)\,\mathrm{d}t\mathrm{d}s.$$

By the assumption $\xi \in W^l\,H_s^{\alpha}([0,1])$, using Proposition 3.4, we have $\widetilde{h} \in W^l\,H_s^{\alpha}(\mathbb{R})$. Again, by the Schwarz's inequality in the probability theory and (3.4), we have

$$\begin{aligned}
\left| E\left[\widetilde{h}^{(l)}(t)\,\widetilde{h}^{(l)}(s)\right] - E\left[\widetilde{h}^{(l)}(t+\tfrac{1}{n})\widetilde{h}^{(l)}(s)\right]\right| &= \left| E\left[\widetilde{h}^{(l)}(t) - \widetilde{h}^{(l)}(t+\tfrac{1}{n})\right]\widetilde{h}^{(l)}(s)\,\right| \\
&\le \left(E\left[\left(\widetilde{h}^{(l)}(t) - \widetilde{h}^{(l)}(t+\tfrac{1}{n})\right)^2\right]\right)^{\frac{1}{2}} \left(E[\,(\widetilde{h}^{(l)}(s))^2\,]\right)^{\frac{1}{2}} \\
&\le K\tfrac{1}{n^{\alpha}},
\end{aligned}$$

where K is a constant. From this, we deduce that

$$E[\,c_n^2(h)\,] \le \frac{2}{(n\pi)^{2l}} \int\limits_{[-1,1]\times[0,1]} \left| E\left[\widetilde{h}^{(l)}(t)\,\widetilde{h}^{(l)}(s)\right] - E\left[\widetilde{h}^{(l)}(t+\frac{1}{n})\widetilde{h}^{(l)}(s)\right]\right| \mathrm{d}t\mathrm{d}s = O\left(\frac{1}{n^{2l+\alpha}}\right).$$

By using $\mathrm{Var}(c_n(h)) \le E[\,c_n^2(h)\,]$, we get $\mathrm{Var}(c_n(h)) = O\left(\frac{1}{n^{2l+\alpha}}\right)$.

If l is an odd number, using integration by parts, we have

$$c_n(h) = \frac{2(-1)^{\frac{l+1}{2}}}{(n\pi)^l} \int_0^1 h^{(l)}(t)\,\sin(\pi nt)\,\mathrm{d}t.$$

Similarly, we deduce that expectations, second-order moments, and variances of $c_n(h)$ have same estimates. So, for any n, we have

$$E[\,c_n(h)\,] = O\left(\frac{1}{n^{l+\alpha}}\right), \qquad E[\,c_n^2(h)\,] = O\left(\frac{1}{n^{2l+\alpha}}\right), \qquad \mathrm{Var}(c_n(h)) = O\left(\frac{1}{n^{2l+\alpha}}\right). \tag{4.5}$$

Therefore, we get the following theorem, in which the error term $c_n(h)$ is denoted by r_n.

Theorem 4.1. Let ξ be a stochastic process on $[0,1]$ and $\xi \in W^l\, H_s^\alpha([0,1])$ ($l \geq 2,\ 0 \leq \alpha \leq 1$). Then their Fourier cosine coefficients satisfy an asymptotic representation:

$$c_n(\xi) = -\frac{2}{(n\pi)^2}\left(\sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}}(\xi^{(2i+1)}(0) - (-1)^n \xi^{(2i+1)}(1))\right) + r_n,$$

where the error r_n satisfies

$$E[\,r_n\,] = O\left(\frac{1}{n^{l+\alpha}}\right), \quad E[\,r_n^2\,] = O\left(\frac{1}{n^{2l+\alpha}}\right), \quad \mathrm{Var}(r_n) = O\left(\frac{1}{n^{2l+\alpha}}\right).$$

From Theorem 4.1, the expectation of Fourier cosine coefficients of ξ has the asymptotic representation:

$$E[\,c_n(\xi)\,] = -\frac{2}{(n\pi)^2}\sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}}\left(E[\,\xi^{(2i+1)}(0)\,] - (-1)^n E[\,\xi^{(2i+1)}(1)\,]\right) + O\left(\frac{1}{n^{l+\alpha}}\right). \tag{4.6}$$

By $c_n(\xi) = c_n(g) + c_n(h)$, we get

$$c_n^2(\xi) = c_n^2(g) + \tau_n(\xi),$$

where $\tau_n(\xi) = 2c_n(g)c_n(h) + c_n^2(h)$, and so

$$E[\,c_n^2(\xi)\,] = E[\,c_n^2(g)\,] + E[\,\tau_n(\xi)\,]. \tag{4.7}$$

From (4.1), we get

$$c_n^2(g) = \frac{4}{(n\pi)^4}\sum_{i,j=0}^{[\frac{l}{2}]-1} \frac{(-1)^{i+j}}{(n\pi)^{2(i+j)}}\left(\xi^{(2i+1)}(0) - (-1)^n\xi^{(2i+1)}(1)\right)\left(\xi^{(2j+1)}(0) - (-1)^n\xi^{(2j+1)}(1)\right).$$

A direct computation shows that

$$E[\,c_n^2(g)\,] = \frac{4}{(n\pi)^4}\sum_{0\leq i+j\leq[\frac{l}{2}]-1} \frac{(-1)^{i+j}}{(n\pi)^{2i+2j}}\left(\,\mu'_{i,j} - (-1)^n\lambda'_{i,j}\,\right),$$

where

$$\mu'_{i,j} = E\left[\,\xi^{(2i+1)}(0)\xi^{(2j+1)}(0)\,\right] + E\left[\,\xi^{(2i+1)}(1)\xi^{(2j+1)}(1)\,\right],$$
$$\lambda'_{i,j} = E\left[\,\xi^{(2i+1)}(0)\xi^{(2j+1)}(1)\,\right] + E\left[\,\xi^{(2i+1)}(1)\xi^{(2j+1)}(0)\,\right].$$

By the Schwarz inequality in the probability theory, we have

$$E[\,\tau_n(\xi)\,] \leq 2E[\,|c_n(g)c_n(h)|\,] + E[\,c_n^2(h)\,] \leq 2(\,E[\,c_n^2(g)\,]\,)^{\frac{1}{2}}\cdot(\,E[\,c_n^2(h)\,]\,)^{\frac{1}{2}} + E[\,c_n^2(h)\,]$$

By (4.1) and (4.5), we get $E[\,c_n^2(g)\,] = O\left(\frac{1}{n^4}\right)$ and $E[\,c_n^2(h)\,] = O\left(\frac{1}{n^{2l+\alpha}}\right)$. So

$$E[\,\tau_n(\xi)\,] = O\left(\frac{1}{n^{l+2+\frac{\alpha}{2}}}\right).$$

From this and (4.7), the second-order moment of Fourier cosine coefficients of ξ:

$$E[\,c_n^2(\xi)\,] = \frac{4}{(n\pi)^4}\sum_{0\leq i+j\leq m} \frac{(-1)^{i+j}}{(n\pi)^{2i+2j}}\left(\,\mu'_{i,j} - (-1)^n\lambda'_{i,j}\,\right) + O\left(\frac{1}{n^{l+2+\frac{\alpha}{2}}}\right). \tag{4.8}$$

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By (4.6), we deduce that

$$(E[\,c_n(\xi)\,])^2 = \frac{4}{(n\pi)^4} \sum_{0\le i+j\le m} \frac{(-1)^{i+j}}{(n\pi)^{2(i+j)}} \,(\mu''_{i,j} - (-1)^n \lambda''_{i,j}) + O\left(\frac{1}{n^{l+2+\frac{\alpha}{2}}}\right), \tag{4.9}$$

where

$$\mu''_{i,j} = E[\,\xi^{(2i+1)}(0)\,]\,E[\,\xi^{(2j+1)}(0)\,] + E[\,\xi^{(2i+1)}(1)\,]\,E[\,\xi^{(2j+1)}(1)\,],$$
$$\lambda''_{i,j} = E[\,\xi^{(2i+1)}(0)\,]\,E[\,\xi^{(2j+1)}(1)\,] + E[\,\xi^{(2i+1)}(1)\,]\,E[\,\xi^{(2j+1)}(0)\,].$$

By a known formula, the covariance of stochastic variables η_1 and η_2 is equal to $E[\,\eta_1\eta_2\,] - E[\eta_1]E[\eta_2]$. So we have

$$\mu'_{i,j} - \mu''_{i,j} = \mathrm{Cov}(\xi^{(2i+1)}(0),\, \xi^{(2j+1)}(0)) + \mathrm{Cov}(\xi^{(2i+1)}(1),\, \xi^{(2j+1)}(1)) =: \mu_{i.j}$$

and

$$\lambda'_{i,j} - \lambda''_{i,j} = \mathrm{Cov}(\xi^{(2i+1)}(0),\, \xi^{(2j+1)}(1)) + \mathrm{Cov}(\xi^{(2i+1)}(1),\, \xi^{(2j+1)}(0)) =: \lambda_{i.j}.$$

Again, using a known formula: $\mathrm{Var}(c_n(\xi)) = E[\,c_n^2(\xi)\,] - (\,E[\,c_n(\xi)\,]\,)^2$, by (4.8) and (4.9), and (4.6), we have

Theorem 4.2. Let ξ be a stochastic process on $[0,1]$ and $\xi \in W^l\, H_s^\alpha([0,1])$ ($l \ge 2$, $0 \le \alpha \le 1$). Then its Fourier cosine coefficients satisfy

$$E[\,c_n(\xi)\,] = -\frac{2}{(n\pi)^2} \sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}} \left(\,E[\,\xi^{(2i+1)}(0)\,] - (-1)^n\, E[\,\xi^{(2i+1)}(1)\,]\,\right) + O\left(\frac{1}{n^{l+\alpha}}\right),$$

$$\mathrm{Var}(c_n(\xi)) = \frac{4}{(n\pi)^4} \sum_{0\le i+j\le [\frac{l}{2}]-1} \frac{(-1)^{i+j}}{(n\pi)^{2(i+j)}} \,(\,\mu_{i,j} - (-1)^n \lambda_{i,j}\,) + O\left(\frac{1}{n^{l+2+\frac{\alpha}{2}}}\right).$$

where

$$\mu_{i,j} = \mathrm{Cov}(\xi^{(2i+1)}(0),\, \xi^{(2j+1)}(0)) + \mathrm{Cov}(\xi^{(2i+1)}(1),\, \xi^{(2j+1)}(1)),$$
$$\lambda_{i,j} = \mathrm{Cov}(\xi^{(2i+1)}(0),\, \xi^{(2j+1)}(1)) + \mathrm{Cov}(\xi^{(2i+1)}(1),\, \xi^{(2j+1)}(0))$$

and the covariance $\mathrm{Cov}(\eta_1, \eta_2) = E[\eta_1\eta_2] - E[\eta_1]E[\eta_2]$.

Especially, we have

$$E[\,c_n(\xi)\,] = -\frac{2}{(n\pi)^2}\,(\,E[\,\xi'(0)\,] - (-1)^n\, E[\,\xi'(1)\,]\,) + o\left(\frac{1}{n^2}\right),$$

$$E[\,c_n^2(\xi)\,] = \frac{4}{(n\pi)^4} E[\,(\xi'(0) - (-1)^n \xi'(1))^2\,] + o\left(\frac{1}{n^4}\right). \tag{4.10}$$

From this, we know that

$$E[\,c_n(\xi)\,] = O\left(\frac{1}{n^2}\right), \qquad E[\,c_n^2(\xi)\,] = O\left(\frac{1}{n^4}\right) \tag{4.11}$$

and they cannot be improved.

5. Asymptotic representations of errors of partial sums

Let ξ be a stochastic process on $[0,1]$ and $\xi \in W^l\, H_s^\alpha([0,1])$ ($l \ge 2$, $0 \le \alpha \le 1$). We first consider that the approximation order of partial sums $S_N(\xi)$ of Fourier cosine series. By (4.11), using the Parseval identity, we obtain the approximation order of partial sums:

$$\rho = \left(\,E[\,\|\, S_N(\xi) - \xi\,\|_2^2\,]\,\right)^{\frac{1}{2}} = \left(\frac{1}{2}\sum_{n=N}^{\infty} E[\,c_n^2(\xi)\,]\right)^{\frac{1}{2}} = O\left(\frac{1}{N^{3/2}}\right).$$

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We will see from Theorem 5.1 that the approximation order $O\left(\frac{1}{N^{3/2}}\right)$ cannot be improved.

By (4.10), we obtain that

$$E[\, c_{2n}^2(\xi)\,] + E[\, c_{2n+1}^2(\xi)\,] = \frac{1}{2(n\pi)^4}\, E[\, |\xi'(0)|^2 + |\xi'(1)|^2\,] + o\left(\frac{1}{n^4}\right).$$

Using the Parseval identity, we deduce that

$$2E[\,\|\, S_{2N-1}(\xi) - \xi\,\|_2^2\,] = \sum_{n=N}^{\infty} (E[\, c_{2n}^2(\xi)\,] + E[\, c_{2n+1}^2(\xi)\,]) = \frac{1}{6\pi^4 N^3}\, E[\, |\xi'(0)|^2 + |\xi'(1)|^2\,] + o\left(\frac{1}{N^3}\right).$$

Again, by the equality $E[\,\|\, S_{2N}(\xi) - \xi\,\|_2^2\,] = E[\,\|\, S_{2N-1}(\xi) - \xi\,\|_2^2\,] + O\left(\frac{1}{N^4}\right)$, we get

Theorem 5.1. Let ξ be a stochastic process on $[0,1]$ and $\xi \in W^l\, H_s^{\alpha}([0,1])$ $(l \geq 2,\, 0 < \alpha \leq 1)$. Then, for any $N \in \mathbb{Z}_+$, the approximation error ρ of partial sums $S_N(\xi)$ of its Fourier cosine series satisfies

$$\rho^2 = E\left[\,\|\, S_N(\xi) - \xi\,\|_2^2\,\right] = \frac{2}{3\pi^4 N^3}\, E\left[\, |\xi'(0)|^2 + |\xi'(1)|^2\,\right] + o\left(\frac{1}{N^3}\right).$$

6. Fourier cosine expansions with a polynomial term

Suppose that $\xi \in W^l\, H_s^{\alpha}([0,1])$ $(l \geq 2,\, 0 \leq \alpha \leq 1)$. In the decomposition (3.3): $\xi = g + h$, where $g(t)$ is a stochastic polynomial of degree $\leq l$ and is stated in (3.1), expanding $h(t)$ into the Fourier cosine series, we get

$$\xi(t) = \sum_{i=0}^{[\frac{l}{2}]-1} \left(-\xi^{(2i+1)}(0) p_i(1-t) + \xi^{(2i+1)}(1) p_i(t) \right) + \sum_{n=0}^{\infty} c_n(h)\, \cos(\pi n t), \tag{6.1}$$

where $p_i(t)$ are fundamental polynomials (see Section 3). We call it a Fourier cosine expansion with a stochastic polynomial term. By (4.5), we know that coefficients $c_n(h)$ decay fast and

$$E[\, c_n(h)\,] = O\left(\frac{1}{n^{l+\alpha}}\right), \qquad E[\, c_n^2(h)\,] = O\left(\frac{1}{n^{2l+\alpha}}\right), \qquad \mathrm{Var}(c_n(h)) = O\left(\frac{1}{n^{2l+\alpha}}\right). \tag{6.2}$$

Therefore, we can reconstruct ξ using less coefficients $c_n(h)$ and odd-order derivatives of ξ at the endpoints of $[0,1]$. Take partial sum of Fourier cosine expansion (6.1),

$$U_N(\xi;\, t) = \sum_{i=0}^{[\frac{l}{2}]-1} \left(-\xi^{(2i+1)}(0) p_i(1-t) + \xi^{(2i+1)}(1) p_i(t) \right) + \sum_{n=0}^{N-1} c_n(h)\, \cos(\pi n t).$$

By the Parseval identity, we get

$$E[\,\|\, U_N(\xi) - \xi\,\|_2^2\,] = E\left[\,\|\, \sum_{n=N}^{\infty} c_n(h)\, \cos(\pi n t)\,\|_2^2\,\right] = \frac{1}{2} \sum_{n=N}^{\infty} E[\, c_n^2(h)\,] = O\left(\frac{1}{N^{2l-1+\alpha}}\right).$$

Theorem 6.1. Let ξ be a stochastic process on $[0,1]$ and $\xi \in W^l\, H_s^{\alpha}([0,1])$ $(l \geq 2,\, 0 \leq \alpha \leq 1)$, and let $U_N(\xi)$ be the partial sum of Fourier cosine expansion (6.1) with a stochastic polynomial term. Then approximation errors ρ satisfy

$$\rho^2 = E[\,\|\, \xi - U_N(\xi)\,\|_2^2\,] = O\left(\frac{1}{N^{2l-1+\alpha}}\right).$$

The partial sum $U_N(\xi;\, t)$ is a combination of a stochastic algebraic polynomial of degree $\leq l$ and is a stochastic cosine polynomial of degree N, where l is the smoothness index of ξ and N determines the error of

approximation. Comparing Theorem 6.1 with Theorem 5.1, we see that $U_N(\xi;\, t)$ is a good approximation tool. By using it, we can obtain the best approximation order of $\xi \in W^l\, H_s^\alpha([0,1])$.

For $\xi \in W^l\, H_s^\alpha([0,1])$, comparing its Fourier cosine series

$$\xi(t) = \sum_{n=0}^{\infty} c_n(\xi)\, \cos(\pi n t) \tag{6.3}$$

and its Fourier cosine expansion (6.1) with a stochastic polynomial term, we find that even if smoothness index l is very large, the decay speed of Fourier cosine coefficients $c_n(\xi)$ are only

$$E[\, c_n(\xi)\,] = O\left(\frac{1}{n^2}\right), \qquad \mathrm{Var}(c_n(\xi)) = O\left(\frac{1}{n^4}\right),$$

and they cannot be improved (see Theorem 4.1). However, by (6.2), the Fourier cosine coefficients $c_n(h)$ decay fast as l is large. Therefore, we can reconstruct the stochastic process ξ by the least Fourier cosine coefficients $c_n(h)$ and values of odd-order derivatives of ξ at endpoints of $[0,1]$.

7. The case of deterministic functions

When a stochastic process ξ is a deterministic function, we denote it by f. The family $W^l\, H_s^\alpha([0,1])$ is reduced to an ordinary family $W^l\, H^\alpha([0,1])$, i.e., the family of f satisfying the condition $f^{(l)} \in \mathrm{Lip}\,\alpha$ on $[0,1]$. As a special case of stochastic processes, for deterministic functions, we get the following results. These results are also new.

Let f be a function on $[0,1]$ and $f^{(l)} \in \mathrm{Lip}\,\alpha([0,1])$ ($l \ge 2$, $0 \le \alpha \le 1$). We give the decomposition formula $f(t) = q_l(t) + \gamma(t)$, where

$$q_l(t) = \sum_{i=0}^{[\frac{l}{2}]-1} \left(-f^{(2i+1)}(0)\, p_i(1-t) + f^{(2i+1)}(1)\, p_i(t) \right)$$

is a polynomial of degree $\le l$ and the error $\gamma(t)$ satisfies $\gamma^{(2i+1)}(0) = \gamma^{(2i+1)}(1) = 0$ ($i = 0, 1, ..., [l/2]-1$) (by Proposition 3.2). Let $\widetilde{\gamma}$ be the 2−periodic even extension of γ. Then $\widetilde{\gamma}^{(l)} \in \mathrm{Lip}\,\alpha(\mathbb{R})$ (by Proposition 3.4).

Based on this decomposition, we study the Fourier cosine series of f [3,5,6]:

$$f(t) = \sum_{n=0}^{\infty} c_n(f)\, \cos(\pi n t), \tag{7.1}$$

where $c_0(f) = \int_0^1 f(t)\, \mathrm{d}t$ and $c_n(f) = 2\int_0^1 f(t)\, \cos(\pi n t)\, \mathrm{d}t$ ($n = 1, 2, ...$). We get the asymptotic representation of Fourier cosine coefficients of f:

$$c_n(f) = -\frac{2}{(n\pi)^2} \left(\sum_{i=0}^{[\frac{l}{2}]-1} \frac{(-1)^i}{(n\pi)^{2i}} \left(f^{(2i+1)}(0) - (-1)^n\, f^{(2i+1)}(0)\right) \right) + O\left(\frac{1}{n^{l+\alpha}}\right) \quad \text{(by Theorem 4.1)}$$

and the asymptotic representation of squares of partial sum errors of the Fourier cosine series:

$$\|\, S_N(f) - f\,\|_2^2 = \int_0^1 |S_N(f;\, t) - f(t)|^2\, \mathrm{d}t = \frac{2}{3\pi^4 N^3}\left(|f'(0)|^2 + |f'(1)|^2\right) + o\left(\frac{1}{N^3}\right) \quad \text{(by Theorem 5.1).}$$

In this decomposition: $f(t) = q_l(t) + \gamma(t)$, expanding $\gamma(t)$ into the Fourier cosine series, we obtain a Fourier cosine expansion with a polynomial term:

$$f(t) = q_l(t) + \sum_{n=0}^{\infty} c_n(\gamma)\, \cos(\pi n t), \tag{7.2}$$

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where $c_n(\gamma) = O\left(\frac{1}{n^{l+\alpha}}\right)$. Its partial sum $U_N(f;\, t) = q_l(t) + \sum\limits_{n=0}^{N-1} c_n(\gamma)\, \cos(\pi n t)$ satisfies

$$\| U_N(f) - f \|_2 = \left(\frac{1}{2} \sum_{n=N}^{\infty} c_n^2(\gamma) \right)^{\frac{1}{2}} = O\left(\frac{1}{N^{l-\frac{1}{2}+\alpha}} \right).$$

Comparing the series (7.1) with the series (7.2), we find that we can reconstruct f by the least Fourier cosine coefficients $c_n(\gamma)$ in the Fourier cosine expansion (7.2) with a polynomial term as the smoothness index l is large.

For example, let f be a function on $[0, 1]$ and $f^{(4)} \in \mathrm{Lip}\, 1$. We give a decomposition

$$f(t) = q_4(t) + \gamma(t) \qquad (0 \le t \le 1),$$

where

$$\begin{aligned} q_4(t) &= -f'(0)\left(\tfrac{1}{2}t^2 - t + \tfrac{1}{3}\right) + f'(1)\left(\tfrac{1}{2}t^2 - 6\right) \\ &- f'''(0)\left(\tfrac{1}{24}t^4 - \tfrac{1}{6}t^3 + \tfrac{1}{6}t^2 - \tfrac{1}{45}\right) + f'''(1)\left(\tfrac{1}{24}t^4 - \tfrac{1}{12}t^2 + \tfrac{7}{360}\right) \end{aligned}$$

is a polynomial of degree 4 and the residual $\gamma(t)$ satisfies $\gamma'(0) = \gamma'(1) = \gamma'''(0) = \gamma'''(1) = 0$. The Fourier cosine expansion with a polynomial term is

$$f(t) = q_4(t) + \sum_{n=0}^{\infty} c_n(\gamma)\, \cos(\pi n t) \qquad (0 \le t \le 1), \tag{7.3}$$

where

$$c_n(\gamma) = 2 \int_0^1 \gamma(t)\, \cos(\pi n t)\, \mathrm{d}t = O\left(\frac{1}{n^9} \right).$$

We can reconstruct f by N Fourier cosine coefficients $\{c_n(\gamma)\}_{n=0}^{N-1}$ and four derivatives $f'(0)$, $f'(1)$, $f'''(0)$, and $f'''(1)$ with the help of the partial sum of the series (7.3):

$$U_N(f) = q_4(t) + \sum_{n=0}^{N-1} c_n(\gamma)\, \cos(\pi n t),$$

i.e., $f \approx U_N(f)$. The approximation error is

$$\rho = \| U_N(f) - f \|_2 = O\left(\frac{1}{N^{\frac{9}{2}}} \right).$$

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q-poly-Cauchy numbers associated with Jackson integral

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Abstract

As generalizations of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\widehat{c}_n^{(k)}$, we introduce the concept about q-analogues or extensions of the poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ and of the second kind $\widehat{c}_{n,q}^{(k)}$, and investigate their properties. We also study q-analogues or extensions of the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ and of the second kind $\widehat{c}_n^{(k)}(z)$.

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1 Introduction

The second author introduced poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ ([9]) by the integral of the falling factorial:

$$\begin{aligned} c_n^{(k)} &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)_n dx_1 \dots dx_k \\ &= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k}{n} dx_1 \dots dx_k\,, \end{aligned}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \ge 1)$ with $(x)_0 = 1$. If $k = 1$, then $c_n^{(1)} = c_n$ is the classical Cauchy number ([3, 15]). The number $c_n/n!$ is sometimes referred to as the Bernoulli number of the second kind ([2, 6, 16]). The poly-Cauchy numbers of the first kind $c_n^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind.

$$c_n^{(k)} = \sum_{m=0}^{n} \frac{s(n,m)}{(m+1)^k} \quad (n \ge 0,\ k \ge 1)$$

([9, Theorem 1]), where $s(n,m)$ is the (signed) Stirling number of the first kind, determined by the rising factorial:

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n} (-1)^{n-m} s(n,m) x^m\,.$$

The generating function of the poly-Cauchy numbers $c_n^{(k)}$ is given by

$$\mathrm{Lif}_k\big(\ln(1+t)\big) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!}$$

([9, Theorem 2]), where $\mathrm{Lif}_k(z)$ is called *polylogarithm factorial* function (or simply, *poly-factorial* function) defined by

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}\,.$$

By this definition, k is not restricted to positive integers in $c_n^{(k)}$.

Similarly, define the poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ ([9]) by

$$\begin{aligned} \widehat{c}_n^{(k)} &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 \cdots x_k)_n dx_1 \dots dx_k \\ &= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 \cdots x_k}{n} dx_1 \dots dx_k\,. \end{aligned}$$

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If $k = 1$, then $\widehat{c}_n^{(1)} = \widehat{c}_n$ is the classical Cauchy number of the second kind ([3, 15]). The poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind.

$$\widehat{c}_n^{(k)} = \sum_{m=0}^{n} \frac{(-1)^m s(n,m)}{(m+1)^k} \quad (n \ge 0,\ k \ge 1)$$

([9, Theorem 4]). The generating function of the poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ is given by

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr) = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)} \frac{t^n}{n!}$$

([9, Theorem 5]).

The poly-Cauchy numbers have been considered as analogues of poly-Bernoulli numbers $B_n^{(k)}$, generalizing the classical Bernoulli numbers B_n, defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\,.$$

The poly-Cauchy numbers (of the both kind) can be extended in various ways (e.g. [7, 10]). A different direction of generalizations of Cauchy numbers is about *Hypergeometric Cauchy numbers* ([12]). Arithmetical and combinatorial properties including sums of products have been studied ([11, 13, 14]).

In this paper, as essential generalizations of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\widehat{c}_n^{(k)}$, we introduce the concept about q-analogues or extensions of the poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ and of the second kind $\widehat{c}_{n,q}^{(k)}$ by using Jackson's q-integrals, and investigate their properties. We also study q-analogues or extensions of the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ and of the second kind $\widehat{c}_n^{(k)}(z)$. Note that similar notations are used in [10, 12], but have completely different meanings.

2 Jackson's q-derivative and q-integral

Let q be a real number with $0 \le q < 1$. Jackson's q-derivative (see e.g. [1, (10.2.3)], [4], [5, (1.1)]) is defined by

$$D_q f = \frac{d_q f}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}$$

and Jackson's q-integral ([1, (10.1.3)], [4], [5, (1.3)]) is defined by

$$\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^{\infty} f(q^n x) q^n\,.$$

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For example, when $f(x)=x^m$ for some nonnegative integer m,

$$
\begin{aligned}
D_q f &= \frac{x^m - q^m x^m}{(1-q)x}\\
&= [m]_q x^{m-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^x t^m d_q t &= (1-q)x \sum_{n=0}^\infty q^{mn} x^m q^n\\
&= (1-q)x^{m+1} \sum_{n=0}^\infty q^{n(m+1)}\\
&= \frac{x^{m+1}}{[m+1]_q}\,,
\end{aligned}
$$

where

$$
[x]_q = \frac{1-q^x}{1-q}
$$

is the q-number with $[0]_q=0$. Note that $\lim_{q\to 1}[x]_q = x$.

3 q-poly-Cauchy numbers of the first kind

Let n and k be integers with $n\ge 0$. Define q-analogue of poly-Cauchy numbers of the first kind (or simply, *q-poly-Cauchy numbers of the first kind*) as

$$
\begin{aligned}
c_{n,q}^{(k)} &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1\cdots x_k)_n d_q x_1 \cdots d_q x_k\\
&= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1\cdots x_k}{n} d_q x_1 \cdots d_q x_k\,.
\end{aligned}
$$

The q-poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ can be written in terms of the Stirling numbers of the first kind.

Theorem 1

$$
c_{n,q}^{(k)} = \sum_{l=0}^n \frac{s(n,l)}{[l+1]_q^k} \quad (n\ge 0)\,,
$$

where $s(n,l)$ is the (signed) Stirling number of the first kind.

Remark. If $q\to 1$, then by $c_{n,q}^{(k)} \to c_n^{(k)}$ and $[l+1]_q \to l+1$, Theorem 1 is reduced to Theorem 1 in [9].

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Proof. By

$$(x)_n = \sum_{l=0}^{n} s(n,l)x^l \,,$$

we have

$$\begin{aligned} c_{n,q}^{(k)} &= \sum_{l=0}^{n} s(n,l) \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1^l \cdots x_k^l d_q x_1 \cdots d_q x_k \\ &= \sum_{l=0}^{n} \frac{s(n,l)}{[l+1]_q^k} \,. \end{aligned}$$

∎

Define q-polyfactorial function $\mathrm{Lif}_{k,q}(z)$ by

$$\mathrm{Lif}_{k,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n![n+1]_q^k} \,.$$

Note that by $[n+1]_q \to n+1$ as $q \to 1$, we get $\mathrm{Lif}_{k,q}(z) \to \mathrm{Lif}_k(z)$, which is the polyfactorial function.

Theorem 2 *The generating function of the q-poly-Cauchy numbers $c_{n,q}^{(k)}$ is given by the following:*

$$\sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{t^n}{n!} = \mathrm{Lif}_{k,q}\big(\ln(1+t)\big) \,.$$

Remark. As $q \to 1$, then by $c_{n,q}^{(k)} \to c_n^{(k)}$ and $\mathrm{Lif}_{k,q}(z) \to \mathrm{Lif}_k(z)$, Theorem 2 is reduced to Theorem 2 in [9].

Proof. By the definition

$$\begin{aligned} \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{t^n}{n!} &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{n=0}^{\infty} \binom{x_1 \cdots x_k}{n} t^n d_q x_1 \cdots d_q x_k \\ &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (1+t)^{x_1 \cdots x_k} d_q x_1 \cdots d_q x_k \\ &= \sum_{n=0}^{\infty} \frac{\big(\ln(1+t)\big)^n}{n!} \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1^n \cdots x_k^n d_q x_1 \cdots d_q x_k \\ &= \sum_{n=0}^{\infty} \frac{\big(\ln(1+t)\big)^n}{n![n+1]_q^k} = \mathrm{Lif}_{k,q}\big(\ln(1+t)\big) \,. \end{aligned}$$

∎

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Let $S(n,m)$ be the Stirling number of the second kind, determined by

$$S(n,m)=\frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n\,.$$

Corollary 1 *For $m\ge 0$ we have*

$$\sum_{m=0}^{n}S(n,m)c_{m,q}^{(k)}=\frac{1}{[n+1]_q^k}\,.$$

Remark. If $q\to 1$, then by $c_{m,q}^{(k)}\to c_m^{(k)}$ and $[n+1]_q\to n+1$, Corollary 1 is reduced to Theorem 3 in [9].

Proof. Replace t by e^t-1 in Theorem 2. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n![n+1]_q^k}&=\mathrm{Lif}_{k,q}(t)\\&=\sum_{m=0}^{\infty}c_{m,q}^{(k)}\frac{(e^t-1)^m}{m!}\\&=\sum_{m=0}^{\infty}c_{m,q}^{(k)}\sum_{n=0}^{\infty}S(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}c_{m,q}^{(k)}S(n,m)\frac{t^n}{n!}\,.\end{aligned}$$

By comparing the coefficients of the both sides, we get the desired result. ∎

Denote $\zeta_q(s)$ be the q-zeta functions defined by

$$\sum_{n=1}^{\infty}\frac{1}{[n]_q^s}\,.$$

Note that $\lim_{q\to 1}\zeta_q(s)=\zeta(s)$, which is the Riemann zeta function.

Proposition 1 *For $k\ge 1$ we have*

$$\begin{aligned}\zeta_q(k)&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}S(n,m)c_{m,q}^{(k)}\,,\\\zeta(k)&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}S(n,m)c_{m}^{(k)}\,.\end{aligned}$$

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Proof. By Corollary 1

$$\zeta_q(k)=\sum_{n=0}^\infty\frac{1}{[n+1]_q^k}=\sum_{n=0}^\infty\sum_{m=0}^n S(n,m)c_{m,q}^{(k)}\,,$$

which is the first identity. As $q\to 1$, we get the second identity. ∎

Example. If $k=2$ in the second identity of Proposition 1, then

$$\frac{\pi^2}{6}=\sum_{n=0}^\infty\sum_{m=0}^n S(n,m)c_m^{(2)}\,.$$

4 q-poly-Cauchy numbers of the second kind

Let n and k be integers with $n\ge 0$. Define q-analogue of poly-Cauchy numbers of the second kind (or simply, *q-poly-Cauchy numbers of the second kind*) as

$$\begin{aligned}\widehat{c}_{n,q}^{(k)}&=\underbrace{\int_0^1\cdots\int_0^1}_{k}(-x_1\cdots x_k)_n d_qx_1\cdots d_qx_k\\&=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{-x_1\cdots x_k}{n}d_qx_1\cdots d_qx_k\,.\end{aligned}$$

The q-poly-Cauchy numbers of the second kind $\widehat{c}_{n,q}^{(k)}$ can be written in terms of the Stirling numbers of the first kind.

Theorem 3

$$\widehat{c}_{n,q}^{(k)}=\sum_{l=0}^n\frac{(-1)^l s(n,l)}{[l+1]_q^k}\quad(n\ge 0)\,.$$

Remark. If $q\to 1$, then by $\widehat{c}_{n,q}^{(k)}\to\widehat{c}_n^{(k)}$ and $[l+1]_q\to l+1$, Theorem 3 is reduced to Theorem 4 in [9].

Proof. We have

$$\begin{aligned}\widehat{c}_{n,q}^{(k)}&=\sum_{l=0}^n(-1)^l s(n,l)\underbrace{\int_0^1\cdots\int_0^1}_{k}x_1^l\cdots x_k^l d_qx_1\cdots d_qx_k\\&=\sum_{l=0}^n\frac{(-1)^l s(n,l)}{[l+1]_q^k}\,.\end{aligned}$$

∎

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Similarly to Theorem 2, we have the following.

Theorem 4 *The generating function of the q-poly-Cauchy numbers of the second kind* $\widehat{c}_{n,q}^{(k)}$ *is given by the following:*

$$\sum_{n=0}^{\infty} \widehat{c}_{n,q}^{(k)} \frac{t^n}{n!} = \mathrm{Lif}_{k,q}\bigl(-\ln(1+t)\bigr)\,.$$

Remark. As $q \to 1$, then by $\widehat{c}_{n,q}^{(k)} \to \widehat{c}_n^{(k)}$ and $\mathrm{Lif}_{k,q}(z) \to \mathrm{Lif}_k(z)$, Theorem 4 is reduced to Theorem 5 in [9].

Replace t by $e^t - 1$ in Theorem 4. Then we have the following. As $q \to 1$, Corollary 2 is reduced to Theorem 6 in [9]. The proof is similar to that of Corollary 1 and is omitted.

Corollary 2 *For* $m \geq 0$ *we have*

$$\sum_{m=0}^{n} S(n,m)\widehat{c}_{m,q}^{(k)} = \frac{(-1)^n}{[n+1]_q^k}\,.$$

5 Some relations between two kinds of q-poly-Cauchy numbers

There are some relations between q-poly-Cauchy numbers of the first kind and those of the second kind.

Theorem 5 *For* $n \geq 1$ *we have*

$$(-1)^n \frac{c_{n,q}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{c}_{m,q}^{(k)}}{m!}\,,$$

$$(-1)^n \frac{\widehat{c}_{n,q}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_{m,q}^{(k)}}{m!}\,.$$

Remark. If $q \to 1$, then by $c_{n,q}^{(k)} \to c_n^{(k)}$ and $\widehat{c}_{n,q}^{(k)} \to \widehat{c}_n^{(k)}$, Theorem 5 is reduced to Theorem 7 in [9].

Proof. We shall prove the first identity. The second one is proven similarly and omitted. By the definition of $c_{n,q}^{(k)}$

$$(-1)^n \frac{c_{n,q}^{(k)}}{n!} = (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k}{n} d_q x_1 \dots d_q x_k$$

$$= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 \cdots x_k + n - 1}{n} d_q x_1 \dots d_q x_k\,.$$

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We use the fact

$$\binom{x+y}{n}=\sum_{l=0}^{n}\binom{x}{l}\binom{y}{n-l} \tag{1}$$

as $x=-x_1\cdots x_k$ and $y=n-1$. The identity (1) holds because

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{x+y}{n}t^n&=(1+t)^{x+y}=(1+t)^x(1+t)^y\\&=\left(\sum_{l=0}^{\infty}\binom{x}{l}t^l\right)\left(\sum_{k=0}^{\infty}\binom{y}{k}t^k\right)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{x}{l}\binom{y}{n-l}t^n\,.\end{aligned}$$

Then

$$\begin{aligned}(-1)^n\frac{c_{n,q}^{(k)}}{n!}&=\underbrace{\int_0^1\cdots\int_0^1}_{k}\sum_{m=0}^{l}\binom{-x_1\cdots x_k}{m}\binom{n-1}{n-m}d_qx_1\dots d_qx_k\\&=\sum_{m=0}^{n}\binom{n-1}{m-1}\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{-x_1\cdots x_k}{m}d_qx_1\dots d_qx_k\\&=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{c}_{m,q}^{(k)}}{m!}\,.\end{aligned}$$

Note that $\binom{n-1}{-1}=0$. ∎

6 q-poly-Cauchy polynomials of the first kind

Let n and k be integers with $n\ge 0$. In [7, 10], we introduced *poly-Cauchy polynomials of the first kind* $c_n^{(k)}(z)$ by

$$c_n^{(k)}(z)=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{x_1\cdots x_k-z}{n}d_qx_1\cdots d_qx_k\,.$$

Note that z is replaced by $-z$ in [7]. If $z=0$, then $c_n^{(k)}(0)=c_n^{(k)}$ is the poly-Cauchy number of the first kind.

Now, we generalize this polynomial by using Jackson's q-integral. Define q-analogue of poly-Cauchy polynomials of the first kind (or simply, *q-poly-Cauchy polynomials of the*

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first kind) as

$$c_{n,q}^{(k)}(z) = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k - z)_n d_q x_1 \cdots d_q x_k$$
$$= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k - z}{n} d_q x_1 \cdots d_q x_k \, .$$

The q-poly-Cauchy polynomials of the first kind $c_{n,q}^{(k)}(z)$ can be written in terms of the Stirling numbers of the first kind.

Theorem 6

$$c_{n,q}^{(k)}(z) = \sum_{m=0}^{n} s(n,m) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{[m-i+1]_q^k} \quad (n \geq 0)\, ,$$

where $s(n,m)$ is the (signed) Stirling number of the first kind.

Remark. If $q \to 1$, then by $c_{n,q}^{(k)}(z) \to c_n^{(k)}(z)$ and $[l-i+1]_q \to l+1$, Theorem 6 is reduced to Theorem 2.1 in [7]. Note that z is replaced by $-z$.

Proof. By

$$(x)_n = \sum_{l=0}^{n} s(n,l) x^l \, ,$$

we have

$$c_{n,q}^{(k)}(z) = \sum_{m=0}^{n} s(n,m) \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k - z)^m d_q x_1 \cdots d_q x_k$$
$$= \sum_{m=0}^{n} s(n,m) \sum_{i=0}^{m} \binom{m}{i} (-z)^{m-i} \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1^i \cdots x_k^i d_q x_1 \cdots d_q x_k$$
$$= \sum_{m=0}^{n} s(n,m) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{[m-i+1]_q^k} \, .$$

∎

Remember that q-polyfactorial function $\mathrm{Lif}_{k,q}(z)$ is defined by

$$\mathrm{Lif}_{k,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n![n+1]_q^k} \, .$$

Theorem 7 *The generating function of the q-poly-Cauchy numbers $c_{n,q}^{(k)}(z)$ is given by the following:*

$$\sum_{n=0}^{\infty} c_{n,q}^{(k)}(z)\frac{t^n}{n!} = \frac{\mathrm{Lif}_{k,q}\bigl(\ln(1+t)\bigr)}{(1+t)^z}\,.$$

Remark. If $q \to 1$, then by $c_{n,q}^{(k)} \to c_n^{(k)}$ and $\mathrm{Lif}_{k,q}(z) \to \mathrm{Lif}_k(z)$, Theorem 7 is reduced to Theorem 2.2 in [7]. Note that z is changed by $-z$.

Corollary 3 *For $m \ge 0$ we have*

$$\sum_{m=0}^{n} S(n,m)c_{m,q}^{(k)}(z) = \sum_{l=0}^{n}\binom{n}{l}\frac{(-z)^l}{[n-l+1]_q^k}\,.$$

Remark. If $q \to 1$, then by $c_{m,q}^{(k)}(z) \to c_m^{(k)}(z)$ and $[n-l+1]_q \to n-l+1$, Corollary 3 is reduced to Theorem 2.4 in [7]. Note that z is changed by $-z$.

7 q-poly-Cauchy polynomials of the second kind

Let n and k be integers with $n \ge 0$. Similarly to the q-poly-Cauchy polynomials of the first kind, define q-analogue of poly-Cauchy polynomials of the second kind (or simply, *q-poly-Cauchy polynomials of the second kind*) as

$$\begin{aligned}\widehat{c}_{n,q}^{(k)}(z) &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1\cdots x_k + z)_n d_qx_1\cdots d_qx_k\\ &= n!\underbrace{\int_0^1 \cdots \int_0^1}_{k}\binom{-x_1\cdots x_k+z}{n} d_qx_1\cdots d_qx_k\,.\end{aligned}$$

The q-poly-Cauchy polynomials of the second kind $\widehat{c}_{n,q}^{(k)}$ can be written in terms of the Stirling numbers of the first kind.

Theorem 8

$$\widehat{c}_{n,q}^{(k)}(z) = \sum_{m=0}^{n}(-1)^m s(n,m)\sum_{i=0}^{m}\binom{m}{i}\frac{(-z)^i}{[m-i+1]_q^k}\quad (n\ge 0)\,.$$

Remark. If $q \to 1$, then by $\widehat{c}_{n,q}^{(k)}(z) \to \widehat{c}_n^{(k)}(z)$ and $[m-i+1]_q \to m-i+1$, Theorem 8 is reduced to Theorem 3.1 in [7]. Note that z is replaced by $-z$.

Similarly to Theorem 7, we have the following.

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Theorem 9 *The generating function of the q-poly-Cauchy polynomials of the second kind* $\widehat{c}_{n,q}^{(k)}(z)$ *is given by the following:*

$$\sum_{n=0}^{\infty} \widehat{c}_{n,q}^{(k)}(z) \frac{t^n}{n!} = (1+t)^z \mathrm{Lif}_{k,q}\bigl(-\ln(1+t)\bigr)\,.$$

Remark. As $q \to 1$, then by $\widehat{c}_{n,q}^{(k)}(z) \to \widehat{c}_n^{(k)}(z)$ and $\mathrm{Lif}_{k,q}(z) \to \mathrm{Lif}_k(z)$, Theorem 9 is reduced to Theorem 3.2 in [9]. Note that z is changed by $-z$.

Replace t by $e^t - 1$ in Theorem 9. Then we have the following. As $q \to 1$, Corollary 4 is reduced to Theorem 3.4 in [7]. The proof is similar to that of Corollary 1 and is omitted.

Corollary 4 *For* $m \ge 0$ *we have*

$$\sum_{m=0}^{n} S(n,m) \widehat{c}_{m,q}^{(k)}(z) = \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^{n-l} z^l}{[n-l+1]_q^k}\,.$$

8 Some relations between two kinds of q-poly-Cauchy polynomials

There are some relations between q-poly-Cauchy numbers of the first kind and those of the second kind.

Theorem 10 *For* $n \ge 1$ *we have*

$$(-1)^n \frac{c_{n,q}^{(k)}(z)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{c}_{m,q}^{(k)}(z)}{m!}\,,$$
$$(-1)^n \frac{\widehat{c}_{n,q}^{(k)}(z)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_{m,q}^{(k)}(z)}{m!}\,.$$

Remark. Since $c_{n,q}^{(k)}(z) \to c_n^{(k)}(z)$ and $\widehat{c}_{n,q}^{(k)}(z) \to \widehat{c}_n^{(k)}(z)$ as $q \to 1$, Theorem 10 is reduced to Theorem 4.2 in [7].

Proof. We shall prove the second identity. The first one is proven similarly and omitted.

By the definition of $\widehat{c}_{n,q}^{(k)}(z)$

$$
\begin{aligned}
(-1)^n \frac{\widehat{c}_{n,q}^{(k)}(z)}{n!} &= (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 \cdots x_k + z}{n} d_q x_1 \dots d_q x_k \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k - z + n - 1}{n} d_q x_1 \dots d_q x_k \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{m=0}^{l} \binom{x_1 \cdots x_k - z}{m} \binom{n-1}{n-m} d_q x_1 \dots d_q x_k \\
&= \sum_{m=0}^{n} \binom{n-1}{m-1} \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k - z}{m} d_q x_1 \dots d_q x_k \\
&= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_{m,q}^{(k)}(z)}{m!} .
\end{aligned}
$$

Note that $\binom{n-1}{-1} = 0$. ∎

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HIGHER-ORDER DAEHEE OF THE FIRST KIND AND POLY-CAUCHY OF THE FIRST KIND MIXED TYPE POLYNOMIALS

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Abstract. In this paper, we investigate some new identities and properties of the higher-order Daehee of the first kind and poly-Cauchy of the first kind mixed type polynomials arising from umbral calculus.

1. Introduction

The Daehee polynomials of the first kind of order $r(\in \mathbb{N})$, $D_n^{(r)}(x)$, are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \ (see\ [8,13,22]). \tag{1.1}$$

When $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers. The poly-Cauchy polynomials of the first kind (of index $k \in \mathbb{Z}$) are given by the generating function to be

$$Lif_k(log(1+t))(1+t)^{-x} = \sum_{n=0}^{\infty} C_n^{(k)}(x)\frac{t^n}{n!}, \ (see\ [10,14]). \tag{1.2}$$

where $Lif_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$.
When $x = 0$, $C_n^{(k)} = C_n^{(k)}(0)$ are called the poly-Cauchy numbers. The higher-order Cauchy of the first kind and the poly-Cauchy of the first kind mixed type polynomials are also defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r \frac{Lif_k(\log(1+t))}{(1+t)^x} = \sum_{n=0}^{\infty} A_n^{(r,k)}(x)\frac{t^n}{n!}, \ (\text{see } [10]). \tag{1.3}$$

When $x = 0$, $A_n^{(r,k)} = A_n^{(r,k)}(0)$ are called higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type numbers.

For $\lambda \neq 1 \in \mathbb{C}$, and $s \in \mathbb{Z}_{>0}$, the Frobenius-Euler polynomials defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\lambda)\frac{t^n}{n!}, \quad (\text{see [1-22]}). \tag{1.4}$$

As is well known, the higher order Bernoulli polynomials are also given by the generating function to be

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$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x)\frac{t^n}{n!}, \quad \text{(see [1-22])}. \tag{1.5}$$

When $x=0$, $B_n^{(s)} = B_n^{(s)}(0)$ are called the Bernoulli numbers of order s.
As is well known, the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad \text{(see [9,10,12])}, \tag{1.6}$$

and

$$(\log(1+t))^n = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_1(l+n,n)t^{l+n}, \tag{1.7}$$

where $n \geq 0$.
In this paper, we consider the higher-order Daehee of the first kind and the poly-Cauchy of the first kind mixed-type polynomials which are given by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(1+t)^x = \sum_{n=0}^{\infty} D_n^{(r,k)}(x)\frac{t^n}{n!}, \tag{1.8}$$

where $k, r \in \mathbb{Z}_{\geq 0}$.
When $x=0$, $D_n^{(r,k)} = D_n^{(r,k)}(0)$ are called the higher-order Daehee of the first kind and the poly-Cauchy of the first kind mixed-type numbers.
Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{1.9}$$

Let $\mathbb{P}$ be the space of polynomials over $\mathbb{C}$ in variable x and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional L on the polynomial $p(x)$ is denoted by $\langle L \mid p(x)\rangle$, and the addition and the scalar multiplication on $\mathbb{P}^*$ are respectively given by $\langle L+M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle$, and $\langle cL \mid p(x)\rangle = c\langle L \mid p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. Let $f(t) \in \mathcal{F}$. Then we define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t) | x^n \rangle = a_n, \ (n \geq 0), \text{(see [20])}. \tag{1.10}$$

From (1.9) and (1.10), we can derive the following equation:

$$\langle t^k | x^n \rangle = n!\delta_{n,k}, \tag{1.11}$$

where $\delta_{n,k}$ is the Kronecker's symbol and $n, k \in \mathbb{Z}_{\geq 0}$.
For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t) | x^n\rangle = \langle L | x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be throughout of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra.

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The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series; if $o(f(t)) = 0$, then $g(t)$ is called an invertible series. For $o(f(t)) = 1$, $o(g(t)) = 0$, there exists a unique sequence $S_n(x)$ ($\deg S_n(x) = n$) such that $\left\langle g(t)f(t)^k \middle| S_n(x)\right\rangle = n!\delta_{n,k}$ for $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$. For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|\, p(x)\rangle = \langle f(t)|\, g(t)p(x)\rangle = \langle g(t)|\, f(t)p(x)\rangle\,, \tag{1.12}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)\middle|\, x^k\right\rangle \frac{t^k}{k!},\ p(x) = \sum_{k=0}^{\infty} \left\langle t^k \middle|\, p(x)\right\rangle \frac{x^k}{k!},\ \text{(see [20])}. \tag{1.13}$$

From (1.13), we easily get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k},\ \textit{and}\ e^{yt}p(x) = p(x+y),\ \text{(see [20])}. \tag{1.14}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we have the following equations (see [10,20]):

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!},\ \textit{for all}\ x \in \mathbb{C}, \tag{1.15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $f(\bar{f}(t)) = 1$.

$$f(t)S_n(x) = nS_{n-1}(x),\ (n \geq 0),\quad S_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j \middle|\ x^n\right\rangle x^j, \tag{1.16}$$

$$S_n(x+y) = \sum_{l=0}^{n} \binom{n}{l} S_l(x) P_{n-l}(y),\ \textit{where}\ p_n(x) = g(t)S_n(x), \tag{1.17}$$

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} S_n(x), \tag{1.18}$$

and

$$\langle f(t)|\, xp(x)\rangle = \langle \partial_t f(t)|\, p(x)\rangle\,,\ \frac{dS_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \bar{f}(t)\middle|\, x^{n-l}\right\rangle S_l(x). \tag{1.19}$$

For $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

$$S_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x),\ (n \geq 0), \tag{1.20}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle|\, x^n \right\rangle,\ (\textit{see}\ [10, 20]).$$

In this paper, we investigate some properties of the higher-order Daehee of the first kind and poly-Cauchy of the first kind mixed type polynomials and we give some identities of those polynomials which are derived from umbral calculus.

2. Higher-order Daehee of the first kind and poly-Cauchy of the first kind mixed type polynomials

From (1.8) and (1.15), we note that

$$D_n^{(r,k)}(x) \sim \left(\left(\frac{e^t-1}{t} \right)^r \frac{1}{Lif_k(t)},\ e^t-1 \right). \tag{2.1}$$

Thus, by (2.1), we get

$$\left(\frac{e^t-1}{t} \right)^r \frac{1}{Lif_k(t)} D_n^{(r,k)}(x) \sim (1, e^t-1), \tag{2.2}$$

and

$$(x)_n = x(x-1)\cdots(x-n+1) \sim (1, e^t-1).$$

By (2.2), we see that

$$\begin{aligned} D_n^{(r,k)}(x) &= \left(\frac{t}{e^t-1} \right)^r Lif_k(t)(x)_n = \sum_{m=0}^{n} S_1(n,m) \left(\frac{t}{e^t-1} \right)^r Lif_k(t) x^m \\ &= \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{\binom{m}{l}}{(l+1)^k} \left(\frac{t}{e^t-1} \right)^r x^{m-l} \\ &= \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} B_{m-l}^{(r)}(x). \end{aligned} \tag{2.3}$$

Threrfore, by (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_n^{(r,k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{\binom{m}{l}}{(m-l+1)^k} B_l^{(r)}(x).$$

Now, we observe that

$$\begin{aligned} &\left\langle \left(\frac{\log(1+t)}{t} \right)^r Lif_k(\log(1+t))(\log(1+t))^j \middle| x^n \right\rangle \\ &= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \left\langle \left(\frac{\log(1+t)}{t} \right)^r \middle| Lif_k(\log(1+t)) x^{n-l-j} \right\rangle \\ &= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \sum_{m=0}^{n-l-j} C_m^{(k)} \frac{1}{m!} (n-l-j)_m \left\langle \left(\frac{\log(1+t)}{t} \right)^r \middle| x^{n-l-j-m} \right\rangle \\ &= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \sum_{m=0}^{n-l-j} C_m^{(k)} \frac{1}{m!} (n-l-j)_m D_{n-l-j-m}^{(r)} \\ &= \sum_{l=0}^{n-j} \sum_{m=0}^{l} j! \binom{n}{l} \binom{l}{m} S_1(n-l,j) C_m^{(k)} D_{l-m}^{(r)}. \end{aligned} \tag{2.4}$$

Therefore, by (1.16), (2.1), and (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_n^{(r,k)}(x) = \sum_{j=0}^{n} \left\{ \sum_{l=0}^{n-j} \sum_{m=0}^{l} \binom{n}{l} \binom{l}{m} S_1(n-l,j) C_m^{(k)} D_{l-m}^{(r)} \right\} x^j$$

By (1.11), (1.14), (1.15), and (2.1), we get

$$\begin{aligned} D_n^{(r,k)}(y) &= \left\langle \sum_{i=0}^{\infty} D_i^{(r,k)}(y) \frac{t^i}{i!} \middle| x^n \right\rangle = \left\langle \left(\frac{\log(1+t)}{t} \right)^r Lif_k(\log(1+t))(1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{\log(1+t)}{t} \right)^r \middle| Lif_k(\log(1+t))(1+t)^y x^n \right\rangle \\ &= \sum_{l=0}^{n} C_l^{(k)}(-y) \frac{(n)_l}{l!} \left\langle \left(\frac{\log(1+t)}{t} \right)^r \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(r)} C_l^{(k)}(-y). \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$D_n^{(r,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(r)} C_l^{(k)}(-x).$$

From $D_n^{(r,k)}(x) \sim \left(\left(\frac{e^t-1}{t} \right)^r \frac{1}{Lif_k(t)}, e^t - 1 \right)$, we note that

$$P_n(x) = \left(\frac{e^t - 1}{t} \right)^r \frac{1}{Lif_k(t)} D_n^{(r,k)}(x) = (x)_n. \tag{2.6}$$

By (1.17) and (2.6), we get

$$D_n^{(r,k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} D_j^{(r,k)}(x)(y)_{n-j}. \tag{2.7}$$

It is not difficult to show that

$$D_n^{(r,k)}(x+1) - D_n^{(r,k)}(x) = (e^t - 1) D_n^{(r,k)}(x) = n D_{n-1}^{(r,k)}(x). \tag{2.8}$$

From (1.18) and (2.1), we have

$$D_{n+1}^{(r,k)}(x) = x D_n^{(r,k)}(x-1) - e^{-t} \frac{g'(t)}{g(t)} D_n^{(r,k)}(x), \tag{2.9}$$

where $g(t) = \left(\frac{e^t-1}{t} \right)^r \frac{1}{Lif_k(t)}$.
Here, we observe that

$$\begin{aligned} \frac{g'(t)}{g(t)} &= (\log(g(t)))' = (r \log(e^t - 1) - r \log t - \log(Lif_k(t)))' \\ &= r \frac{te^t - e^t + 1}{t(e^t - 1)} - \frac{Lif_k'(t)}{Lif_k(t)}. \end{aligned} \tag{2.10}$$

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It is easy to show that

$$\begin{aligned}\frac{te^t-e^t+1}{e^t-1}&=\frac{t}{e^t-1}e^t-1=\sum_{n=0}^{\infty}B_n(1)\frac{t^n}{n!}-1\\&=B_1(1)t+\frac{1}{2!}B_2(1)t^2+\cdots=\frac{1}{2}t+\cdots.\end{aligned}\tag{2.11}$$

From (2.11), we note that

$$\begin{aligned}&\frac{te^t-e^t+1}{t(e^t-1)}D_n^{(r,k)}(x)=\frac{te^t-e^t+1}{t(e^t-1)}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(m-l+1)^k}B_l^{(r)}(x)\right)\\&=\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(m-l+1)^k}\frac{te^t-e^t+1}{e^t-1}\left(\frac{B_{l+1}^{(r)}(x)}{l+1}\right)\\&=\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(m-l+1)^k}\left(\frac{-t}{e^{-t}-1}-1\right)\frac{B_{l+1}^{(r)}(x)}{l+1}\\&=\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(l+1)(m-l+1)^k}\sum_{j=0}^{l+1}(-1)^j\binom{l+1}{j}B_jB_{l+1-j}^{(r)}(x)\\&\quad-\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(l+1)(m-l+1)^k}B_{l+1}^{(r)}(x).\end{aligned}\tag{2.12}$$

By Theorem 2, we get

$$\begin{aligned}&\frac{te^t-e^t+1}{t(e^t-1)}D_n^{(r,k)}(x)\\&=\frac{te^t-e^t+1}{t(e^t-1)}\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}S_1(n-l,j)C_m^{(k)}D_{l-m}^{(r)}\right\}x^j\\&=\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}S_1(n-l,j)C_m^{(k)}D_{l-m}^{(r)}\left(\frac{-t}{e^{-t}-1}-1\right)x^{j+1}\\&=\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}S_1(n-l,j)C_m^{(k)}D_{l-m}^{(r)}\left(\sum_{i=0}^{j+1}(-1)^i\binom{j+1}{i}B_ix^{j+1-i}-x^{j+1}\right).\end{aligned}\tag{2.13}$$

Now, we observe that

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$$
\begin{aligned}
&\frac{Lif_k'(t)}{Lif_k(t)} D_n^{(r,k)}(x) \\
&= Lif_k'(t)\left(\frac{1}{Lif_k(t)} D_n^{(r,k)}(x)\right) = Lif_k'(t)\left(\frac{t}{e^t-1}\right)^r (x)_n \\
&= \sum_{m=0}^{n} S_1(n,m) Lif_k'(t)\left(\frac{t}{e^t-1}\right)^r x^m \\
&= \sum_{m=0}^{n} S_1(n,m) Lif_k'(t) B_m^{(r)}(x) \\
&= \sum_{m=0}^{n} S_1(n,m)\left(Lik_{k-1}(t) - Lif_k(t)\right) \frac{B_{m+1}^{(r)}(x)}{m+1} \qquad (2.14)\\
&= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^{k-1}} B_{m+1}^{(r)}(x) - \sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^k} B_{m+1}^{(r)}(x)\right) \\
&= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^{k-1}} B_{m+1-l}^{(r)}(x) - \sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^k} B_{m+1-l}^{(r)}(x)\right) \\
&= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} \frac{l}{(l+1)^k} B_{m+1-l}^{(r)}(x).
\end{aligned}
$$

Therefore, by (2.9), (2.10), (2.12), (2.13), and (2.14), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$
\begin{aligned}
D_{n+1}^{(r,k)}(x) &= x D_n^{(r,k)}(x-1) - r \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{\binom{m}{l}}{(l+1)(m-l+1)^k} \\
&\quad \times \left\{ \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} B_j B_{l+1-j}^{(r)}(x-1) - B_{l+1}^{(r)}(x-1) \right\} \\
&\quad + \sum_{m=0}^{n} \sum_{l=1}^{m+1} \frac{\binom{m}{l-1}}{(l+1)^k} S_1(n,m) B_{m+1-l}^{(r)}(x-1) \\
&= x D_n^{(r,k)}(x-1) - r \sum_{j=0}^{n} \frac{1}{j+1} \sum_{l=0}^{n-j} \sum_{m=0}^{l} \binom{n}{l} \binom{l}{m} S_1(n-l,j) C_m^{(k)} D_{l-m}^{(r)} \\
&\quad \times \left\{ \sum_{i=0}^{j+1} (-1)^i \binom{j+1}{i} B_i (x-1)^{j+1-i} - (x-1)^{j+1} \right\} \\
&\quad + \sum_{m=0}^{n} \sum_{l=1}^{m+1} \frac{\binom{m}{l-1}}{(l+1)^k} S_1(n,m) B_{m+1-l}^{(r)}(x-1).
\end{aligned}
$$

Now, we note that

$$
\begin{aligned}
\left\langle \log(1+t) \middle| x^{n-l} \right\rangle &= \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m} \middle| x^{n-l} \right\rangle = \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(m-1)!}{m!} t^m \middle| x^{n-l} \right\rangle \\
&= (-1)^{n-l-1}(n-l-1)!.
\end{aligned} \tag{2.15}
$$

From (1.19), (2.1), and (2.15), we have

$$
\begin{aligned}
\frac{d}{dx} D_n^{(r,k)}(x) &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1}(n-l-1)! D_l^{(r,k)}(x) \\
&= n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l^{(r,k)}(x).
\end{aligned} \tag{2.16}
$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geq 1$, we have*

$$
\frac{d}{dx} D_n^{(r,k)}(x) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l^{(r,k)}(x).
$$

We note that

$$
\begin{aligned}
D_n^{(r,k)}(y) &= \left\langle \sum_{l=0}^{\infty} D_l^{(r,k)}(y) \frac{t^l}{l!} \middle| x^n \right\rangle = \left\langle \left(\frac{\log(1+t)}{t} \right)^r Lif_k(\log(1+t))(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \partial_t \left(\left(\frac{\log(1+t)}{t} \right)^r Lif_k(\log(1+t))(1+t)^y \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \left(\frac{\log(1+t)}{t} \right)^r \right) Lif_k(\log(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\
&+ \left\langle \left(\frac{\log(1+t)}{t} \right)^r \left(\partial_t Lif_k(\log(1+t)) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
&+ \left\langle \left(\frac{\log(1+t)}{t} \right)^r Lif_k(\log(1+t)) \left(\partial_t (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
&= y D_{n-1}^{(r,k)}(y-1) + \left\langle \left(\partial_t \left(\frac{\log(1+t)}{t} \right)^r \right) Lif_k(\log(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\
&+ \left\langle \left(\frac{\log(1+t)}{t} \right)^r \left(\partial_t Lif_k(\log(1+t)) \right) (1+t)^y \middle| x^{n-1} \right\rangle .
\end{aligned} \tag{2.17}
$$

Now, we observe that

$$\begin{aligned}
&\left\langle \left(\partial_t \left(\frac{\log(1+t)}{t}\right)^r\right) Lif_k(\log(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\
&= r \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_k(\log(1+t))(1+t)^y \middle| \frac{\frac{1}{1+t} - \frac{\log(1+t)}{t}}{t} x^{n-1} \right\rangle \\
&= \frac{r}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_k(\log(1+t))(1+t)^y \middle| \frac{1}{1+t} x^n \right\rangle \\
&\quad - \frac{r}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r} Lif_k(\log(1+t))(1+t)^y \middle| x^n \right\rangle \\
&= -\frac{r}{n} D_n^{(r,k)}(y) + \frac{r}{n} D_n^{(r-1,k)}(y-1),
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\left\langle \left(\frac{\log(1+t)}{t}\right)^r (\partial_t Lif_k(\log(1+t)))\,(1+t)^y \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^r \frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{(1+t)\log(1+t)} (1+t)^y \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} (1+t)^{y-1} \middle| \frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{t} x^{n-1} \right\rangle \\
&= \frac{1}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_{k-1}(\log(1+t))(1+t)^{y-1} \middle| x^n \right\rangle \\
&\quad - \frac{1}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_{k}(\log(1+t))(1+t)^{y-1} \middle| x^n \right\rangle \\
&= \frac{1}{n} D_n^{(r-1,k-1)}(y-1) - \frac{1}{n} D_n^{(r-1,k)}(y-1).
\end{aligned} \tag{2.19}$$

Thus, by (2.17), (2.18) and (2.19), we get

$$\begin{aligned}
D_n^{(r,k)}(y) &= y D_{n-1}^{(r,k)}(y-1) - \frac{r}{n} D_n^{(r,k)}(y) + \frac{r}{n} D_n^{(r-1,k)}(y-1) \\
&\quad + \frac{1}{n} D_n^{(r-1,k-1)}(y-1) - \frac{1}{n} D_n^{(r-1,k)}(y-1).
\end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$D_n^{(r,k)}(x) = \frac{n}{n+r} x D_{n-1}^{(r,k)}(x-1) + \frac{r-1}{n+r} D_n^{(r-1,k)}(x-1) + \frac{1}{n+r} D_n^{(r-1,k-1)}(x-1).$$

Now, we compute $\left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \middle| x^n \right\rangle$ in two different ways.

On the one hand,

$$\begin{aligned}
&\left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \middle| x^n \right\rangle \\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t)) \middle| (\log(1+t))^m x^n \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t)) \middle| x^{n-l-m} \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} D_{n-l-m}^{(r,k)} \\
&= \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l,m) D_l^{(r,k)}.
\end{aligned} \tag{2.21}$$

On the other hand,

$$\begin{aligned}
&\left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \middle| x^n \right\rangle \\
&= \left\langle \partial_t \left(\left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \left(\frac{\log(1+t)}{t}\right)^r \right) Lif_k(\log(1+t))(\log(1+t))^m \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{\log(1+t)}{t}\right)^r \left(\partial_t Lif_k(\log(1+t))\right) (\log(1+t))^m \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t)) \left(\partial_t (\log(1+t))^m\right) \middle| x^{n-1} \right\rangle.
\end{aligned} \tag{2.22}$$

Here, we observe that

$$\begin{aligned}
&\left\langle \left(\partial_t \left(\frac{\log(1+t)}{t}\right)^r \right) Lif_k(\log(1+t))(\log(1+t))^m \middle| x^{n-1} \right\rangle \\
&= r \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} \frac{\frac{1}{1+t} - \frac{\log(1+t)}{t}}{t} Lif_k(\log(1+t))(\log(1+t))^m \middle| x^{n-1} \right\rangle \\
&= \frac{r}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_k(\log(1+t))(\log(1+t))^m \middle| \left(\frac{1}{1+t} - \frac{\log(1+t)}{t}\right) x^n \right\rangle \\
&= \frac{r}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_k(\log(1+t))(1+t)^{-1} \middle| (\log(1+t))^m x^n \right\rangle \\
&\quad - \frac{r}{n} \left\langle \left(\frac{\log(1+t)}{t}\right)^{r} Lif_k(\log(1+t)) \middle| (\log(1+t))^m x^n \right\rangle \\
&= \frac{r}{n} \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l,m) D_l^{(r-1,k)}(-1) - \frac{r}{n} \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l,m) D_l^{(r,k)};
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\left\langle \left(\frac{\log(1+t)}{t}\right)^r (\partial_t Lif_k(\log(1+t)))\,(\log(1+t))^m \middle| x^{n-1} \right\rangle\\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^r \frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{(1+t)\log(1+t)} (\log(1+t))^m \middle| x^{n-1} \right\rangle\\
&= \frac{1}{n}\left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_{k-1}(\log(1+t))(1+t)^{-1} \middle| (\log(1+t))^m x^n \right\rangle\\
&\quad - \frac{1}{n}\left\langle \left(\frac{\log(1+t)}{t}\right)^{r-1} Lif_{k}(\log(1+t))(1+t)^{-1} \middle| (\log(1+t))^m x^n \right\rangle\\
&= \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r-1,k-1)}(-1) - \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r-1,k)}(-1),
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))\,(\partial_t(\log(1+t))^m) \middle| x^{n-1} \right\rangle\\
&= m\left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(1+t)^{-1} \middle| (\log(1+t))^{m-1} x^{n-1} \right\rangle\\
&= m\sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1}\\
&\qquad \times \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(1+t)^{-1} \middle| x^{n-l-m} \right\rangle\\
&= m\sum_{l=0}^{n-m} (m-1)!\binom{n-1}{l} S_1(n-l-1,m-1) D_l^{(r,k)}(-1).
\end{aligned} \tag{2.25}$$

Thus by (2.21), (2.22), (2.23), (2.24), and (2.25), we get

$$\begin{aligned}
&\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r,k)}\\
&= \frac{r}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r-1,k)}(-1) - \frac{r}{n}\sum_{l=0}^{n-m} m!\binom{n}{l}\\
&\quad \times S_1(n-l,m) D_l^{(r,k)} + \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r-1,k-1)}(-1)\\
&\quad - \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m) D_l^{(r-1,k)}(-1)\\
&\quad + m\sum_{l=0}^{n-m} (m-1)!\binom{n-1}{l} S_1(n-l-1,m-1) D_l^{(r,k)}(-1).
\end{aligned} \tag{2.26}$$

Therefore, by (2.26), we obtain the following theorem.

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Theorem 2.7. *For $n \geq m \geq 1$, we have*

$$\begin{aligned}&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l^{(r,k)}\\&=\frac{r-1}{n+r}\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l^{(r-1,k)}(-1)\\&+\frac{1}{n+r}\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l^{(r-1,k-1)}(-1)\\&+\frac{n}{n+r}\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l^{(r,k)}(-1).\end{aligned}$$

Let us consider the following two Sheffer sequences:

$$D_n^{(r,k)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r\frac{1}{Lif_k(t)}, e^t-1\right),\ (x)_n \sim (1, e^t-1). \tag{2.27}$$

Then, from (1.20), we have

$$D_n^{(r,k)}(x)=\sum_{m=0}^{n}C_{n,m}(x)_m, \tag{2.28}$$

where

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))\middle| t^m x^n\right\rangle\\&=\binom{n}{m}\left\langle\left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))\middle| x^{n-m}\right\rangle\\&=\binom{n}{m}D_{n-m}^{(r,k)}.\end{aligned} \tag{2.29}$$

Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$D_n^{(r,k)}(x)=\sum_{m=0}^{n}\binom{n}{m}D_{n-m}^{(r,k)}(x)_m=\sum_{m=0}^{n}m!\binom{n}{m}\binom{x}{m}D_{n-m}^{(r,k)}.$$

For $D_n^{(r,k)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r\frac{1}{Lif_k(t)}, e^t-1\right)$, $H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right)$, let us assume that

$$D_n^{(r,k)}(x)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\lambda). \tag{2.30}$$

Then, from (1.20), we note that

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!(1-\lambda)^s} \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \middle| (1-\lambda+t)^s x^n \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{j=0}^{n} \binom{s}{j} (1-\lambda)^{s-j} (n)_j \\
&\quad \times \left\langle \left(\frac{\log(1+t)}{t}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \middle| x^{n-j} \right\rangle \\
&= \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} (1-\lambda)^{-j} (n)_j \binom{n-j}{l+m} S_1(l+m,m) D_{n-j-l-m}^{(r,k)} \\
&= \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-l}{l} (1-\lambda)^{-j} (n)_j S_1(n-j-l,m) D_l^{(r,k)}.
\end{aligned} \tag{2.31}$$

Therefore, by (2.30) and (2.31), we obtain the following theorem.

Theorem 2.9. *For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, $n \geq 0$, we have*

$$\begin{aligned}
&D_n^{(r,k)}(x) \\
&= \sum_{m=0}^{n} \left\{ \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-l}{l} (n)_j (1-\lambda)^{-j} S_1(n-j-l,m) D_l^{(r,k)} \right\} H_m^{(s)}(x|\lambda).
\end{aligned}$$

Let us consider the following Sheffer sequences:

$$D_n^{(r,k)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r \frac{1}{Lif_k(t)}, e^t-1 \right), \ B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t \right). \tag{2.32}$$

Then, from (1.20), we have

$$D_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x), \tag{2.33}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{\left(\frac{t}{\log(1+t)}\right)^s}{\left(\frac{t}{\log(1+t)}\right)^r} Lif_k(\log(1+t)) \middle| (\log(1+t))^m x^n \right\rangle. \tag{2.34}$$

Case1. For $s > r$, we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{t}{\log(1+t)}\right)^{s-r} Lif_k(\log(1+t)) \middle| \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m) t^{l+m} x^n \right\rangle \\ &= \frac{1}{m!}\sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\ &\quad \times \left\langle \left(\frac{t}{\log(1+t)}\right)^{s-r} Lif_k(\log(1+t)) \middle| x^{n-l-m} \right\rangle \\ &= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) A_{n-l-m}^{(s-r,k)} \\ &= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) A_{n-l}^{(s-r,k)}. \end{aligned} \tag{2.35}$$

Case2. For $s = r$, we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle Lif_k(\log(1+t)) \middle| (\log(1+t))^m x^n \right\rangle \\ &= \frac{1}{m!}\sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \left\langle Lif_k(\log(1+t)) \middle| x^{n-l-m} \right\rangle \\ &= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) C_{n-l-m}^{(k)} = \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) C_l^{(k)}. \end{aligned} \tag{2.36}$$

Case3. For $s < r$, we note that

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{\log(1+t)}{t}\right)^{r-s} Lif_k(\log(1+t)) \middle| (\log(1+t))^m x^n \right\rangle \\ &= \frac{1}{m!}\sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\ &\quad \times \left\langle \left(\frac{\log(1+t)}{t}\right)^{r-s} Lif_k(\log(1+t)) \middle| x^{n-l-m} \right\rangle \\ &= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) D_{n-l-m}^{(r-s,k)} = \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l^{(r-s,k)}. \end{aligned} \tag{2.37}$$

Therefore, by (2.33), (2.34), (2.35), (2.36), and (2.37), we obtain the following theorem.

Theorem 2.10. *Let $n \geq 0$. Then we have*

$$\begin{aligned} D_n^{(r,k)}(x) &= \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) C_l^{(k)} \right\} B_m^{(s)}(x), \ \textit{if } s=r \\ &= \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l^{(r-s,k)} \right\} B_m^{(s)}(x), \ \textit{if } s<r \\ &= \sum_{m=0}^{n} \left\{ \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) A_{n-l}^{(s-r,k)} \right\} B_m^{(s)}(x), \ \textit{if } s>r. \end{aligned}$$

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ON A NEW OPERATOR FROM HARDY SPACE TO n-TH WEIGHTED-TYPE SPACE ON THE UPPER HALF-PLANE

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ABSTRACT. Motivated by some recent results of operators on analytic function spaces, the boundedness of operator defined by $W_{u,\varphi}^{(m)} f = u \cdot f^{(m)} \circ \varphi$ from the Hardy space to the n-th weighted-type space on the upper half-plane $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ is characterized.

1. INTRODUCTION

Let $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ be the upper half-plane in the complex plane $\mathbb{C}$ and $H(\Pi_+)$ the space of all analytic functions in Π_+. For $p > 0$, the Hardy space $H^p(\Pi_+)$ consists of all $f \in H(\Pi_+)$ such that

$$\|f\|_{H^p(\Pi_+)}^p = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx < \infty.$$

When $p \geq 1$, the Hardy space with the norm $\|\cdot\|_{H^p(\Pi_+)}$ becomes a Banach space (a Hilbert space when $p = 2$), and when $0 < p < 1$,

$$d(f,g) := \|f-g\|_{H^p(\Pi_+)}^p$$

defines a Fréchet space distance on $H^p(\Pi_+)$. For some details of Hardy space and some operators on it see, e.g. [2], [3], [9] and [11].

Let $\mu(z)$ be a positive continuous function on a domain $X \subseteq \mathbb{C}$ and $H(X)$ the space of all analytic functions in X. For fixed $n \in \mathbb{N}_0$, the n-th weighted-type space on X denoted by $\mathcal{W}_\mu^{(n)}(X)$ consists of all $f \in H(X)$ such that

$$b_{\mathcal{W}_\mu^{(n)}(X)}(f) := \sup_{z \in X} \mu(z) |f^{(n)}(z)| < \infty.$$

When $n = 0$, $n = 1$ and $n = 2$, the space is called the weighted-type space, the Bloch-type space and the Zygmund-type space, respectively. Some information of these spaces on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and some operators can be found, e.g., in [4], [7], [8], [10], [13] and [15]. This considerable interest in Zygmund-type spaces, as well as a necessity for unification of weighted-type, Bloch-type and Zygmund-type spaces, motivated people to define the n-th weighted-type space.

The quantity $b_{\mathcal{W}_\mu^{(n)}(X)}(f)$ is a semi-norm on the n-th weighted-type space $\mathcal{W}_\mu^{(n)}(X)$. A natural norm on the n-th weighted-type space $\mathcal{W}_\mu^{(n)}(X)$ is defined by

$$\|f\|_{\mathcal{W}_\mu^{(n)}(X)} = \sum_{j=0}^{n-1} |f^{(j)}(a)| + b_{\mathcal{W}_\mu^{(n)}(X)}(f),$$

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where a is an element in X. Under this norm, $\mathcal{W}_\mu^{(n)}(X)$ becomes a Banach space. For $X = \Pi_+$, we obtain the n-th weighted-type space $\mathcal{W}_\mu^{(n)}(\Pi_+)$, and on this space the norm is

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\Pi_+)} = \sum_{j=0}^{n-1} |f^{(j)}(i)| + \sup_{z\in\Pi_+} \mu(z)|f^{(n)}(z)|.$$

Let φ be an analytic self-mapping of X and u the analytic function on X. For $m \in \mathbb{N}_0$, we define the operator $W_{u,\varphi}^{(m)}$ on $H(X)$ by

$$W_{u,\varphi}^{(m)} f(z) = u(z) \cdot f^{(m)}(\varphi(z)),\ z \in X.$$

It is easy to see that if $m = 0$, then $W_{u,\varphi}^{(m)} := W_{u,\varphi}$ is the weighted composition operator. When $m = 0$ and $u(z) = 1$ for all $z \in X$, the operator $W_{u,\varphi}^{(m)} := C_\varphi$ is the composition operator. So, the operator $W_{u,\varphi}^{(m)}$ can be regarded as a generalization of the weighted composition and composition operators.

For the operator $W_{u,\varphi}^{(m)}$, a natural problem is how to characterize the boundedness or compactness in terms of function theoretic properties of u and φ. During the past few decades, composition operators have been studied extensively on analytic function spaces on the unit disc or the unit ball. One can consult [1] and [12] for the general theory of this operator. As a consequence of the Littlewood's subordination theorem, it is well-known that every composition operator is bounded on Hardy spaces. However, when people consider the Hardy space on the upper half-plane, they find that the situation is entirely different. There do exist unbounded composition operators on this space. Matache [9] proved that there didn't exist compact composition operators on Hardy space of the upper half-plane. Because of these facts of composition operators, many authors recently have begun to study them on analytic functions spaces of the upper half-plane. The present author in [4] characterized the boundedness of composition operators from the weighted Bergman spaces to the weighted-type, Bloch-type and Zymund-type spaces with the weight $\mu(z) = \mathrm{Im}z$ on the upper half-plane. In [15], Stević generalized the result of [13]. In [5], the present author characterized the boundedness of composition operator from the weighted Bergman space to n-th weighted-type space with $\mu(z) = \mathrm{Im}z$ and $n = 4$.

Motivated by [4], [5], [13] and [15], in this paper we characterize the bounded operator $W_{u,\varphi}^{(m)}$ from the Hardy space to the n-th weighted-type space on the upper half-plane. This paper can be regarded as a generalization of results in [13] and [15].

By the definition of the norm of linear operator, we have

$$\|W_{u,\varphi}^{(m)}\|_{H^p(\Pi_+)\to\mathcal{W}_\mu^{(n)}(\Pi_+)} = \sup_{\|f\|_{H^p(\Pi_+)}\le 1} \|W_{u,\varphi}^{(m)} f\|_{\mathcal{W}_\mu^{(n)}(\Pi_+)}.$$

Obviously, this quantity is finite if and only if the operator $W_{u,\varphi}^{(m)} : H^p(\Pi_+) \to \mathcal{W}_\mu^{(n)}(\Pi_+)$ is bounded. In this paper, we shall give an asymptotically expression to the norm of this operator.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. Main results

First, the following lemma was proved in [15].

Lemma 2.1. *Suppose that $p \geq 1$, $n \in \mathbb{N}$ and $w \in \Pi_+$, then the function*

$$f_{w,n}(z) = \frac{(\mathrm{Im}w)^{n-\frac{1}{p}}}{(z-\overline{w})^n}$$

belongs to $H^p(\Pi_+)$ and $\sup_{w\in\Pi_+} \|f_{w,n}\|_{H^p(\Pi_+)} \leq \pi^{\frac{1}{p}}$.

Lemma 2.2. *Suppose that $p \geq 1$ and $n \in \mathbb{N}_0$, then there exists a constant $C > 0$ independent of f such that for all $z \in \Pi_+$ it follows that*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H^p(\Pi_+)}}{(\mathrm{Im}z)^{n+\frac{1}{p}}}.$$

Proof. For each $f \in H^p(\Pi_+)$, it follows from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-z}dt. \tag{1}$$

Differentiating in (1) under the integral sign n times , we have

$$f^{(n)}(z) = \frac{n!}{2\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(t-z)^{n+1}}dt.$$

Then

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{[(t-x)^2+y^2]^{(n+1)/2}}dt. \tag{2}$$

By the change $t - x = sy$, we have

$$\int_{-\infty}^{+\infty} \frac{y^n}{[(t-x)^2+y^2]^{(n+1)/2}}dt = \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^{(n+1)/2}} =: c_n < \infty. \tag{3}$$

From (3) and applying Jensen's inequality in (2), we get

$$\begin{aligned}
|f^{(n)}(z)|^p &\leq d_n \int_{-\infty}^{+\infty} \frac{|f(t)|^p}{y^{np}} \frac{y^n}{[(t-x)^2+y^2]^{(n+1)/2}}dt \\
&\leq d_n \int_{-\infty}^{+\infty} \frac{|f(t)|^p}{y^{np+1}}dt \\
&= d_n \frac{\|f\|^p_{H^p(\Pi_+)}}{y^{np+1}},
\end{aligned}$$

where $d_n = (c_n n!/2\pi)^p$. The desired result is obtained.

The following lemma was obtained in [14].

Lemma 2.3. *Suppose that $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n-1 \\ \cdots & \cdots & \cdots & \cdots \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix},$$

then $D_n(a) = \prod_{j=1}^{n-1} j!$.

Before formulating and proving the main result of this paper, we need introduce the following classical Faàdi Bruno's formula

$$(f\circ\varphi)^{(n)}(z)=\sum\frac{n!}{k_1!\cdots k_n!}f^{(k)}(\varphi(z))\prod_{j=1}^{n}\Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j},$$

where $k=k_1+k_2+\cdots+k_n$, and the sum is over all non-negative integers k_1, k_2, ..., k_n satisfying $k_1+2k_2+\cdots+nk_n=n$. For the information related to this formula see [6].

Theorem 2.4. *Suppose that $p\geq 1$, φ is an analytic self-mapping of Π_+ and $u\in H(\Pi_+)$, then the operator $W^{(m)}_{u,\varphi}:H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)$ is bounded if and only if for each fixed $k\in\{0,1,...,n\}$ it follows that*

$$I_k=\sup_{z\in\Pi_+}\mu(z)\Big|C_n^k u^{(n-k)}(z)\sum\frac{k!}{j_1!\cdots j_k!}\frac{1}{(\mathrm{Im}\varphi(z))^{l+m+\frac{1}{p}}}\prod_{i=1}^{k}\Big(\frac{\varphi^{(i)}(z)}{i!}\Big)^{j_i}\Big|<\infty,\tag{4}$$

where $l=j_1+j_2+\cdots+j_k$, and the sum is over all non-negative integers j_1, j_2,..., j_k satisfying $j_1+2j_2+\cdots+kj_k=k$.

Moreover, if the operator $W^{(m)}_{u,\varphi}:H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)/\mathbb{P}_{n-1}$ is bounded, then

$$\|W^{(m)}_{u,\varphi}\|_{H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)/\mathbb{P}_{n-1}}\asymp\sum_{k=1}^{n}I_k,\tag{5}$$

where $\mathbb{P}_{n-1}$ is the set of all polynomials whose degrees are less than or equal to $n-1$.

Proof. First assume that the operator $W^{(m)}_{u,\varphi}:H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)$ is bounded. For any fixed $w\in\Pi_+$ and constants c_1, c_2, ..., c_{m+n}, set the function

$$f_w(z)=\sum_{j=1}^{m+n}\frac{c_j}{m+n-2+j+\frac{2}{p}}\frac{(2i\mathrm{Im}w)^{m+n-2+j+\frac{1}{p}}}{(z-\overline{w})^{m+n-2+j+\frac{2}{p}}}.$$

Then Lemma 2.1 shows that $f_w\in H^p(\Pi_+)$ and $\sup_{w\in\Pi_+}\|f_w\|_{H^p(\Pi_+)}\leq C$.

Now we prove the following claim.

Claim: For each fixed $k\in\{1,...,n\}$, there are constants c_1, c_2, ..., c_n such that

$$f_w^{(k+m)}(w)=\frac{1}{(2i\mathrm{Im}w)^{k+m+\frac{1}{p}}},\quad f_w^{(l)}(w)=0,\quad l\in\{1,...,n\}\setminus\{k\}.\tag{6}$$

Proof of the claim. By differentiating function f_w and letting $z=w$, we have

$$c_1+c_2+\cdots+c_{m+n}=0$$

$$\Big(m+n+\frac{2}{p}\Big)c_1+\Big(m+n+1+\frac{2}{p}\Big)c_2+\cdots+\Big(2(m+n)-1+\frac{2}{p}\Big)c_{m+n}=0$$

$$\cdots$$

$$\prod_{j=0}^{k-2}\Big(m+n+j+\frac{2}{p}\Big)c_1+\prod_{j=0}^{k-2}\Big(m+n+j+1+\frac{2}{p}\Big)c_2+\cdots$$
$$+\prod_{j=0}^{k-2}\Big(2(m+n)-1+\frac{2}{p}\Big)c_{m+n}=1$$

$$\cdots$$

$$\prod_{j=0}^{m+n-2}\Big(m+n+j+\frac{2}{p}\Big)c_1+\prod_{j=0}^{m+n-2}\Big(m+n+j+1+\frac{2}{p}\Big)c_2+\cdots$$
$$+\prod_{j=0}^{m+n-2}\Big(2(m+n)-1+\frac{2}{p}\Big)c_{m+n}=0. \tag{7}$$

Applying Lemma 2.3 with $a=m+n+2/p>0$, we see that the determinant of system (7) is different from zero, from which the claim holds.

For each fixed $k\in\{1,...,n\}$, we choose the corresponding function which satisfies (6), and write it by $f_{w,k}$, and for $k=0$, we write f_w by $f_{w,0}$. Then by the boundedness of the operator $W^{(m)}_{u,\varphi}:H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)$, we have

$$\big|W^{(m)}_{u,\varphi}f_{\varphi(w),k}(w)\big|\le\|W^{(m)}_{u,\varphi}f_{\varphi(w),k}\|_{\mathcal{W}^{(n)}_\mu(\Pi_+)}. \tag{8}$$

Faàdi Bruno's formula and an easy calculation make the inequality (8) become

$$\mu(w)\Big|C^k_n u^{(n-k)}(w)\sum\frac{k!}{j_1!\cdots j_k!}\frac{1}{(\mathrm{Im}\varphi(w))^{l+m+\frac{1}{p}}}\prod_{i=1}^{k}\Big(\frac{\varphi^{(i)}(w)}{i!}\Big)^{j_i}\Big|<\infty, \tag{9}$$

where $j_1+j_2+\cdots+j_k=l$, and the sum is over all non-negative integers j_1, j_2,..., j_k satisfying $j_1+2j_2+\cdots+kj_k=k$. This shows that the condition in (4) holds.

Conversely, assume that the condition in (4) holds. By Faàdi Bruno's formula and Lemma 2.2, we have

$$\|W^{(m)}_{u,\varphi}f\|_{\mathcal{W}^{(n)}_\mu(\Pi_+)}=\sum_{j=0}^{n-1}\big|(f\circ\varphi)^{(j)}(i)\big|+\sup_{z\in\Pi_+}\mu(z)\big|(W^{(m)}_{u,\varphi}f)^{(n)}(z)\big|$$
$$=\sum_{j=0}^{n-1}\Big|\sum\frac{j!}{l_1!\cdots l_j!}f^{(l)}(\varphi(i))\prod_{s=1}^{j}\Big(\frac{\varphi^{(s)}(i)}{s!}\Big)^{l_s}\Big|+\sup_{z\in\Pi_+}\mu(z)\Big|(u\cdot f^{(m)}\circ\varphi)^{(n)}(z)\Big|$$
$$=\sum_{j=0}^{n-1}\Big|\sum\frac{j!}{l_1!\cdots l_j!}f^{(l)}(\varphi(i))\prod_{s=1}^{j}\Big(\frac{\varphi^{(s)}(i)}{s!}\Big)^{l_s}\Big|$$
$$+\sup_{z\in\Pi_+}\mu(z)\Big|\sum_{k=0}^{n}C^k_n u^{(n-k)}(z)\sum\frac{k!}{j_1!\cdots j_k!}f^{(m+l)}(\varphi(z))\prod_{i=1}^{k}\Big(\frac{\varphi^{(i)}(z)}{i!}\Big)^{j_i}\Big|$$
$$\le\sum_{j=0}^{n-1}\sum_{l=0}^{j}|f^{(l)}(\varphi(i))|\Big|\sum\frac{j!}{l_1!\cdots l_j!}\prod_{s=1}^{j}\Big(\frac{\varphi^{(s)}(i)}{s!}\Big)^{l_s}\Big|$$
$$+C\sup_{z\in\Pi_+}\mu(z)\sum_{k=0}^{n}\Big|C^k_n u^{(n-k)}(z)\sum\frac{k!}{j_1!\cdots j_k!}\frac{1}{(\mathrm{Im}\varphi(z))^{l+m+\frac{1}{p}}}\prod_{i=1}^{k}\Big(\frac{\varphi^{(i)}(z)}{i!}\Big)^{j_i}\Big|\|f\|_{H^p(\Pi_+)}$$
$$\le\sum_{j=0}^{n-1}\sum_{l=0}^{j}|f^{(l)}(\varphi(i))|\Big|\sum\frac{j!}{l_1!\cdots l_j!}\prod_{s=1}^{j}\Big(\frac{\varphi^{(s)}(i)}{s!}\Big)^{l_s}\Big|$$
$$+C\sum_{k=0}^{n}\sup_{z\in\Pi_+}\Big|C^k_n u^{(n-k)}(z)\sum\frac{k!}{j_1!\cdots j_k!}\frac{1}{(\mathrm{Im}\varphi(z))^{l+m+\frac{1}{p}}}\prod_{i=1}^{k}\Big(\frac{\varphi^{(i)}(z)}{i!}\Big)^{j_i}\Big|\|f\|_{H^p(\Pi_+)}.$$

From this and by the condition in (4), we prove that $W^{(m)}_{u,\varphi}:H^p(\Pi_+)\to\mathcal{W}^{(n)}_\mu(\Pi_+)$ is bounded.

If we consider the bounded operator $W_{u,\varphi}^{(m)} : H^p(\Pi_+) \to \mathcal{W}_\mu^{(n)}(\Pi_+)/\mathbb{P}_{n-1}$, then from the above we have proved that

$$\|W_{u,\varphi}^{(m)}\|_{H^p(\Pi_+)\to\mathcal{W}_\mu^{(n)}(\Pi_+)/\mathbb{P}_{n-1}} \asymp \sum_{k=0}^{n} I_k.$$

This is the asymptotically expression in (5), and the proof is end.

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SOLUTIONS OF THE PELL EQUATION $x^2-\left(a^2+2a\right)y^2=N$ VIA GENERALIZED FIBONACCI AND LUCAS NUMBERS

BILGE PEKER AND HASAN SENAY

ABSTRACT. In this study, we find continued fraction expansion of $\sqrt{d}$ when $d=a^2+2a$ where a is positive integer. We consider the integer solutions of the Pell equation $x^2-\left(a^2+2a\right)y^2=N$ when $N\in\{\pm1,\pm4\}$. We formulate the n-th solution (x_n,y_n) by using the continued fraction expansion. We also formulate the n-th solution (x_n,y_n) via the generalized Fibonacci and Lucas sequences.

1. Introduction and Preliminaries

The equation $x^2-dy^2=N$, with given integers d and N, unknowns x and y, is called as Pell equation. In the literature, there are several methods for finding the integer solutions of Pell equation such as the Lagrange-Matthews-Mollin algorithm, the cyclic method, Lagrange's system of reductions, use of binary quadratic forms, etc.

If d is negative, the equation can have only a finite number of solutions. If d is a perfect square, i.e. $d=a^2$, the equation reduces to $(x-ay)(x+ay)=N$ and there is only a finite number of solutions. If d is a positive integer but not a perfect square, then simple continued fractions are very useful. The simple continued fraction expansion of $\sqrt{d}$ has the form $\sqrt{d}=\left[a_0,\overline{a_1,a_2,a_3,...,a_{m-1},2a_0}\right]$ with $a_0=\left[\sqrt{d}\right]$. If the fundamental solution of $x^2-dy^2=1$ is $x=x_1$ and $y=y_1$, then all nontrivial solutions are given by $x=x_n$ and $y=y_n$, where $x_n+y_n\sqrt{d}=\left(x_1+y_1\sqrt{d}\right)^n$. If a single solution $(x,y)=(g,h)$ of the equation $x^2-dy^2=N$ is known, other solutions can be found. Let (r,s) be a solution of the unit form $x^2-dy^2=1$. Then $(x,y)=(gr\pm dhs,gs\pm hr)$ are solutions of the equation $x^2-dy^2=N$.

Given a continued fraction expansion of $\sqrt{d}$, where all the a_i's are real and all except possibly a_0 are positive, define sequences $\{p_n\}$ and $\{q_n\}$ by $p_{-2}=0$, $p_{-1}=1$, $p_k=a_kp_{k-1}+p_{k-2}$ and $q_{-2}=1$, $q_{-1}=0$, $q_k=a_kq_{k-1}+q_{k-2}$ for $k\geq0$. Let m be the length of the period of continued fraction. Then the fundamental solution of $x^2-dy^2=1$ is

$$(x_1,y_1)=\begin{cases}(p_{m-1},q_{m-1}) & \text{if } m \text{ is even}\\ (p_{2m-1},q_{2m-1}) & \text{if } m \text{ is odd.}\end{cases}$$

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.

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If the length of the period of continued fraction is even, then the equation $x^2 - dy^2 = -1$ has no integer solutions. If m is odd, the fundamental solution of $x^2 - dy^2 = -1$ is given by $(x_1, y_1) = (p_{m-1}, q_{m-1})$ [4].

Now let us give the following known theorems [5] that will be needed for the next section.

Theorem 1. *Let $d \equiv 2 \pmod 4$ or $d \equiv 3 \pmod 4$. Then the equation $x^2 - dy^2 = -4$ has positive integer solutions if and only if the equation $x^2 - dy^2 = -1$ has positive integer solutions.*

Theorem 2. *Let $d \equiv 0 \pmod 4$. If fundamental solution of the equation $x^2 - (d/4)\, y^2 = 1$ is $x_1 + y_1\sqrt{d/4}$, then fundamental solution of the equation $x^2 - dy^2 = 4$ is $(2x_1, y_1)$.*

Theorem 3. *Let $d \equiv 1 \pmod 4$ or $d \equiv 2 \pmod 4$ or $d \equiv 3 \pmod 4$. If fundamental solution of the equation $x^2 - dy^2 = 1$ is $x_1 + y_1\sqrt{d}$, then fundamental solution of the equation $x^2 - dy^2 = 4$ is $(2x_1, 2y_1)$.*

The next two theorems can be found in [2] and [5].

Theorem 4. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution of the equation $x^2 - dy^2 = 4$. Then all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n / 2^{n-1}$$

with $n \geq 1$.

Theorem 5. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution of the equation $x^2 - dy^2 = -4$. Then all positive integer solutions of the equation $x^2 - dy^2 = -4$ are given by*

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^{2n-1} / 2^{2n-2}$$

with $n \geq 1$.

In this study [3], since generalized Fibonacci and Lucas sequences related solutions of the forthcoming Pell equation are going to be taken into consideration, let us briefly recall the generalized Fibonacci sequences $\{U_n(k,s)\}$ and Lucas sequences $\{V_n(k,s)\}$. Let k and s be two non-zero integers with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by

$$U_0(k,s) = 0, U_1(k,s) = 1$$

and

$$U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$$

for $n \geq 1$. Generalized Lucas sequence is defined by

$$V_0(k,s) = 2, V_1(k,s) = k$$

and

$$V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$$

for $n \geq 1$. It is also well-known from the literature that generalized Fibonacci and Lucas numbers have many interesting and significant properties. Binet's formulas are probably the most important one among them. For generalized Fibonacci and Lucas sequences, Binet's formulas are given by $U_n(k,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n(k,s) = \alpha^n + \beta^n$ where $\alpha = \left(k + \sqrt{k^2 + 4s}\right)/2$ and $\beta = \left(k - \sqrt{k^2 + 4s}\right)/2$ [7].

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There are a large number of studies concerning Pell equation in the literature. Güney [1] solved the Pell equations $x^2-(a^2b^2+2b)\,y^2=N$ when $N\in\{\pm1,\pm4\}$.

In this study, we consider the integer solutions of the Pell equation

$$x^2-\left(a^2+2a\right)y^2=N$$

in terms of the generalized Fibonacci and Lucas numbers.

2. Main Results

We consider the integer solutions of the Pell equation

$$E: x^2-\left(a^2+2a\right)y^2=1. \tag{1}$$

Theorem 6. *Let E be the Pell equation in* (1)*. Then the followings hold:*

(i) *The continued fraction expansion of $\sqrt{a^2+2a}$ is*

$$\sqrt{a^2+2a}=\left[a;\overline{1,2a}\right].$$

(ii) *The fundamental solution is*

$$(x_1,y_1)=(a+1,1)$$

(iii) *The n-th solution (x_n,y_n) can be find by*

$$\frac{x_n}{y_n}=\left[a;(1,2a)_{n-1},1\right]$$

where $(1,2a)_{n-1}$ means that there are $n-1$ successive terms $(1,2a)$.

Proof. **(i)**

$$\begin{aligned}\sqrt{a^2+2a}&=a+\left(\sqrt{a^2+2a}-a\right)=a+\frac{1}{\frac{\sqrt{a^2+2a}+a}{2a}}\\&=a+\frac{1}{1+\frac{\sqrt{a^2+2a}-a}{2a}}=a+\frac{1}{1+\frac{1}{\sqrt{a^2+2a}+a}}\\&=a+\frac{1}{1+\frac{1}{2a+\left(\sqrt{a^2+2a}-a\right)}}.\end{aligned}$$

Therefore $\sqrt{a^2+2a}=\left[a;\overline{1,2a}\right]$. This completes the proof.

(ii) The period length of the continued fraction expansion of $\sqrt{a^2+2a}$ is 2. Therefore, the fundamental solution of the equation $x^2-\left(a^2+2a\right)y^2=1$ is $p_1+q_1\sqrt{a^2+2a}$. It is easily seen that $p_1=a_1p_0+p_{-1}=a+1$ and $q_1=a_1q_0+q_{-1}=1$. That is, the fundamental solution of $x^2-\left(a^2+2a\right)y^2=1$ is $(x_1,y_1)=(a+1,1)$.

(iii) For $n=1$, we get $\frac{x_1}{y_1}=[a;1]=a+\frac{1}{1}=\frac{a+1}{1}$. Hence it is true for $n=1$.

We assume that (x_n,y_n) is a solution of $x^2-\left(a^2+2a\right)y^2=1$. That is, $\frac{x_n}{y_n}=[a;(1,2a)_{n-1},1]$.

Now we must show that it holds for (x_{n+1},y_{n+1}).

$$\begin{aligned}\frac{x_{n+1}}{y_{n+1}}&=a+\frac{1}{1+\frac{1}{2a+\frac{1}{1+\frac{1}{2a+\frac{1}{\dots 2a+1}}}}}\\&=a+\frac{1}{1+\frac{1}{a+a+\frac{1}{1+\frac{1}{2a+\frac{1}{\dots 2a+1}}}}}\\&=a+\frac{1}{1+\frac{1}{a+\frac{x_n}{y_n}}}\\&=\frac{(a+1)x_n+\left(a^2+2a\right)y_n}{x_n+(a+1)y_n}.\end{aligned}$$

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(x_{n+1}, y_{n+1}) is a solution of $x^2 - \left(a^2+2a\right)y^2 = 1$ since $x_{n+1}^2 - \left(a^2+2a\right)y_{n+1}^2 = \left((a+1)x_n + \left(a^2+2a\right)y_n\right)^2 - \left(a^2+2a\right)\left(x_n + (a+1)y_n\right)^2 = x_n^2 - \left(a^2+2a\right)y_n^2 = 1$. □

Theorem 7. *All positive integer solutions of the equation $x^2 - \left(a^2+2a\right)y^2 = 1$ are given by*

$$(x_n, y_n) = \left(\left(V_n\left(2a+2,-1\right)\right)/2, U_n\left(2a+2,-1\right)\right)$$

with $n \geq 1$.

Proof. By Theorem 6-ii, all positive integer solutions of the equation $x^2 - \left(a^2+2a\right)y^2 = 1$ are given by

$$x_n + y_n\sqrt{a^2+2a} = \left(a+1+\sqrt{a^2+2a}\right)^n$$

with $n \geq 1$. Assume that $\alpha = a+1+\sqrt{a^2+2a}$ and $\beta = a+1-\sqrt{a^2+2a}$. Then $\alpha - \beta = 2\sqrt{a^2+2a}$.

$$x_n + y_n\sqrt{a^2+2a} = \alpha^n$$

and

$$x_n - y_n\sqrt{a^2+2a} = \beta^n.$$

Therefore $x_n = \frac{\alpha^n+\beta^n}{2} = \frac{V_n(2a+2,-1)}{2}$ and $y_n = \frac{\alpha^n-\beta^n}{2\sqrt{a^2+2a}} = \frac{\alpha^n-\beta^n}{\alpha-\beta} = U_n\left(2a+2,-1\right)$. That is, $(x_n, y_n) = \left(\frac{V_n(2a+2,-1)}{2}, U_n\left(2a+2,-1\right)\right)$. □

Theorem 8. *The Pell equation $x^2 - \left(a^2+2a\right)y^2 = -1$ has no positive integer solutions.*

Proof. The length of the period of continued fraction $\sqrt{a^2+2a}$ is 2, that is even, then the equation $x^2 - \left(a^2+2a\right)y^2 = -1$ has no positive integer solutions. □

Theorem 9. *The fundamental solution of the Pell equation $x^2 - \left(a^2+2a\right)y^2 = 4$ is*

$$(x_1, y_1) = (2a+2, 2).$$

Proof. If a is odd, then $a^2+2a \equiv 3 \pmod 4$. It is obvious from Theorem 3 and Theorem 6-ii.

If a is even, then $a^2+2a \equiv 0 \pmod 4$. Let $a = 2t$. Then the equation becomes $x^2 - \left(4t^2+4t\right)y^2 = 4$. Therefore we consider the Theorem 2. That is, we must check the equation $x^2 - \left(t^2+t\right)y^2 = 1$.

The continued fraction expansion of $\sqrt{t^2+t}$ is

$$\sqrt{t^2+t} = \begin{cases} \left[1;\overline{2}\right] & , \text{ if } t=1 \\ \left[b;\overline{2,2b}\right] & \text{ if } t>1. \end{cases}$$

Then the fundamental solution of the equation $x^2 - \left(t^2+t\right)y^2 = 1$ is $(2t+1, 2)$. Therefore, from the Theorem 2, the fundamental solution of the equation $x^2 - \left(4t^2+4t\right)y^2 = 4$ is $(4t+2, 2)$ i.e. $(x, y) = (2a+2, 2)$. This completes the proof. □

Theorem 10. *All positive integer solutions of the equation $x^2 - \left(a^2+2a\right)y^2 = 4$ are given by*

$$(x_n, y_n) = \left(V_n\left(2a+2,-1\right), 2U_n\left(2a+2,-1\right)\right)$$

with $n \geq 1$.

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Proof. Using Theorem 4 and Theorem 9, all positive integer solutions of the equation $x^2 - \left(a^2 + 2a\right) y^2 = 4$ are given by

$$x_n + y_n\sqrt{a^2 + 2a} = \left(2a + 2 + 2\sqrt{a^2 + 2a}\right)^n / 2^{n-1} = 2\left(\left(2a + 2 + 2\sqrt{a^2 + 2a}\right)/2\right)^n$$

with $n \geq 1$. Assume that $\alpha = \left(2a + 2 + 2\sqrt{a^2 + 2a}\right)/2$ and $\beta = \left(2a + 2 - 2\sqrt{a^2 + 2a}\right)/2$. Then $\alpha - \beta = 2\sqrt{a^2 + 2a}$.

$$x_n + y_n\sqrt{a^2 + 2a} = 2\alpha^n$$

and

$$x_n - y_n\sqrt{a^2 + 2a} = 2\beta^n.$$

Therefore $x_n = \alpha^n + \beta^n = V_n\left(2a + 2, -1\right)$ and $y_n = \frac{\alpha^n - \beta^n}{\sqrt{a^2+2a}} = 2\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2U_n\left(2a + 2, -1\right)$. That is, $\left(x_n, y_n\right) = \left(V_n\left(2a + 2, -1\right), 2U_n\left(2a + 2, -1\right)\right)$. □

Theorem 11. *Let $a > 2$. The Pell equation $x^2 - \left(a^2 + 2a\right) y^2 = -4$ has no positive integer solutions.*

Proof. If a is odd, then $a^2 + 2a \equiv 3 \pmod 4$. From Theorem 1, we know that the equation $x^2 - \left(a^2 + 2a\right) y^2 = -4$ has positive integer solutions if and only if the equation $x^2 - \left(a^2 + 2a\right) y^2 = -1$ has positive integer solutions. But, from Theorem 8, the equation $x^2 - \left(a^2 + 2a\right) y^2 = -1$ has no positive integer solutions. Therefore, $x^2 - \left(a^2 + 2a\right) y^2 = -4$ has no positive integer solutions.

If a is even, then $a^2 + 2a$ is even. Assume by way of contradiction that there are positive integers m and n such that $m^2 - \left(a^2 + 2a\right) n^2 = -4$. a and $a^2 + 2a$ are even. Therefore, m is even. Let $a = 2k$. Then $m^2 - \left(4k^2 + 4k\right) n^2 = -4$ and we get $\left(m/2\right)^2 - \left(k^2 + k\right) n^2 = -1$.

The continued fraction expansion of $\sqrt{k^2 + k}$ is

$$\sqrt{k^2 + k} = \begin{cases} \left[1; \overline{2}\right] & \text{if } k = 1 \\ \left[k; \overline{2, 2k}\right] & \text{if } k > 1. \end{cases}$$

We know that $a > 2$. Therefore, $k > 1$. Thus, the length of the period of continued fraction $\sqrt{k^2 + k}$ is 2, that is even, then the equation $\left(m/2\right)^2 - \left(k^2 + k\right) n^2 = -1$ has no positive integer solutions. So this is a contradiction. Then the equation $x^2 - \left(a^2 + 2a\right) y^2 = -4$ has no positive integer solutions. □

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Improving ARMA-GARCH forecasts for high frequency data with regime-switching ARMA-GARCH

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Abstract

We propose a new regime switching autoregressive model for financial time series that improves ARMA-GARCH (the hybrid model of autoregressive moving average and generalized autoregressive conditional heteroskedasticity) performance for high frequency data. Although ARMA - GARCH model has been widely used to characterize and model observed financial time series in stock markets at daily and lower frequencies (e.g., weekly, monthly), few studies occur on models which can characterize stock prices at very high sampling frequencies. We aim to improve ARMA-GARCH performance for high frequency data. Here, we attempt to incorporate autoregressive hidden Markov model (HMM) into GARCH style models to generate better backtesting results than existing models. We estimate and test ARMA-GARCH, ARIMA-IGARCH (autoregressive integrated moving average and integrated GARCH model), and FARIMA-FIGARCH (autoregressive fractionally integrated moving average and fractionally integrated GARCH) first and find that the white noise series is not i.i.d. Gaussian. Thus we apply the autoregressive HMM driven model to the white noise series and test it's forecasting effect. The results indicate that the standard ARMA-GARCH and our autoregressive-HMM-noises model can both performs good in daily S&P 500 log returns, while the autoregressive-HMM-noise model can do better in high frequency data.

Key words: regime-switching autoregressive model, hidden Markov, high frequency financial data, ARMA-GARCH model, FARIMA-FIGARCH model

1 Introduction

There are two essential aspects of financial returns for many financial decisions: observed process and volatility process. Knowing the dynamics of asset returns plays an important role in decision making. Volatility is essential in option trading. For example, volatility of exchange rates is an important element for pricing current options.

It is a widely observed phenomenon that volatility is time-dependent in financial data and that volatility has the tendency to cluster. The ARCH model and its generalized autoregressive conditional heteroskedasticity (GARCH) model describe the underlying time-varying volatility behavior so that they allow for both

volatility clustering and unconditional heavy tails. Consider an GARCH(p,q) model for observations r_t,

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= \sigma_t u_t, \qquad u_t \sim \mathbf{N}(0,1) \\ \sigma_t^2 &= \sum_{i=1}^{p} \alpha_i \sigma_{t-1}^2 + \sum_{j=1}^{q} \beta_j \varepsilon_{t-1}^2 + \gamma, \end{aligned}$$

where σ_t is the variance of ε_t conditional on the information available at time t. σ_t is called the conditional variance of ε_t. ARCH model was introduced by Engle [1982] to model inflation rates, and extended to GARCH model by Bollerslev [1987]. Since then, a large number of variants of the initial ARCH and GARCH models have been developed. See Bollerslev [2001] for an overview of the GARCH literature. While the purpose of ARCH/GARCH models is to model volatilities in financial data, the modeling of the conditional mean is still a major concern. Without an adequately specified conditional mean model, the statistical inference and empirical analysis might be wrong. A joint estimation of the conditional-mean and conditional-variance model is established. The joint ARMA-GARCH model is defined by writing

$$\begin{aligned} r_t &= \sum_{i=1}^{p} a_i r_{t-1} + \sum_{j=1}^{q} b_j \varepsilon_{t-1} + \varepsilon_t \\ \varepsilon_t &= \sigma_t u_t, \qquad u_t \sim \mathbf{N}(0,1) \\ \sigma_t^2 &= \sum_{i=1}^{p} \alpha_i \sigma_{t-1}^2 + \sum_{j=1}^{q} \beta_j \varepsilon_{t-1}^2 + \gamma, \end{aligned}$$

The stationary issue of ARMA-GARCH model is studied in Ling and McAleer [2003b]. They also provide adaptive estimation in nonstationary ARMA models with GARCH errors in Ling and McAleer [2003a].

It is well observed that the GARCH forecasts are too high during volatile periods. This phenomenon may be due to a high persistence of individual shocks in those forecasts. Lamoureux and Lastrapes [1990] demonstrates that this persistence may arise from the variance process varying between different models. For instance, the volatility persists in the high position for some time, and persists in the low position for some time. A GARCH model, which cannot capture this constancy of volatility, explains this volatility persistence as persistence of individual shocks. This example is illustrated in Timmermann [2000]. A parallel explanation for the persistence is Perron [1989] and Perron [1990]'s work that shows the structural change in the mean renders it more difficult to reject null hypothesis of unit root, which means permanent persistence of shocks occurs in the mean.

To solve the issue of excessive GARCH forecasts in the volatile period, we extend the GARCH model by introducing regimes of different volatility levels. A well known econometric regime-switching model is presented in Hamilton [1989]. The model is a nonlinear generalization of an unobserved components trend and cycle model, and parameter estimation can be calculated as a by-product of an iterative algorithm similar in spirit to the Kalman filter. It is observed that the usual numerical maximum of the likelihood functions is subject to computational difficulties associated with the often ill-behaved likelihood surface (multiple local maxima, essential singularities, and local increases as boundary conditions are approached). Expectation-Maximization (EM) algorithm(Karlis and Xekalaki [2003], McLachlan and Peel [2000]) is used to overcome the numerical difficulties. Neftci [1984] suggests a model with transition probabilities that are duration dependent. It is clear that understanding duration dependence in business cycle is important to understand and forecast business cycle and their economic nature.

A regime switching model with no autoregressive elements has been first investigated by Lindgren [1978] who proves a consistency property of maximum-likelihood estimators obtained for the model which assumes

an independent sequence of hidden states from a finite mixture distribution. Lindgren's result states that, in case y_t actually follows a hidden Markov model, the maximum-likelihood estimators obtained under the independence model are consistent for the stationary distribution of y_t. Regime-switching models that incorporate autoregressive elements can be located in the speech recognition literature Rabiner [1990] and Juang and Rabiner [1985].

Most models assume a stationary Markov transition process, and also assume only two or three regimes. Calvet and Fisher [2004] suggests a model with a much larger number of regimes. These multifractal models afford another approach for incorporating long-memory into volatility forecasting. Sims and Zha [2006] also advocates a model with a much larger number of regimes. This model's parameters are estimated with prior Bayesian information.

Formal tests of the null hypothesis of no Markov switching have been proposed by Garcia [1998], Hansen [2006], Hamilton and Perez-Quiros [1996] and Carrasco et al. [2004]. The problem is to test the null hypothesis that there are K regimes against the alternative of $K+1$. When $K=1$, the null hypothesis is to test whether there are any shifts in regimes at all. The parameters driving the dynamic of the underlying Markov chain are not identified under the null hypothesis. As a result, the testing problem is non-standard and the likelihood ratio test does not converge to a chi-square distribution. Garcia [1998], studies the asymptotic distribution of a sup-type Likelihood ratio test. Hansen [2006] treats the likelihood as a empirical process indexed by all the parameters (those identified and those unidentified under the null). His test relies on taking the supremum of the likelihood ratio over the nuisance parameters. Both papers require estimating the model under the alternatives, which may be cumbersome. Carrasco et al. [2004] derives a class of information matrix-type tests and show that they are equivalent to the likelihood ratio test. Hence, our tests are asymptotically optimal. Moreover these tests are easy to implement as they do not require the estimation of the model under the alternative.

We propose a new regime switching autoregressive model for financial time series that improves ARMA-GARCH performance for high frequency data. Although the hybrid model of autoregressive moving average and generalized autoregressive conditional heteroskedasticity (ARMA-GARCH model) has been widely used to characterize and model observed financial time series in stock markets at daily and lower frequencies (e.g., weekly, monthly), few studies demonstrate models that can characterize stock prices at very high sampling frequencies. Here, we aim to improve ARMA-GARCH performance for high frequency data by attempting to incorporate autoregressive HMM driven models into GARCH style models to generate better backtesting results than existing models.

The remainder of this paper is organized as follows. Section 2 starts with presenting our autoregressive hidden Markov model (HMM). Next, we discusses the limitation of the existing p.d.f. function for the autoregressive HMM, and put forth improved density functions under different assumptions. Then, we apply the EM-algorithm to estimate the transition matrix of the autoregressive HMM and explore the performance of this estimation method in different settings. Finally, we provide an enhanced estimation method for autoregressive coefficients. Section 3 focuses on empirical studies. We estimate and test ARMA-GARCH, ARIMA-IGARCH, and FARIMA-FIGARCH by using S&P 500 index data. Based on the statistics results that the white noise series is not i.i.d. Gaussian, we incorporate autoregressive HMM driven model into ARMA-GARCH style models by adding the autoregressive HMM driven element to the white noise series and test it's forecasting effect. The encouraging statistic results are shown. Section 4 concludes.

2 The model

2.1 Regime-switching autoregressive models with Gaussian innovations

In this article, we propose a model for modeling univariate time series. This model is employed for short-term prediction of asset prices or returns. This model is based on regime-switching and autoregressive models. Consider an observation window of length K moving along the time series data with overlapping length M. For example, for time series $\{s_t\}$, the observation window at time n is vector $\vec{s}_n$ with components $\{s_n, s_{n+1}, \ldots, s_{n+K}\}$, and the next window is $\{s_{n+M}, x_{n+M+1}, \ldots, x_{n+M+K}\}$. First and second windows share M components. Components within each window have autoregressive structure, furthermore, the coefficients and volatilities of the autoregressive process follow a two state hidden Markov process (HMP). We present the model by writing

$$\begin{aligned} x_n &= -\sum_{i=1}^{p} a_i(\omega_n) x_{n-i} + e_n \\ s_n &= x_n \sigma(\omega_n) \qquad \omega_n \in \{1, \ldots, N\}, \end{aligned} \tag{2.1}$$

where x_n is normalized s_n, ω_n denotes a random state, N is the number of states, e_k are Gaussian i.i.d. random variables with mean 0 and variance 1, and p is the order of autoregression (Nelson et al. [2001], Meyers [2010] and Lai and Xing [2008]). Autoregressive coefficients a_i and variance σ follows discrete Markov process. In other words, the state of the Markov chain ω_n affects the values of the autoregression coefficient a_i and the scaling parameter σ_i. For example, if this is a two state Markov process and the order of autoregression $p = 2$, then $a_1 \in \{a_1(1), a_1(2)\}$, $a_2 \in \{a_2(1), a_2(2)\}$, $\sigma \in \{\sigma(1), \sigma(2)\}$. This process is driven by transition matrix $\begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}$.

Rabiner [1990] introduces the density function of the Gaussian autoregressive source in his section on autoregressive HMMs. The density function for $\vec{s}$ is

$$f(\vec{s}) = (2\pi\sigma^2)^{-\frac{K}{2}} \exp\left(-\frac{1}{2\sigma^2}\delta(\vec{s}, a)\right), \tag{2.2}$$

where

$$\begin{aligned} \delta(\vec{s}, a) &= r_a(0) r(0) + 2\sum_{i=1}^{p} r_a(i) r(i) \\ a' &= [1, a_1, \ldots, a_p] \\ r_a(i) &= \sum_{n=0}^{p-i} a_n a_{n+i} \\ r(i) &= \sum_{n=0}^{K-i-1} x_n x_{n+i}. \end{aligned}$$

Nevertheless, this probability density function has an significant approximation that are mentioned in the literature. We discover the assumption for this approximation, which doesn't hold with respect to short sequence of financial or economic data. Moreover, we introduce two accurate density functions those are more applicable for financial time series.

The first improved density function imposes no structure on first p sample data. Given the first p samples $\{x_1 \ldots x_p\}$, we may rewrite (2.1) as

$$H\vec{x} = \vec{e},$$

where $\vec{x} = \{x_1, \cdots, x_K\}$, $\vec{e} = \{\varepsilon_1, \cdots, \varepsilon_p, e_1, \cdots, e_{K-p}\}$, $\varepsilon \sim \mathbf{N}(0,1)$ and

$$H = \left(\begin{array}{cccc|cc} h_{11} & 0 & \cdots & 0 & 0 & 0 \\ h_{21} & h_{22} & \cdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{p1} & h_{p2} & \cdots & h_{pp} & 0 & 0 \\ \hline a_p & a_{p-1} & \cdots & a_1 & 1 & 0 \\ 0 & a_p & \cdots & a_2 & a_1 & 1 \end{array}\right) = \begin{array}{c|c} H_{11} & 0 \\ \hline H_{21} & H_{22} \end{array} . \tag{2.3}$$

We propose a method to calculate the accurate form of density function for $\vec{s}$ as follow:

1. Compute $\Sigma_{\vec{s}} = \mathbf{E}(\vec{s}\vec{s}^T)$ with $\vec{s} = \{s_1, \ldots, s_K\}$
2. Use Eigenvalue Decomposition to get

$$B^T \Sigma_{\vec{s}} B = \beta = \begin{pmatrix} \beta_0 & & \cdots & 0 \\ & \beta_1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \beta_{K-1} \end{pmatrix}$$

   where B is an upper triangular matrix, the diagonal elements of which are all unity.

3. Get the probability density

$$f(x \mid \Sigma_{\vec{s}}) = (2\pi)^{-K/2} (\sigma^2)^{-(K-p)/2} \left. \prod_{i=0}^{p-1} \frac{\beta_i}{\sigma^2} \right)^{-\frac{1}{2}} \exp\{-\vec{s}^t H^t H \vec{s}/(2\sigma^2)\}. \tag{2.4}$$

The second improved density function imposes linear autoregressive structure on first p sample data. Aim to address the problem of (2.1) being undefined for the first p samples, we may introduce some ghost variables and rewrite (2.1) as

$$\begin{aligned} x_1 + a_1 x_0 + a_2 x_{-1} &= \varepsilon_1 \\ x_2 + a_1 x_1 + a_2 x_0 &= \varepsilon_2 \\ x_3 + a_1 x_2 + a_2 x_1 &= \varepsilon_3 \\ &\vdots \\ x_K + a_1 x_{K-1} + a_2 x_{K-2} &= \varepsilon_K \end{aligned}$$

where ε_i are i.i.d. $\mathbf{N}(0,1)$. The parameters are $\{a_1, a_2, \ldots, a_p, x_0, x_{-1}, \ldots, x_{-p}, \sigma\}$. Variables x_0 and x_{-1} are ghost variables, which can be treated as scalars. Then, the following steps for calculation of the density function is similar to the first density function. See details in the references. Comparison of the degrees of freedom between (2.3) and (2.5) shows the degree of freedom for the method with linear structure increases much slower than the one without linear structure as autoregressive order p increases. We prefer a parsimonious model with small number of degree of freedom, which is the case here.

2.2 Estimation of transition matrix

The BaumWelch algorithm is a particular case of a generalized expectation-maximization (GEM) algorithm. It can compute maximum likelihood estimates and posterior mode estimates for the parameters (transition and emission probabilities) of an HMM, when given only emissions as training data. This algorithm has been applied to estimate parameters of hidden Markov models with discrete and Gaussian observations (Baum et al. [1970], Welch [2003], Baggenstoss [2001], Jelinek [1998] and Bilmes et al. [1998]). Our enhanced Baum-Welch algorithm for the autoregressive hidden Markov model that was introduced in last section is discussed in this section. We start by introducing notations, then introduce the algorithm, finally, we give a brief description of the performance.

Notation The Markov system can be described at any time by two variables: observation o_t and state S_t. Variable o_t is observable and S_t is latent. The parameters for the HMM are $\{A,B,\pi\}$, where $A=\{a_{ij}\}$ is the transition matrix, $B=\{b_i(x_t)\}$ is emission probability and π_i is the initial distribution. Two intermediate variables need to be defined first:

$$\alpha_t(i)=P(o_1=x_1,\dots,o_t=x_t,S_t=i)$$

and

$$\beta_t(j)=P(o_{t+1}=x_{t+1},\dots,o_T=x_T\mid S_t=i).$$

Variable $\alpha_t(i)$ can be calculated with the forward method (Rabiner [1990], Frey [2010]) with two steps: initialization step

$$\alpha_1(i)=\pi(i)b_i(x_1)$$

and induction step

$$\alpha_{t+1}(j)=\Big(\sum_{i=1}^{N}\alpha_t(i)a_{ij}\Big)b_j(x_{t+1}) \qquad \text{for} \quad t=1,\dots,T-1. \tag{2.5}$$

Variable $\beta_t(j)$ can be calculated with backward method also with two steps: initialization step

$$\beta_T(i)=1$$

and induction step

$$\beta_t(j)=\Big(\sum_{i=1}^{N}\beta_{t+1}(i)a_{ij}\Big)b_j(x_{t+1}) \qquad \text{for} \quad t=T-1,\dots,0. \tag{2.6}$$

E-step and M-step Then with the E-step, we can get

$$\begin{aligned}\xi_t(i,j)&=P(q_t=i,q_{t+1}=j\mid X,\theta)\\&=\frac{P(q_t=i,q_{t+1}=j,X\mid\theta)}{P(X\mid\theta)}\\&=\frac{\alpha_t(i)\beta_{t+1}(j)a_{ij}b_j(x_{t+1})}{\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_t(i)\beta_{t+1}(i)a_{ij}b_j(x_{t+1})}.\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\gamma_t(i) &= P(q_t = i \mid X, \theta) \\ &= \sum_{j=1}^{N} \xi_t(i,j) \\ &= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}.\end{aligned} \tag{2.8}$$

The second equation holds because

$$\beta_t(j) = (\sum_{i=1}^{N} \beta_{t+1}(i)a_{ij})b_j(x_{t+1}).$$

After computing those intermediate variables, we estimate mean vector μ_i and covariance matrix Σ_i from sample data as

$$\mu_i = \frac{\sum_{t=1}^{T} x_t \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}$$

and

$$\Sigma_i = \frac{\sum_{t=1}^{T}(x_t - \mu_i)(x_t - \mu_i)' \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}, \tag{2.9}$$

then use algorithm (4) to estimate H_i from Σ_i. So the emission probability for observation $\vec{x}$ is

$$b_t(i) = (2\pi)^{-K/2} |\Sigma_i|^{-1/2} \exp\{-\frac{1}{2}x^t \Sigma_i^{-1} x\},$$

where

$$\Sigma_i^{-1} = H_i^t H_i \qquad \text{and} \qquad |\Sigma_i| = |H_i|^{-2}.$$

A reasonable re-estimate formula for transition matrix is

$$\begin{aligned}\bar{a}_{ij} &= \frac{\text{expected number of transitions from state } S_i \text{ to state } S_j}{\text{expected number of transitions from state } S_i} \\ &= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}\end{aligned} \tag{2.10}$$

The pseudocode below presents the algorithms of applying Balm Welch to autoregressive HMM.

Algorithm 1 HMM Forward.

1: Initialize: $t \leftarrow 0$, a_{ij}, b_j, visible sequence $\vec{o}$, $\alpha_j(0)$
2: **repeat**
3: $\quad t \leftarrow t+1$
4: $\quad \alpha_j(t) \leftarrow b_j(o_t)\sum_{i=1}^{M} \alpha_i(t-1)a_{ij}$
5: **until** $t = T$
6: **return** $\alpha_j(T)$ for the final state

Algorithm 2 HMM Backward.

```
Initialize: t ← T, a_ij, b_j, visible sequence o⃗, β_j(T)
repeat
    t ← t − 1
    β_i(t) ← Σ_{j=1}^{M} β_j(t+1) a_ij b_j o_{t+1}
until t = 1
return β_i(0) for the known initial state.
```

Algorithm 3 HMM EM algorithm.

```
Initialize: a_ij, b_j, training sequence x⃗, convergence criterion θ, z ← 0
repeat
    z ← z + 1
    Compute α_i(t) by forward algorithm (1).
    Compute β_i(t) by backward algorithm (1).
    Compute sufficient statistics ξ_{i,j}(t) from α_i(t), β_i(t) and b(z−1) by Eq. (2.7)
    Compute sufficient statistics γ_i(t) from α_i(t), β_i(t) and b(z−1) by Eq. (2.8)
    Update transition matrix a(z) from a(z−1), ξ_{i,j}(t) and γ_i(t) by Eq. (2.10)
    Estimate covariance matrix Σ_i for each state i by Eq. (2.9)
    Compute H_i for each state i by Algorithm (4)
    Compute emission probabilities b(z) by Eq (2.2)
until z = T
return
```

Performance We illustrate the performance of Baum-Welch estimator by showing a numerical example. Given a transition matrix $\begin{matrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{matrix}$. The parameters for our testing cases are: $p_1 = (0.95, 0.8, 0.5)$, $p_2 = (0.8, 0.5, 0.3)$, $\mu = (0.5, -0.3)$, $\sigma = (0.5, 0.8)$ and sample size $T = 10000$.

Figure (1) shows a boxplot of Balm-Welch estimates for the hidden Markov driven Gaussian mixture model from replications 100 with sample size $T = 10000$ with parameters $\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\mu_1 = 0.5$, $\mu_2 = -0.3$, $\{p_1, p_2\} \in P$. P is a set of combinations of $\{p_1, p_2\}$. Let

$$\begin{aligned} P :=& \{0.95, 0.8\}, \{0.95, 0.5\}, \{0.95, 0.3\}, \{0.8, 0.8\}, \{0.8, 0.5\}, \\ & \{0.8, 0.3\}, \{0.5, 0.8\}, \{0.5, 0.5\}, \{0.5, 0.3\}. \end{aligned}$$

The results demonstrates that estimator performs well for p_1, p_2, σ_1 and σ_2, but tends to have some bias for μ_1 and μ_2. Although this bias exists, it still delivers reasonable estimates for a number of observation as small as 100. We can also see cases with symmetric parameters, for example, $p_1 = 0.5$, $p_2 = 0.5$, have superior estimations. We also draw boxplots of cases with number of observations = $(25, 50, 200, 3000)$ for each combination. The results also demonstrates that models with symmetric parameters converge better under our algorithm.

2.3 Estimation of autoregressive coefficients

In our study, three estimation methods are derived for autoregressive coefficients. First method is an MLE method that needs approximation of the p.d.f. to some extent. Thus we employ the MLE method for

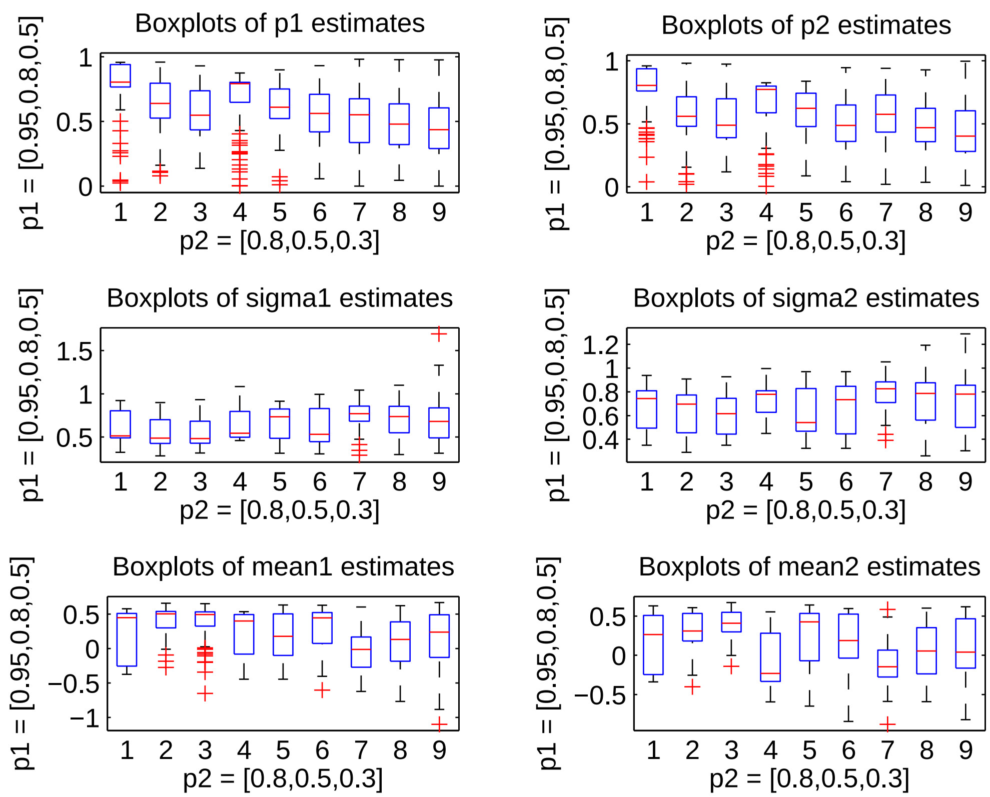

Figure 1: Boxplots of estimated p_1, p_2, σ_1, σ_2, μ_1, μ_2 for number of observations = 100 with parameters $\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\mu_1 = 0.5$, $\mu_2 = -0.3$, $\{p_1, p_2\} \in P$.

problems with large sample size T. The second method is an ordinary least square (OLS) method. Since it does not give information about the distribution of data, we only use it to generate an initial guess. The third method is the Frobenius norm minimization method that is parsimonious, stable, and shows high resolution precision according to our tests. We choose Frobenius norm minimization method as the major estimation method for our model. In addition to detailing this method, we present two examples to demonstrate the accuracy and stability of this method.

Estimation Method The assumption for this method is that $\vec{x}$ are correlated multivariate normal variables and $\vec{e}$ are i.i.d. $\mathbf{N}(0,1)$ random variables. We have

$$H\vec{x} = \vec{e}$$

with

$$H = \left(\begin{array}{cc|ccc} h_{11} & 0 & & 0 & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & 0 \\ 0 & a_2 & a_1 & 1 & \\ 0 & 0 & a_2 & a_1 & 1 \end{array}\right).$$

First, we estimate covariance matrix Σ_x from sample data x.

Then, based on equation(2.2), we know

$$\Sigma_x^{-1} = H^t H,$$

so we use Cholesky decomposition to get Σ_x

$$\begin{aligned} U^t U &= \mathrm{Chol}(\Sigma_x) \\ \tilde{H}^{-1} &= U^t. \end{aligned}$$

Last, we minimize the Frobenius norm of $\tilde{H} - H$

$$\begin{aligned} \varepsilon_{ij} &= \tilde{H}_{ij} - H_{ij} \\ a^* &= \min_{a_1,a_2}\left(\sum\nolimits_{1<i,\,j<K} \varepsilon_{ij}^2\right) \end{aligned}$$

Algorithm 4 Estimation of autocorrelation coefficients.

1: Given $\vec{x} = x_1^{(i)}, \cdots, , x_K^{(i)}$.
2: Estimate covariance matrix $\vec{\Sigma}_x$ from $\vec{x}$.
3: $U \leftarrow \mathrm{Chol}(\vec{\Sigma}_x)$
4: $\tilde{H} = (U^{-1})^t$
5: $\varepsilon_{ij} = \tilde{H}_{ij} - H_{ij}$
6: $a^* \leftarrow \min_{a_1,a_2}(\sum_{1<i,\,j<K} \varepsilon_{ij}^2)$

Accuracy We begin with

$$H = \left(\begin{array}{cc|ccc} h_{11} & 0 & & 0 & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & 0 \\ 0 & a_2 & a_1 & 1 & \\ 0 & 0 & a_2 & a_1 & 1 \end{array} \right).$$

After taking

$$\Sigma_x = H^{-1} H^{-t},$$

we generate scenarios $\vec{x}$ with covariance Σ_x and expectation $\mathbf{E}(\vec{x}) = 0$. Having $\vec{x}$, we use our method Algorithm (4) to find $\tilde{H}$, then make a comparison of Frobenius norms between $\tilde{H}_{ij}$ and H_{ij} to see if

$$\|\tilde{H} - H\|_F^2 \simeq 0,$$

where

$$\Delta_F := \|\tilde{H} - H\|_F^2 = \sum_{i,j} (\tilde{H}_{i,j} - H_{i,j})^2.$$

Table (1) shows estimation errors measured by Δ_F with respect to 6 sets of parameters. We can see when $\vec{a} = \{a_0, a_1, a_2\}$ with $a_0 = 1$, $|a_1| < 1$ and $|a_2| < 1$, errors Δ_F is less than 0.02. When $a_0 = 1$, $|a_1| > 1$ and $|a_2| > 1$, errors Δ_F is larger than 0.03. From the model definition, we know $a_0 = 1$, $|a_1| < 1$ and $|a_2| < 1$ is a reasonable assumption, which means yesterday's price has less impact than today's price, the day before yesterday's price has less impact than both yesterday's and today's price. Table (2) shows when $|a_1| < 1$ and $|a_2| < 1$, this method can get errors less than 0.02 with simulation number 5000.

Table 1: Errors Δ_F with respect to different parameters, with 10000 simulations.

$\vec{a}$	$\{h_{11}, h_{21}, h_{22}\}$ $\{0.02, 0.01, 0.05\}$	$\{0.2, 0.1, 0.5\}$
$\{1, 5, 3\}$	0.0548	0.0345
$\{1, 0.5, 0.3\}$	0.0015	0.0046
$\{1, 0.5, -0.3\}$	0.0030	0.0065

Table 2: Errors Δ_F with respect to different simulation numbers.

number of simulations	$\{h_{11}, h_{21}, h_{22}\} = \{0.2, 0.1, 0.5\}$ $\vec{a} = \{1, 5, 3\}$	$\vec{a} = \{1, 0.5, 0.3\}$
100	0.4146	0.0871
1000	0.1050	0.0381
5000	0.0845	0.0197
10000	0.0319	0.0056

Stability We add a small value ι to zero entries in H with $|\iota_i| \leq \frac{|h_{ij}|}{100}$ to make it more realistic

$$H = \left(\begin{array}{cc|ccc} h_{11} & \iota & & \iota & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & \\ \iota & a_2 & a_1 & 1 & \\ \iota & \iota & a_2 & a_1 & 1 \end{array} \right).$$

With new H, we apply the same algorithm as in Section (2.3). Compared with Table (2), results in Table (3) show that with perturbation, this method can also get errors less than 0.02 with simulation number 5000 when $|a_1| < 1$ and $|a_2| < 1$. To summarize, this is a stable method.

Table 3: Errors Δ_F with respect to different simulation numbers with perturbations.

	$\{h_{11}, h_{21}, h_{22}\} = \{0.2, 0.1, 0.5\}$	
numer of simuation	$\vec{a} = \{1, 5, 3\}$	$\vec{a} = \{1, 0.5, 0.3\}$
100	0.7477	0.1595
1000	0.1657	0.0334
5000	0.0759	0.0178
10000	0.0278	0.0083

3 Empirial analysis

We aim to improve the ARMA-GARCH performance for high frequency data by incorporating autoregressive HMM driven model that was presented in previous section into GARCH style models to generate better backtesting results than existing models. Our empirical study is divided into two parts. First, we estimate and test ARMA-GARCH, ARIMA-IGARCH, and FARIMA-FIGARCH by using S&P 500 index data. The statistics result shows the white noise series is not i.i.d. Gaussian. Second, we apply the autoregressive HMM driven model to the white noise series and test it's forecasting performance. The results indicate that standard ARMA-GARCH and our autoregressive-HMM-noises model both perform well in daily S&P 500 log returns, while the autoregressive-HMM-noise model does better in high frequency data.

3.1 Market estimation

The time series data consists of historical closing index values of the S&P 500 index until October 24, 2012, obtained from Bloomberg L.P. The daily, 1 hour, 30 minute, 5 minute, and 1 minute log-returns are calculated. The size of this data set, 3600 observations, is large enough for ARMA-GARCH model fitting. Smaller sizes of 1200 and 2400 observations are also tested.

3.1.1 Global MLE and quasi MLE estimation

The ARMA(p,q)-GARCH(m,n) model (Bollerslev [1986], Brockwell [2005]) for$\{x_t\}$ is given by

$$\begin{aligned} x_t &= c + \sum_i^p a_i x_{t-i} + \sum_i^q b_i \varepsilon_{t-i} + \varepsilon_t \\ \varepsilon_t &= \sigma_t u_t, u_t \sim \mathbf{N}(0,1) \\ \sigma_t^2 &= \gamma + \sum_i^m \alpha_i \sigma_{t-i}^2 + \sum_i^m \beta_i \varepsilon_{t-i}^2. \end{aligned} \tag{3.1}$$

We estimate parameters using the classical maximum likelihood estimation (MLE) procedure. The log-likelihood of an ARMA-GARCH model in the form of (3.1) is given by

$$L(\theta \mid u_1, \ldots, u_T) = \Sigma_{t=1}^T \log \quad \frac{u_t \mid \theta}{\sigma_t} \quad ,$$

where $\theta = (c, a_1, \ldots, a_p, b_1, \ldots, b_q, \gamma, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ is the vector of parameters to be estimated and $f(x)$ is the probability density function of the distribution assumed for u_t with $t = 1, \ldots, T$. There are two types of MLE we use

- Quasi-MLE: Given an observed univariate time series, estimate the parameters of a conditional mean specification of ARMA form first. The estimation process also infers the residuals ε_t from the input series, then fits the conditional variance specification of GARCH via maximum likelihood to residuals ε_t(Tse and Tsui [2002], Chan et al. [2005]). Thus Quasi-MLE requires two maximizations of two different likelihood functions: one for the mean process fit and another for the conditional variance process.
- Global-MLE: only one maximization of the global likelihood function is performed.
- The Matlab garchfit function does global-MLE estimation, our programs armaxfilter.m and tarch.m do quasi-MLE estimation.

Table (4) shows estimation results of the ARMA(1,1)-GARCH(1,1) for S&P 500 data with sample size 1200. Based on our study, global-MLE has better performance for large sample sizes, while quasi-MLE estimates are more reliable for small data sizes. We use quasi-MLE for our later studies.

3.1.2 GARCH models

The parameter estimates of ARMA(1,1)-GARCH(1,1) models for S&P 500 data are reported in Table (5). Standard deviations are given in parentheses. This table shows different patterns between daily and hourly returns and high frequency returns. We can see daily returns, hourly returns, and 30 minutes returns have a unit root phenomena with almost $\alpha = 1$, which means this time series is not stationary. (A time series $x_1, x_2, x_3, \ldots$ is said to be covariance stationary if $\mathbf{E}(x_t)$ and $\mathbf{Cov}(x_t, x_{t+k})$ do not depend on t.) The unit root phenomenon is especially obvious for 30 minute returns, where $\alpha = 1$. We do not observe a unit root phenomenon in the 5 minute and 1 minute return series for the ARMA(1,1)-GARCH(1,1) model.

We choose three time series with 1200, 2400 and 3600 historical data points before October 24, 2012. Table (5) also shows that parameter estimates give similar results for different sample sizes, which means the last 1200 data points of this time series are essential to determine model parameters.

Table 4: Parameter estimates of ARMA(1,1)-GARCH(1,1) for S&P 500 data with sample size 1200.

	c	**a(AR)**	**b(MA)**	γ	α	β
			global MLE			
daily return	0.02	1.00	-0.09	0.00	0.87	0.12
5mins return	0.18	0.99	0.01	0.00	0.00	1.00
1min return	1.86	0.87	0.57	0.00	0.40	0.57
			quasi MLE			
daily return	0.05	1.00	-0.14	0.00	0.89	0.10
5mins return	0.11	0.99	0.03	0.00	0.75	0.25
1min return	0.03	1.00	0.10	0.00	0.91	0.09

The estimated parameters α and β observed in Table (4) and Table (5) sum up to values close to 1. Based on this observation, we apply the integrated GARCH(1,1), which is named IGARCH(1,1) (Mikosch and Stărică [2004], Morana [2002]). IGARCH is a process for which

$$\gamma > 0$$

and

$$\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{n} \beta_i = 1.$$

Table (6) shows that the IGARCH(1,1) model captures the integrated feature.

Since the clear distinction between GARCH and IGARCH models has been criticized, we consider the generalized fractional integrated GARCH(FIGARCH) model (Beine et al. [2002], Baillie et al. [1996a]) and the corresponding mean process FARIMA (Ling and Li [1997], Baillie et al. [1996b]) to capture fractional features of a time-series of index returns. FARIMA processes are more specifically ARIMA(p, d, q) process with $0 < |d| < 0.5$, that satisfy difference equations of the form

$$(1-L)^d a(L) x_t = b(L)\varepsilon_t$$

where $a(L)$ and $b(L)$ are polynomials of degree p and q, respectively, satisfying, $a(z) \neq 0$, $b(z) \neq 0$, for all z such that $|z| \leq 1$. L is the lag operator, and ε_t is a white noise sequence with mean 0 and finite variance σ^2.

The conditional variance, h_t, of a FIGARCH(p, d, q) process is modeled as follows:

$$\sigma_t^2 = \omega + [1 - \beta(L) - \phi(L)(1-L)^d]\varepsilon_t^2 + \beta(L)\sigma_t.$$

The parameter estimates of FARIMA(1,1)-FIGARCH(1,1) for S&P 500 data are reported in Table (7). Standard deviations are given in parentheses. This table shows that for a FARIMA(1,1) mean process, every time series has fractional d. For a FIGARCH(1,1) variance process, daily, hourly, 30 minute, and 1 minute returns have fractional d, and only 5 minute returns have $d = 1$.

We also observe negative d in FARIMA, which can be explained by short memory. The long memory can be empirically observed, e.g. by a slowly decaying auto-covariance function (ACF) (Beran [1994]). The classic example of a long-range dependent process is the fractional autoregressive integrated moving average (FARIMA) model with a power-law ACF. It appears that the values of FARIMA with Gaussian noise, for the memory parameter d greater than 0, have such a slowly decaying ACF that it is not absolutely summable.

Table 5: Parameter estimates of ARMA(1,1)-GARCH(1,1) for S&P 500 data.

	c	a(AR)	b(MA)	γ	β	α
	Sample size = 1200					
daily return	-0.0001	0.7818	-0.8186	0.0000	0.0963	0.8363
	(0.0000)	(0.0160)	(0.0127)	(0.0000)	(0.0008)	(0.0026)
1 hour return	0.0001	0.1868	-0.2959	0.0000	0.0380	0.9507
	(0.0000)	(0.0683)	(0.0577)	(0.0000)	(1.5269e-04)	(1.4315e-04)
30 min return	-0.0001	-0.2496	0.2494	0.0000	0.0000	0.9998
	(0.0000)	(0.7116)	(0.6898)	(0.0000)	(0.0000)	(1.9878e-06)
5 mins return	-0.0001	-0.4004	0.4345	0.0000	0.3357	0.6641
	(0.0000)	(0.5252)	(0.5335)	(0.0000)	(0.0109)	(0.0042)
1 min return	0.0000	0.2312	-0.1691	0.0000	0.5261	0.0581
	(0.0000)	(0.0426)	(0.0445)	(0.0000)	(0.1102)	(0.6504)
	Sample size = 2400					
daily return	-0.0001	0.7103	-0.7518	0.0000	0.0762	0.9181
	(0.0000)	(0.0222)	(0.0209)	(0.0000)	(1.7174e-04)	(1.9106e-04)
1 hour return	-0.0000	-0.10	-0.1013	0.0000	0.0225	0.9746
	(0.000)	(6.755e-04)	(5.510e-04)	(0.0000)	(2.6664e-05)	(2.8123e-05)
30 min return	-0.0000	0.3225	-0.3017	0.0000	0.0079	0.9915
	(0.0000)	(0.0186)	(0.0185)	(0.0000)	(4.4229e-06)	(3.1188e-06)
5 mins return	-0.0000	-0.8436	0.8556	0.0000	0.3482	0.6516
	(0.0000)	(0.0924)	(0.0802)	(0.0000)	(0.0093)	(0.0029)
1 min return	-0.0000	0.2008	-0.1249	0.0000	0.4982	0.1858
	(0.0000)	(0.3282)	(0.3420)	(0.0000)	(0.0491)	(0.1046)
	Sample size = 3600					
daily return	-0.0001	0.5223	-0.6015	0.0000	0.0872	0.9056
	(0.0000)	(0.0506)	(0.0446)	(0.0000)	(9.3159e-05)	(1.0331e-04)
1 hour return	-0.0001	-0.9433	0.9256	0.0000	0.0186	0.9772
	(0.0000)	(0.0019)	(0.0027)	(0.0000)	(1.517e-05)	(2.011e-05)
30 min return	-0.0000	0.3124	-0.2842	0.0000	0.0070	0.9922
	(0.0000)	(0.1177)	(0.1195)	(0.0000)	(2.6991e-06)	(2.8575e-06)
5 mins return	-0.0000	-0.7744	0.7938	0.0000	0.3054	0.6287
	(0.0000)	(0.0339)	(0.0298)	(0.0000)	(0.0105)	(0.0091)
1 min return	0.0000	0.3954	-0.3163	0.0000	0.5387	0.1716
	(0.0000)	(0.0166)	(0.0134)	(0.0000)	(0.0322)	(0.0440)

Table 6: Parameter estimates of ARMA(1,1)-IGARCH(1,1) for S&P 500 data

	c	a(AR)	b(MA)	γ	β	α
	Sample size = 1200					
daily return	0.01	1.00	-0.01	0.00	0.12	0.89
1 hour return	0.07	0.99	-0.11	0.00	0.05	0.95
30 min return	0.06	1.00	0.01	0.00	0.00	1.00
5 mins return	-0.02	1.00	0.03	0.00	0.33	0.67
1 min return	0.06	1.00	0.06	0.00	0.66	0.34
	Sample size = 2400					
daily return	0.02	1.00	-0.03	0.00	0.08	0.92
1 hour return	0.06	1.00	-0.04	0.00	0.03	0.97
30 min return	0.01	1.00	0.02	0.00	0.01	0.99
5 mins return	-0.00	1.00	0.01	0.00	0.35	0.65
1 min return	0.01	1.00	0.07	0.00	0.57	0.43
	Sample size = 3600					
daily return	0.03	1.00	-0.08	0.00	0.09	0.91
1 hour return	0.04	1.00	-0.02	0.00	0.02	0.98
30 min return	0.01	1.00	0.03	0.00	0.01	0.99
5 mins return	0.03	1.00	0.02	0.00	0.36	0.64
1 min return	0.02	1.00	0.08	0.00	0.60	0.40

This behavior serves as a classical definition of the long-range dependence (Beran [1994]). When $d < 0$, the ACF still follows a power law, hence exhibiting more significant dependence than any other process with exponentially decaying ACF, such as, e.g. an autoregressive moving average (ARMA) time series, but the rate of decay is slower than for the d-positive case making the ACF absolutely summable. This negative memory phenomenon can be described as follows: increases in the values of the time series are likely to be followed by decreases and, conversely, decreases are more likely to be followed by increases (negative correlation). Such a time series is said to have short memory.

After fitting GARCH style models to time series, we examine the innovations. We fit classic tempered stable (CTS) distributions (Rachev et al. [2011], Rachev and Mittnik [2000]) to the innovations inferred from ARMA(1,1)-GARCH(1,1).

Let $\alpha \in (0,1)\bigcup(1,2)$, C, λ_+, $\lambda_- > 0$, and $m \in R$. X is said to have the classic tempered stable (CTS) distribution if the characteristic function of X is given by

$$\begin{aligned}\phi_x(u) &= \phi_{CTS}(u;\alpha,C,\lambda_+,\lambda_-,m)\\ &= \exp(ium - iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})\\ &\quad + C\Gamma(-\alpha)((\lambda_+ - iu)^{\alpha} - \lambda_+^{\alpha} + (\lambda_- + iu)^{\alpha} - \lambda_-^{\alpha})),\end{aligned}$$

and we denote $X \sim CTS(\alpha,C,\lambda_+,\lambda_-,m)$. Table (8) presents parameter estimates.

Table 7: Parameter estimates of FARIMA(1,d,1)-FIGARCH(1,d,1) for S&P 500 data

	d	a(AR)	b(MA)	ω	ϕ	d	β
				Sample size = 1200			
daily return	-0.6611	0.9629	-0.3293	0.0000	0.0486	0.3465	0.3075
	(0.0289)	(0.0289)	(0.0289)	(0.0000)	(0.1258)	(0.0258)	(0.2019)
1 hour return	-0.5174	0.9299	-0.5010	0.0000	0.0965	0.8070	0.9035
	(0.0283)	(0.0283)	(0.0283)	(0.0000)	(0.0020)	(0.0038)	(0.0006)
30 min return	-0.6483	0.9101	-0.2695	0.0000	0.4327	0.1346	0.5673
	(0.0289)	(0.0289)	(0.0289)	(0.0000)	(0.0194)	(0.0009)	(0.0250)
5 mins return	-0.5945	0.9661	-0.3741	0.0000	0.0000	1.0000	0.5939
	(0.0289)	(0.0289)	(0.0289)	(0.0000)	(0.0000)	(0.0125)	(0.0036)
1 min return	-0.6784	0.9304	-0.1943	0.0000	0.4605	0.0791	0.0000
	(0.0289)	(0.0289)	(0.0289)	(0.0000)	(0.0183)	(0.0436)	(0.0000)
				Sample size = 2400			
daily return	-0.4549	0.9049	-0.5049	0.0000	0.1424	0.4175	0.5218
	(0.0204)	(0.0204)	(0.0204)	(0.0000)	(0.0040)	(0.0048)	(0.0075)
1 hour return	-0.3679	0.8837	-0.5807	0.0000	0.0429	0.9141	0.9571
	(0.0204)	(0.0204)	(0.0204)	(0.0000)	(0.0010)	(0.0015)	(0.0002)
30 min return	-0.4329	0.8715	-0.4305	0.0000	0.4163	0.1674	0.5837
	(0.0204)	(0.0204)	(0.0204)	(0.0000)	(0.0025)	(0.0055)	(0.0153)
5 mins return	-0.3534	0.9063	-0.5919	0.0000	0.0000	1.0000	0.6787
	(0.0218)	(0.0218)	(0.0218)	(0.0000)	(0.0000)	(0.0208)	(0.0023)
1 min return	-0.4061	0.8387	-0.3554	0.0000	0.1933	0.2852	0.0000
	(0.0204)	(0.0204)	(0.0204)	(0.0000)	(0.0135)	(0.0107)	(0.0000)
				Sample size = 3600			
daily return	-0.3313	0.9104	-0.7107	0.0000	0.0430	0.7667	0.7805
	(0.0167)	(0.0167)	(0.0167)	(0.0000)	(0.0006)	(0.0030)	(0.0014)
1 hour return	-0.3260	0.8655	-0.5962	0.0000	0.0454	0.9091	0.9546
	(0.0182)	(0.0182)	(0.0182)	(0.0000)	(0.0007)	(0.0018)	(0.0003)
30 min return	-0.3085	0.7959	-0.4744	0.0000	0.4174	0.1653	0.5826
	(0.0167)	(0.0167)	(0.0167)	(0.0000)	(0.0046)	(0.0044)	(0.0195)
5 mins return	-0.3227	0.8689	-0.5581	0.0000	0.0000	1.0000	0.6517
	(0.0188)	(0.0188)	(0.0188)	(0.0000)	(0.0000)	(0.0293)	(0.0126)
1 min return	-0.3186	0.8058	-0.4072	0.0000	0.1584	0.2438	0.0000
	(0.0167)	(0.0167)	(0.0167)	(0.0000)	(0.0104)	(0.0050)	(0.0000)

Table 8: Fit standard CTS to innovations of ARMA-GARCH with sample size 3600

	α	λ_+	λ_-
daily return	0.0001	2.3900	2.1617
1 hour return	1.1908	0.2620	0.2640
30 min return	1.1057	0.2642	0.2750
5 mins return	1.3754	0.2851	0.2123
1 min return	0.7627	0.7431	0.6944

3.2 Out-of-Sample performance

One of the essential objectives of financial modeling is forecasting. The ARMA-GARCH style models studied in Chapter (3.1) are some of the most popular for univariate forecasting. Based on our observations in Chapter (3.1), the white noise series u_t are not i.i.d. $\mathbf{N}(0,1)$ in high frequency data. Correlations occur within the white noise series, thus we improve ARMA-GARCH models by introducing our new autoregressive regime-switching model into white noise series, which is named ARMA-GARCH with autoregressive regime-switching noise model. We forecast VaR models with both original ARMA-GARCH and our new ARMA-GARCH model with autoregressive regime-switching noises, and then compare performance with Bernoulli and Berkowitz tests.

3.2.1 Forecasting

The steps for forecasting via ARMA-GARCH models of the form:

$$\begin{aligned} x_t &= c + \sum_i^p a_i x_{t-i} + \sum_i^q b_i \varepsilon_{t-i} + \varepsilon_t \\ \varepsilon_t &= \sigma_t u_t, u_t \sim \mathbf{N}(0,1) \\ \sigma_t^2 &= \gamma + \sum_i^m \alpha_i \sigma_{t-i}^2 + \sum_i^m \beta_i \varepsilon_{t-i}^2, \end{aligned}$$

are as follows:

1. Estimate parameters and corresponding ε_t, σ_t and u_t, where $t = 1, \dots, h$.
2. Get $\sigma_{h+1}^2 = \hat{\gamma} + \hat{\alpha}\sigma_h^2 + \hat{\beta}\varepsilon_h^2$.
3. Generate 100 random variable $u_{h+1} \sim \mathbf{N}(0,1)$.
4. Get $\varepsilon_{h+1} = \sigma_{h+1} u_{h+1}$.
5. Get $x_{h+1} = \hat{c} + \hat{a} x_h + \hat{b}\varepsilon_h + \varepsilon_{h+1}$.
6. Move estimation window one step forward, estimate parameters and corresponding ε_t, σ_t and u_t, where $t = 2, \dots, h+1$.
7. Loop is closed.

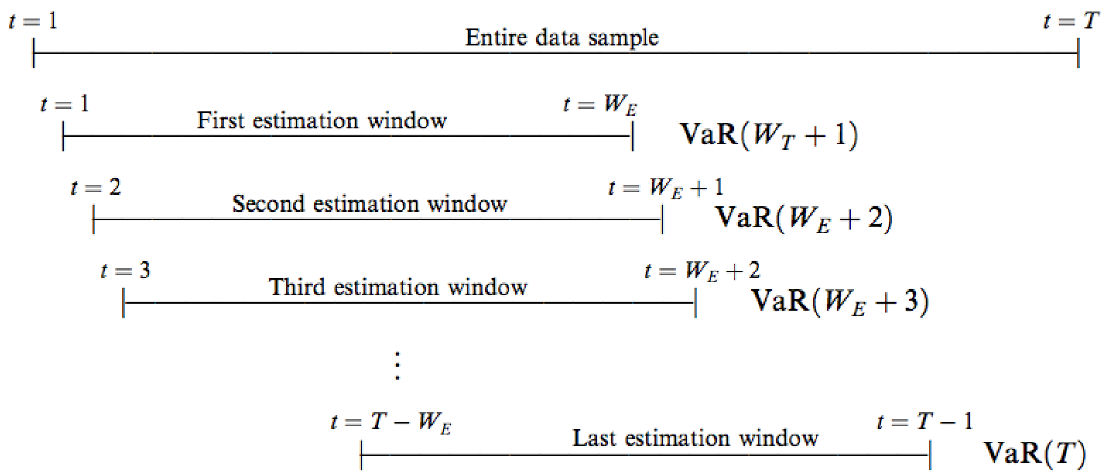

Figure 2: Compute VaR.

The steps for forecasting via ARMA-GARCH models with autoregressive regime-switching noises model are:

1. Estimate parameters and corresponding ε_t, σ_t and u_t, where $t=1,\dots,h$.
2. Fit autoregressive regime-switching model to noise series u_t, get autoregressive coefficients matrix $\{\hat{H}_1,\hat{H}_2\}$ and state transition matrix $\hat{A}$.
3. Simulate noise process $\tilde{u}_t$ with estimated autoregressive coefficients matrix $\{\hat{H}_1,\hat{H}_2\}$ and state transition matrix $\hat{A}$.
4. Get $\sigma^2_{h+1}=\hat{\gamma}+\hat{\alpha}\sigma^2_h+\hat{\beta}\varepsilon^2_h$.
5. Get $\varepsilon_{h+1}=\sigma_{h+1}\tilde{u}_{h+1}$.
6. Get $x_{h+1}=\hat{c}+\hat{a}x_h+\hat{b}\varepsilon_h+\varepsilon_{h+1}$.
7. Move estimation window one step forward, estimate parameters and corresponding ε_t, σ_t and u_t, where $t=2,\dots,h+1$.
8. Loop is closed.

3.2.2 Backtesting

Backtesting aims to take ex ante value-at-risk (VaR) (Tsay [2005]) forecasts from a particular model and compare them with ex post realized returns (i.e., historical observations). Whenever losses exceed VaR, a VaR violation is said to have occurred. There are several methods to backtest models. We discuss the Bernoulli and Berkowitz tests (Christoffersen [1998], Berkowitz [2001], Engle and Manganelli [2004], Berkowitz et al. [2011]) here.

To conduct Bernoulli test, we introduce specific notations: W_E is estimation window size; T denotes number of observations in a sample; η_t indicates whether a VaR violation occurs (i.e. $\eta = 1$); $v_i, i = 0, 1$ is number of violations $(i = 1)$ and number of no violation $(i = 0)$ observed.

We estimate parameters of the model from first estimation window W_E, then forecast VaR for day $E+1$, The estimation window is then moved forward by one step to get the risk forecast for day $E+2$. The estimation window is moved forward by one day until $T-1$ (Figure 2), then we have $T-E$ VaR forecasts. As the data from day $E+1$ to day T are already known, VaR forecasts can be compared with the actual outcome. If the actual return on a particular day exceeds the VaR forecast percentile limit, then the VaR limit is said to have been violated. We denote the violations as η_t, which has the value 1 when a violation occurs and 0 when a violation doesn't occur. The number of violations are stored in the variable v_1 and v_0, where v_1 is the number of days with violations and v_0 is the number of days without violations(Danielsson [2011]).

We then use the Bernoulli coverage test to find out the proportion of violations. The null hypothesis for VaR violations is:

$$H0 : \eta \sim \mathbf{B}(p)$$

where B stands for the Bernoulli distribution.

Table 9: Comparison of ARMA-GARCH(model 1) and HMM-autoregressive noise model(model 2) via Bernoulli test.

	Bernoulli test	
	Test statistics	p-value
	daily return	
Model 1	7.3524	0.0715
Model 2	5.2749	0.2527
	1 hour return	
Model 1	2.7773	0.5273
Model 2	6.5951	0.0960
	5 minute return	
Model 1	4.3962	0.3217
Model 2	3.8844	0.3742
	1 minute return	
Model 1	10.2653	0.0264
Model 2	0.7875	0.7525

The Bernoulli tests for standard ARMA(1,1)-GARCH(1,1) and ARMA(1,1)-GARCH(1,1) with regime-switching noise forecasting VaR for S&P 500 data are reported in Table (9). The length of estimation window is 1000, the forecasting horizon is 1000 steps, and number of paths is 200. We can see for daily, hourly, and 5 minute returns, both models have $p-$values larger than 5 percent. For 1 minute returns, the $p-$value for the standard ARMA(1,1)-GARCH(1,1) model is 0.0264, which means that the null hypothesis is rejected, while the $p-$value of our HMM-autoregressive noise model is 0.7525 and thus not rejected.

Besides the Bernoulli test, the Berkowitz test is also conducted. A test would be needed to verify the $i-$th observed return r_i follows the predicted distribution P_i from the model. The problem is we have only one observed return, r_i, for each sliding-window. We carry out a probability integral transform PIT transformation

and map each r_i to its percentile point on its forecasted density function. For example, if an observed return is equal to the $30th$ percentile on the forecasted density, then its value maps to 0.3. Under the null hypothesis that the model is adequate and the mapped value $\hat{r}_i = P_i^{-1}(r_i)$ follows a uniform distribution $U(0,1)$. So we can evaluate the model by using Kolmogorov's test to test if $\hat{r}_i$ is uniformally distributed. However, we use the Berkowitz test to perform the transformation $\hat{\hat{r}}_i \phi^{-1}(\hat{r}_i) \sim N(0,1)$. Subsequently we can use any test for normality over $\hat{\hat{r}}_i$ to evaluate our model(Lobato et al. [2007] Christodoulakis et al. [2007]).

In practice, daily forecasts can be obtained with the following procedure: First, estimate parameters of the model from first estimation window W_E, then forecast return for day $E+1$. Second, carry out a PIT (probability integral transform) transformation of the forecasted return for day $E+1$. Third, map the observed return r_{E+1} for day $E+1$ to its percentile point on its forecasted density function P_{E+1}. Fourth, the estimation window is then moved up by one day to obtain the forecasted density function P_{E+2}. Next, the estimation window is moved forward by a step of one day until $T-1$ Figure (2), resulting in $T-E$ forecasted density function. Finally, carry out the Berkowitz test for paired observed returns r_i and P_i with $i = \{E+1,\ldots,T\}$ to see if r_i follows P_i.

Table 10: Comparison of ARMA-GARCH(model 1) and HMM-autoregressive noise model(model 2) via Berkowitz test.

	Berkowitz test	
	Test statistics	p-value
	daily return	
Model 1	7.3524	0.0615
Model 2	5.2749	0.1527
	1 hour return	
Model 1	2.7773	0.4273
Model 2	6.5951	0.0860
	5 minute return	
Model 1	4.3962	0.2217
Model 2	3.8844	0.2742
	1 minute return	
Model 1	10.2653	0.0164
Model 2	0.7875	0.8525

The Berkowitz test for standard ARMA(1,1)-GARCH(1,1) and ARMA(1,1)-GARCH(1,1) with regime-switching noise forecasting VaR for S&P 500 data are reported in Table (10). The length of the estimation window is 1000, the forecasting horizon is 100 steps, and the number of paths is 200. We can see for the daily return, hourly return and 5 minutes return that both models have $p-$values larger than 5 percent. For the high frequency data, 1 minute return, the $p-$value of standard ARMA(1,1)-GARCH(1,1) is 0.0164, which means the null hypothesis is rejected. The $p-$value of our HMM-autoregressive noise model is 0.8525, which is not rejected by the null hypothesis.

4 Conclusion

Studies show that the GARCH forecasts are too high in volatile periods. The reason for this error is that a GARCH model, which cannot capture the persistence of volatility at different positions, e.g. low position or high position, explains this volatility persistence as persistence of individual shocks. A parallel explanation for the persistence is that the structural change in the mean makes it more difficult to reject null hypothesis of the unit root, which means that permanent persistence of shocks occurs in the mean. A solution for this issue is to introduce a regime-switching element into the model.

We estimated and studied standard ARMA-GARCH, ARIMA-IGARCH and FARIMA-FIGARCH models by using the historical closing index values of the S&P 500 index until October 24, 2012, obtained from Bloomberg L.P. The daily, 1 hour, 30 minute, 5 minute, and 1 minute log-returns are calculated. Based on our study, global-MLE has better performance for large sample sizes, while quasi-MLE estimates are more reliable for small data sizes. We choose quasi-MLE for our empirical studies.

Parameters estimated for ARMA(1,1)-GARCH(1,1) model shows different patterns between daily and hourly returns and high frequency returns. We can see daily returns, hourly returns, and 30 minutes returns have a unit root phenomenon with almost $\alpha = 1$, which means this time series is not stationary. The unit root phenomena is especially obvious for 30 minute returns, where $\alpha = 1$. We do not observe unit root phenomena in 5 minute and 1 minute return series for the ARMA(1,1)-GARCH(1,1) model. Parameter estimates of ARMA(1,1)-IGARCH(1,1) for S&P 500 data demonstrate the IGARCH(1,1) model can capture the integrated feature. The parameter estimates of FARIMA(1,1)-FIGARCH(1,1) for S&P 500 data show that for a FARIMA(1,1) mean process, every time series has fractional d. For a FIGARCH(1,1) variance process, daily, hourly, 30 minute, and 1 minute returns have fractional d, and only 5 minute returns have $d = 1$. We also observe negative d in FARIMA, which can be explained by short memory.

Based on our examinations of the innovation process for ARMA-GARCH model, the white noise series u_t are not i.i.d. $\mathbf{N}(0,1)$ in high frequency data. Correlations occur within the white noise series, thus we improved the ARMA-GARCH models by introducing our new autoregressive regime-switching model into white noise series, which is named ARMA-GARCH with autoregressive regime-switching noise model. We forecast VaR models with both both original ARMA-GARCH and our new ARMA-GARCH with autoregressive regime-switching noises, then compared performance with Bernoulli and Berkowitz tests.

From the Berkowitz tests results, we can see for the daily, hourly and 5 minutes returns, both models have p-values larger than 5 percent. For high frequency data 1 minute returns, the p-value of standard ARMA(1,1)-GARCH(1,1) is 0.0164, which means the null hypothesis is rejected, while the p-value of our HMM-autoregressive noise model is 0.8525 and thus is not rejected. The Bernoulli test generates similar results. The results indicate that standard the ARMA-GARCH and our autoregressive-HMM-noises model can both perform well in daily S&P 500 log returns, while the autoregressive-HMM-noise model does better in high frequency data.

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